

SIMILARITY SOLUTIONS OF A REPLICATOR DYNAMICS EQUATION ASSOCIATED WITH A CONTINUUM OF PURE STRATEGIES

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ABSTRACT. We introduce a nonlinear degenerate parabolic equation containing a nonlocal term. The equation serves as a replicator dynamics model where the set of strategies is a continuum. In our model the payoff operator (which is the continuous analog of the payoff matrix) is nonsymmetric and, also, evolves with time. We are interested in solutions $u(t, x)$ of our equation which are positive and their integral (with respect to x) over the whole space is 1, for any $t > 0$. These solutions, being probability densities, can serve as time-evolving mixed strategies of a player. We show that for our model there is an one-parameter family of self-similar such solutions $u(t, x)$, all approaching the Dirac delta function $\delta(x)$ as $t \rightarrow 0^+$.

1. INTRODUCTION

Replicator dynamics models are popular models in evolutionary game theory. They have significant applications in economics, population biology, as well as in other areas of science [3, 4, 9, 10].

Replicator dynamics has been studied extensively in the finite dimensional case: Let $A = (a_{ij})$ be an $m \times m$ matrix. The typical replicator dynamics equation is [3]

$$u_t = [Au - (u, Au)]u = (Au)u - (u, Au)u, \quad (1.1)$$

where the subscript t in u_t denotes derivative with respect to the time variable t , (u, Au) denotes the usual inner product, i.e. the dot product, of the vectors u and Au , and $(Au)u$ is the vector whose i -th component is the product of the i -th components of (Au) and u (i.e. the “pointwise product” of two vectors). The matrix A is called the payoff matrix, while $S = \{1, \dots, m\}$ is the set of pure strategies (or options) and the vector

$$u = (u_1(t), \dots, u_m(t))^T,$$

is a probability (mass) function on S , meaning that

$$u_j(t) \geq 0, \quad \text{for } j = 1, \dots, m, \quad \text{and} \quad \sum_{j=1}^m u_j(t) = 1. \quad (1.2)$$

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It is easy to see that if the conditions (1.2) are satisfied for $t = 0$, then they are satisfied for all $t \geq 0$ (under the flow (1.1)). The vector u represents the mixed strategy of one member of the population, i.e. one player, against the rest of the population. The dependence of u in t allows the player to update her strategy, in order to increase her payoff.

Infinite dimensional versions of this evolutionary strategy models have been proposed, e.g., in [1] and [6] (see also [7] and the survey [3]) in connection to certain economic and biological applications. For instance, there are situations where (pure) strategies correspond to geographical points and hence it is natural to model the set of strategies by a continuum. However, the abstract form of the proposed equations does not allow one to obtain much insight, for example on the form of solutions.

To make some progress in this direction, the recent works [5, 8] initiated the study of the case where S is the set \mathbb{R}^d , $d \geq 1$, and the payoff operator A is the Laplacean operator Δ . Then the evolution law (1.1) becomes

$$u_t = [\Delta u - (u, \Delta u)]u, \quad (1.3)$$

where (\cdot, \cdot) denotes the usual inner product of the Hilbert space $L^2(\mathbb{R}^d)$ of the square-integrable functions defined on \mathbb{R}^d . References [5] and [8] deal only with the special problem of constructing an one-parameter family of self-similar solutions for (1.3), namely solutions u of the form

$$u(t, x) = t^{-\kappa} g(rt^{-\lambda}), \quad \text{where } r := |x| = \sqrt{x_1^2 + \cdots + x_d^2}.$$

A peculiar feature of these solutions is that all of them are probability densities on \mathbb{R}^d , for all $t > 0$, and approach the Dirac delta function $\delta(x)$ as $t \rightarrow 0^+$.

One criticism towards (1.3) is that the Laplacean operator Δ is a symmetric operator and, also, time-independent. A payoff operator A which is symmetric with respect to the inner product (\cdot, \cdot) corresponds to the case of a *partnership game*, where interests of both players coincide (see, e.g., [3]). These are unrealistic features for a payoff operator in a replicator dynamics model. For this reason, in the present work we consider a nonsymmetric and time-dependent payoff operator, namely

$$Au = A(t)u = \frac{\partial^2 u}{\partial x^2} + at^\gamma x \frac{\partial u}{\partial x}, \quad (1.4)$$

where γ is a specific constant (we will see later that $\gamma = -2/3$), while

$$a > 0 \quad (1.5)$$

is an arbitrary but fixed constant. Then, our replicator dynamics model is described by the equation

$$u_t = [Au - (u, Au)]u, \quad t > 0, \quad x \in \mathbb{R}, \quad (1.6)$$

with $u = u(t, x)$, where the operator A is given by (1.4).

In order for (1.6) to be a replicator dynamics model, we need to make sure that if we start with an initial condition which is a probability density function, namely

$$u(0, x) = f(x) \geq 0, \quad x \in \mathbb{R}, \quad \text{satisfying } \int_{\mathbb{R}} f(x) dx = 1, \quad (1.7)$$

then the solution $u(t, x)$ will remain a probability density function (as a function of x) for all $t > 0$ (as long as it exists). This can be justified as follows: Set

$$U(t) := \int_{-\infty}^{\infty} u(t, x) dx. \tag{1.8}$$

Then, integrating both sides of (1.6) over \mathbb{R} (with respect to x) gives

$$U'(t) = (u, Au)[1 - U(t)], \tag{1.9}$$

where we have assumed that the interchange of integration with respect to x and differentiation with respect to t is allowed. Now, in view of (1.9), the fact that $U(0) = 1$ (which follows from (1.7)) suggests that $U(t) \equiv 1$; in view of (1.8), this says that the integral of $u(t, x)$, with respect to x , on \mathbb{R} is 1 for every t .

Also, if $u(t, x)$ is a solution of (1.6) which exists for all $t > 0$ and, as a function of x , it is integrable on \mathbb{R} and positive for small t , then, due to the nature of equation (1.6) we have that $u(t, x)$ remains positive for all $t > 0$. We can, thus, conclude that the set of probability densities on \mathbb{R} is invariant under the flow (1.6).

2. SPECIAL SOLUTIONS

2.1. Self-Similar solutions. We consider the equation (1.6), where A is given by (1.4). Let us assume that the solution $u(t, x)$ satisfies

$$u(t, \cdot) \in H^1(\mathbb{R}) \quad \text{and} \quad \lim_{x \rightarrow \pm\infty} xu(t, x)^2 = 0. \tag{2.1}$$

By (1.4) we obtain

$$(Au, u) = \int_{-\infty}^{\infty} (Au)u dx = \int_{-\infty}^{\infty} u_{xx}u dx + at^\gamma \int_{-\infty}^{\infty} xu_xu dx.$$

Hence, in view of (2.1), integration by parts yields

$$(Au, u) = - \int_{-\infty}^{\infty} u_x^2 dx - \frac{a}{2}t^\gamma \int_{-\infty}^{\infty} u^2 dx, \tag{2.2}$$

thus, (1.6) is equivalent to

$$u_t = \left[u_{xx} + at^\gamma xu_x + \int_{-\infty}^{\infty} u_x^2 dx + \frac{a}{2}t^\gamma \int_{-\infty}^{\infty} u^2 dx \right] u. \tag{2.3}$$

We will look for self-similar solutions of (1.6), namely solutions $u(t, x)$ of the form

$$u(t, x) = t^{-\kappa}g(xt^{-\lambda}). \tag{2.4}$$

We set $s = xt^{-\lambda}$ (hence $x = st^\lambda$) so that $u(t, x)$ of (2.4) can be also written as $u(t, x) = t^{-\kappa}g(s)$. It follows that

$$u_x(t, x) = t^{-(\kappa+\lambda)}g'(s), \tag{2.5}$$

$$u_{xx}(t, x) = t^{-(\kappa+2\lambda)}g''(s). \tag{2.6}$$

Also,

$$\begin{aligned} u_t(t, x) &= -\kappa t^{-(\kappa+1)}g(s) - \lambda x t^{-\lambda} t^{-(\kappa+1)}g'(s) \\ &= -\kappa t^{-(\kappa+1)}g(s) - \lambda s t^{-(\kappa+1)}g'(s). \end{aligned} \tag{2.7}$$

Then, (2.2) gives

$$(Au, u) = -t^{-(2\kappa+\lambda)} \int_{-\infty}^{\infty} g'(s)^2 ds - \frac{a}{2}t^{\gamma+\lambda-2\kappa} \int_{-\infty}^{\infty} g(s)^2 ds. \tag{2.8}$$

Setting

$$K[g] := \int_{-\infty}^{\infty} g'(s)^2 ds, \quad \Lambda[g] := \int_{-\infty}^{\infty} g(s)^2 ds, \quad (2.9)$$

equation (2.8) becomes

$$(Au, u) = -t^{-(2\kappa+\lambda)} K[g] - \frac{a}{2} t^{\gamma+\lambda-2\kappa} \Lambda[g]. \quad (2.10)$$

Substituting (2.5), (2.6), (2.7), (2.10) in (2.3), we have

$$\begin{aligned} -\kappa g(s) - \lambda s g'(s) &= t^{1-\kappa-2\lambda} g''(s) g(s) + a s t^{1+\gamma-\kappa} g'(s) g(s) \\ &+ t^{1-2\kappa-\lambda} K[g] g(s) + \frac{a}{2} t^{1+\gamma+\lambda-2\kappa} \Lambda[g] g(s). \end{aligned} \quad (2.11)$$

The only way that the above is a meaningful equation is that it does not contain t , which means that

$$1 - \kappa - 2\lambda = 0, \quad 1 + \gamma - \kappa = 0, \quad 1 - 2\kappa - \lambda = 0, \quad 1 + \gamma + \lambda - 2\kappa = 0. \quad (2.12)$$

This gives

$$\gamma = -\frac{2}{3}, \quad \kappa = \frac{1}{3}, \quad \lambda = \frac{1}{3}. \quad (2.13)$$

Finally, we notice that, under (2.13), (2.4) gives

$$\int_{-\infty}^{\infty} u(t, x) dx = \int_{-\infty}^{\infty} t^{-\kappa} g(xt^{-\lambda}) dx = \int_{-\infty}^{\infty} t^{-1/3} g(xt^{-1/3}) dx = \int_{-\infty}^{\infty} g(s) ds,$$

which is independent of t . Thus, if we set

$$\int_{-\infty}^{\infty} g(s) ds = 1,$$

then

$$\int_{-\infty}^{\infty} u(t, x) dx = 1, \quad \text{for all } t \geq 0.$$

The following lemma summarizes what we have done so far.

Lemma 2.1. *If*

$$u(t, x) = t^{-\kappa} g(xt^{-\lambda}) \quad (2.14)$$

is a probability density in x and satisfies (2.3), then we must have

$$\gamma = -\frac{2}{3}, \quad \kappa = \frac{1}{3}, \quad \lambda = \frac{1}{3}, \quad (2.15)$$

$$g(s) \geq 0, \quad s \in \mathbb{R}, \quad \int_{-\infty}^{\infty} g(s) ds = 1, \quad (2.16)$$

$$g''(s)g(s) + a s g'(s)g(s) + K[g]g(s) + \frac{a}{2} \Lambda[g]g(s) + \frac{1}{3}g(s) + \frac{1}{3} s g'(s) = 0, \quad (2.17)$$

where

$$K[g] = \int_{-\infty}^{\infty} g'(s)^2 ds, \quad (2.18)$$

$$\Lambda[g] = \int_{-\infty}^{\infty} g(s)^2 ds. \quad (2.19)$$

Conversely, if (2.15)–(2.19) hold, then $u(t, x)$ given by (2.14) is a probability density in x and satisfies (2.3).

In view of (1.4), the fact that $\gamma = -2/3$ tells us that in the long run and as long as x stays bounded, the payoff operator $A(t)$ of our model approaches the symmetric operator $\partial^2/\partial x^2$. Next, we need to show that there exist function(s) $g(s)$ satisfying (2.16) and (2.17).

2.2. Auxiliary problem. Consider the problem

$$q''(s)q(s) + asq'(s)q(s) + \mu q(s) + \frac{1}{3}sq'(s) = 0, \tag{2.20}$$

$$q(0) = A > 0, \quad q'(0) = 0, \tag{2.21}$$

where μ is a real parameter satisfying

$$\mu > \frac{1}{3}. \tag{2.22}$$

Equation (2.20) can be written in the form

$$q''(s) + \left[\frac{1}{3q(s)} + a\right]sq'(s) + \mu = 0, \tag{2.23}$$

as long as $q(s) \neq 0$. Since $q(0) = A > 0$, the standard existence and uniqueness theorems for ordinary differential equation imply that there is a $\delta > 0$ such that (2.20)-(2.21) has a unique solution $q(s)$ for $s \in (-\delta, \delta)$. In fact, due to the invariance of (2.20) under the transformation $s \mapsto -s$ and the fact that $q'(0) = 0$, we must have

$$q(-s) = q(s), \quad s \in (-\delta, \delta).$$

Lemma 2.2. *The solution $q(s)$ of (2.20)-(2.21) exists for all $s \in \mathbb{R}$ and it is a strictly positive (even) function which is decreasing on $(0, \infty)$. Also,*

$$\lim_{s \rightarrow \infty} q(s) = \lim_{s \rightarrow \infty} q'(s) = 0, \tag{2.24}$$

$$\int_{-\infty}^{\infty} q'(s)^2 ds < \infty, \tag{2.25}$$

$$\int_{-\infty}^{\infty} q(s)^2 ds < \infty. \tag{2.26}$$

Furthermore,

$$\left(\mu - \frac{1}{3}\right) \int_{-\infty}^{\infty} q(s) ds = \int_{-\infty}^{\infty} q'(s)^2 ds + \frac{a}{2} \int_{-\infty}^{\infty} q(s)^2 ds. \tag{2.27}$$

Proof. Since q is an even function, it is enough to show that $q(s)$ exists for all $s \in [0, \infty)$. If this is not true, then either (i) (due the denominator $q(s)$ appearing in (2.23)) there must be an $s_1 \in (0, \infty)$ such that $q(s_1) = 0$, while $q(s) > 0$ for all $s \in [0, s_1)$, or (ii) by a well-known theorem in the theory of ordinary differential equations [2] there must exist some $b > 0$ such that

$$\lim_{s \rightarrow b^-} [|q'(s)| + |q(s)|] = \infty.$$

Let us first exclude the case (i). Suppose that there is an $s_1 > 0$ such that $q(s_1) = 0$, while $q(s) > 0$ for all $s \in [0, s_1)$. Then, $q'(s)$ is negative in $(0, s_1)$. If this were not true, then there should exist a $s_2 \in (0, s_1)$ such that $q'(s_2) = 0$, while $q'(s) < 0$ for all $s \in (0, s_2)$. This would imply that $q''(s_2) \geq 0$. However, by (2.23)

$$q''(s_2) = -\mu < 0,$$

a contradiction.

Now, if we integrate (2.23) from 0 to $s \in (0, s_1)$ and use the fact that $q'(0) = 0$, we obtain

$$q'(s) + \frac{1}{3} \int_0^s \xi [\ln q(\xi)]' d\xi + asq(s) - a \int_0^s q(\xi) d\xi + \mu s = 0$$

or

$$q'(s) = a \int_0^s q(\xi) d\xi - asq(s) - \mu s + \frac{1}{3} \int_0^s \ln q(\xi) d\xi - \frac{1}{3} s \ln q(s). \quad (2.28)$$

Since $q(s) > 0$, while $q'(s) < 0$ for $s \in (0, s_1)$, $q(s)$ is decreasing in $[0, s_1)$ and, consequently, $\ln q(s)$ is decreasing in $[0, s_1)$. Hence, the function

$$f(s) := -\frac{1}{3} \ln q(s) \quad (2.29)$$

is increasing in $(0, s_1)$ and

$$\lim_{s \rightarrow s_1^-} f(s) = -\frac{1}{3} \lim_{s \rightarrow s_1^-} [\ln q(s)] = \infty.$$

Then, it is not hard to show (see, e.g., [5, Proposition A.1]) that

$$\lim_{s \rightarrow s_1^-} \left(s f(s) - \int_0^s f(\xi) d\xi \right) = \infty,$$

i.e. (recall (2.29))

$$\lim_{s \rightarrow s_1^-} \left(-\frac{1}{3} s \ln q(s) + \frac{1}{3} \int_0^s \ln q(\xi) d\xi \right) = \infty.$$

Hence, (2.28) gives

$$\begin{aligned} \lim_{s \rightarrow s_1^-} q'(s) &= \lim_{s \rightarrow s_1^-} \left[a \int_0^s q(\xi) d\xi - asq(s) - \mu s + \frac{1}{3} \int_0^s \ln q(\xi) d\xi - \frac{1}{3} s \ln q(s) \right] \\ &= \infty, \end{aligned}$$

which is impossible, since, as we have seen, q' stays negative in $(0, s_1)$. Hence such an s_1 cannot exist, i.e. q never vanishes and consequently, q' also never vanishes. In particular, $q(s) > 0$, $q'(s) < 0$ (hence, q is decreasing), and, therefore, $0 < q(s) < q(0) = A$, for all $s > 0$ for which $q(s)$ and $q'(s)$ exist.

Now suppose that there is a $b > 0$ such that

$$\lim_{s \rightarrow b^-} [|q'(s)| + |q(s)|] = \infty. \quad (2.30)$$

By the previous discussion, the only way for (2.30) to happen is

$$\lim_{s \rightarrow b^-} q'(s) = -\infty.$$

Then

$$\liminf_{s \rightarrow b^-} q''(s) = -\infty,$$

which contradicts (2.23). Thus q' remains finite and strictly negative on $(0, \infty)$ while q is strictly positive and strictly decreasing on $(0, \infty)$. Due to the evenness of q , we must have also $q(s) > 0$ for all $s < 0$. Hence, $q(s) > 0$ for all $s \in \mathbb{R}$.

From the previous discussion it follows that $\lim_{s \rightarrow \infty} q(s) = L$, namely

$$q(s) = L + o(1) \quad \text{as } s \rightarrow \infty, \quad (2.31)$$

where $L \in [0, A)$. To continue, let us suppose $L > 0$. Then, the above formula implies that, as $s \rightarrow \infty$,

$$\ln q(s) = \ln(L + o(1)) = \ln L(1 + o(1)) = \ln L + o(1). \tag{2.32}$$

Using (2.31) and (2.32) in (2.28), we obtain

$$\begin{aligned} q'(s) &= a \int_0^s [\ln L + o(1)] d\xi - as[\ln L + o(1)] \\ &\quad - \mu s + \frac{1}{3} \int_0^s [\ln L + o(1)] d\xi - \frac{1}{3}s[\ln L + o(1)], \end{aligned}$$

which implies

$$q'(s) = as \ln L + o(s) - as \ln L + o(s) - \mu s + \frac{1}{3}s \ln L + o(s) - \frac{1}{3}s \ln L + o(s),$$

i.e.

$$q'(s) = -\mu s + o(s) \quad \text{as } s \rightarrow \infty,$$

which contradicts (2.31). Therefore $L = 0$, i.e.,

$$\lim_{s \rightarrow \infty} q(s) = 0. \tag{2.33}$$

We continue by noticing that

$$\int_0^\infty q'(s) ds = \lim_{s \rightarrow \infty} q(s) - q(0) = -A, \tag{2.34}$$

hence $q' \in L_1(\mathbb{R})$ (since q' is odd and negative). Suppose

$$\liminf_{s \rightarrow \infty} q'(s) < 0. \tag{2.35}$$

Then, in view of (2.34) there is a sequence $s_n \rightarrow \infty$ such that q' attains a local minimum at s_n for all n and

$$\lim_{n \rightarrow \infty} q'(s_n) = -\delta, \quad \text{for some } \delta > 0. \tag{2.36}$$

But, since $q'(s_n)$ is a local minimum we must have $q''(s_n) = 0$, hence (2.23) gives

$$\left[\frac{1}{3q(s_n)} + a \right] s_n q'(s_n) = -\mu \quad \text{or} \quad q'(s_n) = -\frac{3\mu q(s_n)}{[1 + 3aq(s_n)]s_n},$$

thus

$$\lim_{n \rightarrow \infty} q'(s_n) = 0,$$

contradicting (2.36) and hence (2.35). We have, thus, established that

$$\lim_{s \rightarrow \infty} q'(s) = 0. \tag{2.37}$$

This, together with the fact that q' is odd and integrable, implies $q' \in L_2(\mathbb{R})$, i.e.

$$\int_{-\infty}^\infty q'(s)^2 ds < \infty. \tag{2.38}$$

Finally, (2.23) implies

$$\int_0^s q(\xi)q''(\xi) d\xi + \frac{1}{3} \int_0^s \xi q'(\xi) d\xi + a \int_0^s \xi q(\xi)q'(\xi) d\xi + \mu \int_0^s q(\xi) d\xi = 0.$$

Integrating by parts the first two terms above and using the fact that $q'(0) = 0$ we have

$$q(s)q'(s) - \int_0^s q'(\xi)^2 d\xi + \frac{1}{3}sq(s) + \frac{a}{2}sq(s)^2 - \frac{a}{2} \int_0^s q(\xi)^2 d\xi + \left(\mu - \frac{1}{3}\right) \int_0^s q(\xi) d\xi = 0.$$

Since $a > 0$, $s \geq 0$, and $q(s) > 0$, the above equation implies

$$\left(\mu - \frac{1}{3}\right) \int_0^s q(\xi) d\xi \leq \int_0^s q'(\xi)^2 d\xi + \frac{a}{2} \int_0^s q(\xi)^2 d\xi - q(s)q'(s). \quad (2.39)$$

(1) If we suppose that

$$\int_0^\infty q(\xi) d\xi = \infty, \quad (2.40)$$

then (2.39), by (2.38), implies

$$\int_0^\infty q(\xi)^2 d\xi = \infty.$$

For typographical convenience we set

$$M := \int_0^\infty q'(\xi)^2 d\xi < \infty.$$

Then, formula (2.39) can be written as

$$\left(\mu - \frac{1}{3}\right) \int_0^s q(\xi) d\xi \leq M + \frac{a}{2} \int_0^s q(\xi)^2 d\xi - q(s)q'(s).$$

Furthermore, $q(\xi) > 0$, for all $\xi \in (0, \infty)$, and hence from the above inequality we have

$$\mu - \frac{1}{3} \leq \frac{M + \frac{a}{2} \int_0^s q(\xi)^2 d\xi - q(s)q'(s)}{\int_0^s q(\xi) d\xi}. \quad (2.41)$$

Now, under (2.40), L'Hopital's Rule together with the fact that $q(s)q'(s)$ is bounded yield

$$\lim_{s \rightarrow \infty} \frac{M + \frac{a}{2} \int_0^s q(\xi)^2 d\xi - q(s)q'(s)}{\int_0^s q(\xi) d\xi} = \lim_{s \rightarrow \infty} \frac{\frac{a}{2}q(s)^2}{q(s)} = \frac{a}{2} \lim_{s \rightarrow \infty} q(s) = 0.$$

But, then, from (2.41) we have $\mu \leq 1/3$, which contradicts the fact that $\mu > 1/3$. Consequently,

$$\int_0^\infty q(\xi) d\xi < \infty. \quad (2.42)$$

(2) The function q is strictly positive and strictly decreasing on $(0, \infty)$ with $0 < q(s) < A$, for all $s > 0$. Since

$$\lim_{s \rightarrow \infty} q(s) = 0,$$

there is a $s_0 > 0$ such that $0 < q(s)^2 < q(s)$ for all $s \geq s_0$. Hence

$$\lim_{s \rightarrow \infty} q(s)^2 = 0 \quad \text{and} \quad 0 < \int_{s_0}^\infty q(s)^2 ds < \int_{s_0}^\infty q(s) ds.$$

Thus, from (2.42) it follows that

$$\int_0^\infty q(s)^2 ds < \infty. \quad (2.43)$$

From (2.38), (2.42), and (2.43) we have

$$\begin{aligned} & \lim_{s \rightarrow \infty} \left[\frac{1}{3}sq(s) + \frac{a}{2}sq(s)^2 \right] \\ &= \int_0^\infty q'(\xi)^2 d\xi + \frac{a}{2} \int_0^\infty q(\xi)^2 d\xi - \left(\mu - \frac{1}{3}\right) \int_0^\infty q(\xi) d\xi < \infty. \end{aligned}$$

Thus,

$$\lim_{s \rightarrow \infty} \left[\frac{1}{3}sq(s) + \frac{a}{2}sq^2(s) \right] = \tilde{L} \in \mathbb{R}.$$

If $\tilde{L} \neq 0$, then the above limit tells us that $T(s) := (1/3)q(s) + (a/2)q(s)^2$ is asymptotic to \tilde{L}/s , contradicting the fact that $T(s)$ is integrable. Therefore, $\tilde{L} = 0$. Therefore,

$$\left(\mu - \frac{1}{3}\right) \int_0^\infty q(s) ds = \int_0^\infty q'(s)^2 ds + \frac{a}{2} \int_0^\infty q(s)^2 ds,$$

from which (2.27) follows immediately. The proof of this key lemma is now complete. □

2.3. Construction of the self-similar solutions.

Lemma 2.3. *Let $q(s)$ be the solution of the problem (2.20)-(2.21). Then*

$$\|q'\|_\infty \leq \mu \sqrt{\frac{3A}{1 + 3aA}}, \tag{2.44}$$

where $\|\cdot\|_\infty$ denotes the sup-norm, as usual. Also

$$\int_0^\infty q(s) ds \geq \frac{A^{3/2}\sqrt{1 + 3aA}}{2\sqrt{3}\mu}, \tag{2.45}$$

$$\int_0^\infty q(s)^2 ds \geq \frac{A^{5/2}\sqrt{1 + 3aA}}{3\sqrt{3}\mu}. \tag{2.46}$$

Proof. The function q' is odd, hence

$$\|q'\|_\infty = \sup\{|q'(s)| : s \geq 0\}.$$

Since $q'(s) < 0$ in $(0, \infty)$ with

$$q'(0) = 0 = \lim_{s \rightarrow \infty} q'(s),$$

it follows that q' attains its absolute minimum at some $s_m \in (0, \infty)$, and hence

$$\|q'\|_\infty = \sup\{-q'(s) : s \geq 0\} = -q'(s_m) = |q'(s_m)|.$$

Also, $q''(s_m) = 0$, thus (2.23) implies

$$q'(s_m) = -\frac{\mu}{\left[\frac{1}{3q(s_m)} + a\right]s_m},$$

therefore

$$\|q'\|_\infty = -q'(s_m) = \frac{\mu}{\left[\frac{1}{3q(s_m)} + a\right]s_m}.$$

But $q(s)$ is decreasing in $[0, \infty)$, while $q(0) = A$ and $s_m \in (0, \infty)$, hence

$$\frac{\mu}{\left[\frac{1}{3q(s_m)} + a\right]s_m} \leq \frac{\mu}{\left[\frac{1}{3A} + a\right]s_m} < \frac{\mu}{as_m},$$

and then

$$\|q'\|_\infty \leq \frac{3\mu A}{(1+3aA)s_m} < \frac{\mu}{as_\mu}. \quad (2.47)$$

Also, by (2.23)

$$q''(s) + \mu = -\left[\frac{1}{3q(s)} + a\right]sq'(s) \geq 0 \quad \text{for all } s \geq 0,$$

while $q'(0) = 0$. Thus, we must have

$$q'(s) \geq -\mu s \quad \text{for all } s \geq 0,$$

in particular

$$\|q'\|_\infty = -q'(s_m) \leq \mu s_m. \quad (2.48)$$

By combining (2.47) and (2.48) we obtain

$$\|q'\|_\infty \leq \min\left\{\frac{3\mu A}{(1+3aA)s_m}, \mu s_m\right\}.$$

But, no matter what s_m is, the quantity $\min\{3\mu A(1+3aA)^{-1}s_m^{-1}, \mu s_m\}$ (since the first term is decreasing in s_m while the second is increasing) is always at most N , where

$$N := \mu s^* = \frac{3\mu A}{(1+3aA)s^*}.$$

Then

$$s^* = \sqrt{\frac{3A}{1+3aA}} \quad \text{and} \quad N = \mu \sqrt{\frac{3A}{1+3aA}}.$$

Thus, $\|q'\|_\infty \leq N$, which is (2.44). Furthermore,

$$\|q'\|_\infty \geq -q'(s) \quad \text{for all } s \geq 0,$$

hence

$$q(s) \geq q(0) - s\|q'\|_\infty = A - s\|q'\|_\infty \quad \text{for all } s \geq 0.$$

Then, by (2.44) we have

$$q(s) \geq A - s\mu \sqrt{\frac{3A}{1+3aA}} \quad \text{for all } s \geq 0, \quad (2.49)$$

in particular for

$$0 \leq s \leq \frac{\sqrt{(1+3aA)A}}{\mu\sqrt{3}},$$

since $q(s) > 0$ for all $s \geq 0$. Then (see (2.49)),

$$\begin{aligned} \int_0^\infty q(s) ds &\geq \int_0^{\frac{\sqrt{(1+3aA)A}}{\mu\sqrt{3}}} q(s) ds \\ &\geq \int_0^{\frac{\sqrt{(1+3aA)A}}{\mu\sqrt{3}}} \left(A - s\mu \sqrt{\frac{3A}{1+3aA}}\right) ds \\ &= \frac{A^{3/2}\sqrt{1+3aA}}{2\sqrt{3}\mu}, \end{aligned}$$

which is (2.45). Finally, from (2.49) we have

$$\int_0^\infty q(s)^2 ds \geq \int_0^{\frac{\sqrt{(1+3aA)A}}{\mu\sqrt{3}}} q(s)^2 ds$$

$$\begin{aligned} &\geq \int_0^{\frac{\sqrt{(1+3aA)A}}{\mu\sqrt{3}}} \left(A - s\mu\sqrt{\frac{3A}{1+3aA}} \right)^2 ds \\ &= \frac{A^{5/2}\sqrt{1+3aA}}{3\sqrt{3}\mu}, \end{aligned}$$

which is (2.46). □

Corollary 2.4. *If $q(s)$ satisfies (2.20)-(2.21), then*

$$\lim_{A \rightarrow \infty} \int_{-\infty}^{\infty} q(s) ds = \infty, \tag{2.50}$$

$$\lim_{A \rightarrow \infty} \int_{-\infty}^{\infty} q(s)^2 ds = \infty, \tag{2.51}$$

$$\lim_{A \rightarrow 0^+} \int_{-\infty}^{\infty} q'(s)^2 ds = 0. \tag{2.52}$$

Proof. By (2.45) and the evenness of $q(s)$ we have

$$\int_{-\infty}^{\infty} q(s) ds = 2 \int_0^{\infty} q(s) ds \geq \frac{2A^{3/2}\sqrt{1+3aA}}{3\sqrt{3}\mu}, \tag{2.53}$$

and since

$$\lim_{A \rightarrow \infty} \frac{2A^{3/2}\sqrt{1+3aA}}{3\sqrt{3}\mu} = \infty,$$

we obtain that (2.53) implies

$$\lim_{A \rightarrow \infty} \int_{-\infty}^{\infty} q(s) ds = \infty.$$

The function q is even, and hence q^2 is even too. Furthermore, (2.46) implies

$$\int_{-\infty}^{\infty} q(s)^2 ds = 2 \int_0^{\infty} q(s)^2 ds \geq \frac{2A^{5/2}\sqrt{1+3aA}}{3\sqrt{3}\mu}, \tag{2.54}$$

hence, from (2.54) we have

$$\lim_{A \rightarrow \infty} \int_{-\infty}^{\infty} q(s)^2 ds = \infty.$$

Recall that $-q'(s) > 0$ (and $-q'(s) \leq \|q'\|_{\infty}$) for all $s \in (0, \infty)$. Thus, by (2.24) we obtain

$$0 \leq \int_0^{\infty} q'(s)^2 ds \leq -\|q'\|_{\infty} \int_0^{\infty} q'(s) ds = A\|q'\|_{\infty},$$

and, consequently, by using (2.44) we have

$$0 \leq \int_0^{\infty} q'(s)^2 ds \leq A\mu\sqrt{\frac{3A}{1+3aA}} = \sqrt{\frac{3A^3\mu^2}{1+3aA}}. \tag{2.55}$$

Finally, since $q'(s)$ is odd and hence $q'(s)^2$ is even, by using (2.55) we obtain

$$0 \leq \int_{-\infty}^{\infty} q'(s)^2 ds = 2 \int_0^{\infty} q'(s)^2 ds \leq 2\sqrt{\frac{3A^3\mu^2}{1+3aA}}, \tag{2.56}$$

which implies (2.52). □

Lemma 2.5. *If $q(s)$ is the solution of (2.20)-(2.21), then*

$$\frac{q(1)e^{3aq(1)}}{s^{3\mu} \exp[3\mu + \frac{3\mu}{s} \sqrt{\frac{3A}{1+3aA}}]} < q(s)e^{3aq(s)} \leq \frac{Ae^{3A(1+a)}}{s^{3\mu} \exp[3\mu - \frac{3\mu}{s} \sqrt{\frac{3A}{1+3aA}}]} \quad (2.57)$$

for all $s \geq 1$.

Proof. We consider the function

$$F(s) := -\frac{1}{3} \int_0^s \ln q(\xi) d\xi, \quad s \in \mathbb{R}.$$

Since $q(s)$ is decreasing in $(0, \infty)$, for $0 \leq \xi \leq 1$ we have

$$q(1) \leq q(\xi) \leq q(0) = A, \quad (2.58)$$

$$\ln q(1) \leq -3F(1) \leq \ln A. \quad (2.59)$$

Furthermore, (2.58) implies

$$aq(1) \leq a \int_0^1 q(\xi) d\xi \leq aA. \quad (2.60)$$

By (2.28), we have

$$q'(s) = a \int_0^s q(\xi) d\xi - asq(s) - \mu s - F(s) + sF'(s), \quad (2.61)$$

i.e.

$$sF'(s) - F(s) = q'(s) + \mu s + asq(s) - a \int_0^s q(\xi) d\xi.$$

Thus, for $s \neq 0$ we have

$$\left(\frac{F(s)}{s}\right)' = \frac{q'(s)}{s^2} + \frac{\mu}{s} + a\left(\frac{1}{s} \int_0^s q(\xi) d\xi\right)'. \quad (2.62)$$

We pick an $s \geq 1$ and integrate both sides of the equation (2.62) from 1 to s . This leads to

$$\frac{F(s)}{s} - F(1) = \int_1^s \frac{q'(\xi)}{\xi^2} d\xi + \mu \ln s + a \frac{1}{s} \int_0^s q(\xi) d\xi - a \int_0^1 q(\xi) d\xi,$$

or

$$\int_1^s \frac{q'(\xi)}{\xi^2} d\xi = \frac{F(s)}{s} - F(1) - \mu \ln s - \frac{a}{s} \int_0^s q(\xi) d\xi + a \int_0^1 q(\xi) d\xi. \quad (2.63)$$

Since $q'(s) < 0$, for all $s \in (0, \infty)$,

$$0 \geq \int_1^s \frac{q'(\xi)}{\xi^2} d\xi \geq \int_1^s q'(\xi) d\xi \geq \int_0^\infty q'(\xi) d\xi = \lim_{s \rightarrow \infty} q(s) - q(0) = -A,$$

hence, (2.63) gives

$$0 \geq \frac{F(s)}{s} - F(1) - \mu \ln s - \frac{a}{s} \int_0^s q(\xi) d\xi + a \int_0^1 q(\xi) d\xi \geq -A,$$

or

$$\begin{aligned}
 & F(1) + \mu \ln s + \frac{a}{s} \int_0^s q(\xi) d\xi - a \int_0^1 q(\xi) d\xi \\
 & \geq \frac{F(s)}{s} \\
 & \geq F(1) + \mu \ln s + \frac{a}{s} \int_0^s q(\xi) d\xi - a \int_0^1 q(\xi) d\xi - A.
 \end{aligned} \tag{2.64}$$

By (2.61) we have

$$\frac{F(s)}{s} = \frac{a}{s} \int_0^s q(\xi) d\xi - aq(s) - \mu + F'(s) - \frac{q'(s)}{s}. \tag{2.65}$$

Then, (2.64) combined with (2.65) implies

$$\begin{aligned}
 F(1) + \mu \ln s - a \int_0^1 q(\xi) d\xi & \geq F'(s) - aq(s) - \mu - \frac{q'(s)}{s} \\
 & \geq F(1) + \mu \ln s - a \int_0^1 q(\xi) d\xi - A,
 \end{aligned}$$

or

$$\begin{aligned}
 & \frac{q'(s)}{s} + F(1) + \mu \ln s - a \int_0^1 q(\xi) d\xi \\
 & \geq F'(s) - aq(s) - \mu \\
 & \geq \frac{q'(s)}{s} + F(1) + \mu \ln s - a \int_0^1 q(\xi) d\xi - A.
 \end{aligned} \tag{2.66}$$

By (2.44) and the fact that $\|q'\|_\infty \geq -q'(s)$ for all $s \in (0, \infty)$, we have

$$-q'(s) \leq \|q'\|_\infty \leq \mu \sqrt{\frac{3A}{1+3aA}}, \quad \text{for all } s > 0,$$

which implies

$$-\frac{\mu}{s} \sqrt{\frac{3A}{1+3aA}} \leq \frac{q'(s)}{s} < 0 < -\frac{q'(s)}{s} \leq \frac{\mu}{s} \sqrt{\frac{3A}{1+3aA}}. \tag{2.67}$$

Hence, by using (2.67) in (2.66) we obtain

$$\begin{aligned}
 & \frac{\mu}{s} \sqrt{\frac{3A}{1+3aA}} + F(1) + \ln(s^\mu) - a \int_0^1 q(\xi) d\xi \\
 & > F'(s) - aq(s) - \mu \\
 & \geq -\frac{\mu}{s} \sqrt{\frac{3A}{1+3aA}} + F(1) + \ln(s^\mu) - a \int_0^1 q(\xi) d\xi - A
 \end{aligned}$$

for all $s > 0$. Now, by invoking (2.60) the inequalities above give

$$\begin{aligned}
 & \frac{\mu}{s} \sqrt{\frac{3A}{1+3aA}} + F(1) + \ln(s^\mu) - aq(1) \\
 & > F'(s) - aq(s) - \mu \\
 & \geq -\frac{\mu}{s} \sqrt{\frac{3A}{1+3aA}} + F(1) + \ln(s^\mu) - aA - A.
 \end{aligned}$$

Using the definition of $F(s)$ the above inequalities can be written in the form

$$\begin{aligned} & \frac{\mu}{s} \sqrt{\frac{3A}{1+3aA}} + F(1) + \ln(s^\mu) - aq(1) \\ & > -\frac{1}{3} \ln q(s) - aq(s) - \mu \\ & \geq -\frac{\mu}{s} \sqrt{\frac{3A}{1+3aA}} + F(1) + \ln(s^\mu) - aA - A. \end{aligned}$$

Recalling (2.59), the above inequalities imply

$$\begin{aligned} & 3aq(1) + \ln q(1) - \frac{3\mu}{s} \sqrt{\frac{3A}{1+3aA}} + \ln(s^{-3\mu}) - 3\mu \\ & < \ln q(s) + 3aq(s) \\ & \leq 3aA + 3A + \frac{3\mu}{s} \sqrt{\frac{3A}{1+3aA}} + \ln A + \ln(s^{-3\mu}) - 3\mu, \end{aligned}$$

which by exponentiation yields (2.57). \square

Corollary 2.6. *Let $q(s)$ satisfy (2.20)-(2.21) (in particular $q(0) = A$). Then, as a function of A , the quantity*

$$I(A) := \int_{-\infty}^{\infty} q(s) ds \quad (2.68)$$

is continuous in $(0, \infty)$.

Proof. Let $q(s; A) := q(s)$ be the unique solution of the problem (2.20)-(2.21). By the standard theorem of ordinary differential equations on continuous dependence on the parameters we have that $q(s; A)$ is continuous in A for all $A > 0$. For fixed A_1, A_2 with $0 < A_1 < A_2 < \infty$, the second inequality in (2.57), the monotonicity of q and the condition $\mu > 1/3$ imply that the family $\{q(\cdot; A) : A \in [A_1, A_2]\}$ is dominated by the integrable function $H(s) = h(|s|)$, $s \in \mathbb{R}$, where

$$h(s) := \begin{cases} A_2 e^{3aA_2}, & 0 \leq s \leq 1; \\ \frac{A_2 e^{3A_2(1+a)}}{s^{3\mu} \exp[3\mu - 3\mu \sqrt{\frac{3A_2}{1+3aA_1}}]}, & s \geq 1. \end{cases}$$

Hence, the continuity of $I(A)$ follows by invoking the dominated convergence Theorem. \square

We are now ready to state our main result.

Theorem 2.7. *Let $\gamma = -2/3$. Then, for each number $\beta \in (0, \infty)$ there is a self-similar solution of (2.3) (which is equivalent to (1.6)), namely a solution u of the form $u(t, x) = t^{-1/3} g(xt^{-1/3})$, where $g(s)$ satisfies (2.16), (2.17), (2.18), and (2.19), such that*

$$\beta = K[g] + \frac{a}{2} \Lambda[g],$$

where $K[g]$ and $\Lambda[g]$ are given by (2.18) and (2.19), respectively.

Proof. As in the proof of Corollary 2.6 let $q(s) = q(s; A)$ be the unique solution of the problem (2.20)-(2.21) with $\mu = \beta + (1/3)$; that is,

$$q''(s)q(s) + a sq'(s)q(s) + \beta q(s) + \frac{1}{3} q(s) + \frac{1}{3} sq'(s) = 0, \quad s \in \mathbb{R},$$

$$q(0) = A > 0, \quad q'(0) = 0,$$

and set

$$Q(A) := \int_{-\infty}^{\infty} q'(s; A)^2 ds + \frac{a}{2} \int_{-\infty}^{\infty} q(s; A)^2 ds. \quad (2.69)$$

Then by (2.27),

$$Q(A) = \beta \int_{-\infty}^{\infty} q(s; A) ds = \beta I(A)$$

(recall (2.68)), hence Corollary 2.6 tells us that $Q(A)$ is continuous on $(0, \infty)$. Furthermore, by (2.50) of Corollary 2.4 we have

$$\lim_{A \rightarrow 0^+} Q(A) = 0, \quad \lim_{A \rightarrow \infty} Q(A) = \infty. \quad (2.70)$$

Thus, $Q(A)$ takes every value between 0 and ∞ . In particular, for each number $\beta \in (0, \infty)$ there is an $A = A_\beta$ such that

$$Q(A_\beta) = \beta.$$

Set $g(s) := q(s; A_\beta)$. Then

$$K[g] + \frac{a}{2} \Lambda[g] = \int_{-\infty}^{\infty} q'(s; A)^2 ds + \frac{a}{2} \int_{-\infty}^{\infty} q(s; A)^2 ds = Q(A_\beta) = \beta,$$

hence $g(s)$ satisfies (2.17)-(2.18). Furthermore, by (2.27),

$$\begin{aligned} \int_{-\infty}^{\infty} g(s) ds &= \int_{-\infty}^{\infty} q(s; A_\beta) ds \\ &= \frac{1}{\beta} \left[\int_{-\infty}^{\infty} q'(s; A_\beta)^2 ds + \frac{a}{2} \int_{-\infty}^{\infty} q(s; A_\beta)^2 ds \right] \\ &= \frac{1}{\beta} Q(A_\beta) = 1, \end{aligned}$$

and, therefore, $g(s)$ also satisfies (2.16). \square

Clearly, all these self-similar solutions $u(t, x)$ are probability density functions on \mathbb{R} . A peculiar feature of these solutions is that they all approach the Dirac delta function $\delta(x)$ as $t \rightarrow 0^+$.

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