Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 216, pp. 1–28.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SOLUTIONS TO SYSTEMS OF EQUATIONS MODELLING COMPRESSIBLE FLUID FLOW

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Abstract. We study a system of nonlinear equations that models the flow of a compressible, inviscid, barotropic fluid. The value of the density is known at a point in the spatial domain and the initial value of the velocity is known. The purpose of this paper is to prove the existence of a unique, classical solution to this system of equations under periodic boundary conditions.

1. Introduction

In this article, we consider the following system of equations which arises from a model of the multi-dimensional flow of a compressible, inviscid, barotropic fluid:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \rho^{-1} \nabla p = 0 \tag{1.1}$$

$$\nabla \cdot \mathbf{v} = -\psi^{-1} \frac{\partial \psi}{\partial t} - \psi^{-1} \mathbf{v} \cdot \nabla \psi \tag{1.2}$$

$$p = \hat{p}(\rho) \tag{1.3}$$

where $\mathbf{v}$ is the velocity, ρ is the density, p is the pressure, and ψ is a given positive smooth function.

The fluid's thermodynamic state is determined by the density ρ, and the pressure p is determined from the density by an equation of state $p = \hat{p}(\rho)$. We assume that p is a given smooth function of ρ, and we assume that p and $\frac{dp}{d\rho}$ are positive functions. It follows from (1.1) and (1.3) that

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \rho^{-1} \hat{p}'(\rho) \nabla \rho = 0 \tag{1.4}$$

The density ρ is a positive function which satisfies the condition that $\rho(\mathbf{x}_0, t) = b(t)$, where $\mathbf{x}_0$ is a given point in the domain Ω and where b is a given positive smooth function of t. The initial condition for the velocity is $\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x})$.

The purpose of this paper is to prove the existence of a unique classical solution $\mathbf{v}$, ρ to the system of equations (1.2), (1.4), where $\rho(\mathbf{x}_0, t) = b(t)$ and $\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x})$, for $0 \le t \le T$ and under periodic boundary conditions. That is, we choose for our domain the N-dimensional torus $\mathbb{T}^N$, where $N = 2$ or $N = 3$.

2010 *Mathematics Subject Classification.* 35A05.
Key words and phrases. Existence; uniqueness; compressible; fluid.

Submitted February 28, 2015. Published August 17, 2015.

We will also present a proof that the standard conservation of mass equation $\frac{\partial\rho}{\partial t}+\mathbf{v}\cdot\nabla\rho+\rho\nabla\cdot\mathbf{v}=0$ is approximately satisfied by the solution ρ, $\mathbf{v}$ to equations (1.2), (1.4). Equations (1.2), (1.4) are an approximation to Euler's equations for a compressible, barotropic fluid (see, e.g., [2]).

In our previous related work, the existence of a unique classical solution to a similar system of equations modelling the flow of an incompressible, barotropic fluid was proven in [3], with the condition that $\rho(\mathbf{x}_0,t)$ was given and where $\nabla\cdot\mathbf{v}=0$. In this paper, the fluid is compressible and $\nabla\cdot\mathbf{v}=-\psi^{-1}\frac{\partial\psi}{\partial t}-\psi^{-1}\mathbf{v}\cdot\nabla\psi$. The proof uses new inequalities which are proven in Section 4.

The proof of the existence theorem is based on the method of successive approximations, in which an iteration scheme, based on solving a linearized version of the equations, is designed and convergence of the sequence of approximating solutions to a unique solution satisfying the nonlinear equations is proven. The framework of the proof follows one used, for example, by A. Majda to prove the existence of a solution to a system of conservation laws [8]. Embid [7] also uses the same general framework to prove the existence of a solution to equations for zero Mach number combustion. Under this framework, the convergence proof is presented in two steps. In the first step, we prove uniform boundedness of the approximating sequence of solutions in a high Sobolev space norm. The second step is to prove contraction of the sequence in a low Sobolev space norm. Standard compactness arguments finish the proof.

The paper is organized as follows. First the main result, Theorem 2.1, is presented and proven in the next section. Next, we present a proof that the standard conservation of mass equation $\frac{\partial\rho}{\partial t}+\mathbf{v}\cdot\nabla\rho+\rho\nabla\cdot\mathbf{v}=0$ is approximately satisfied by the solution ρ, $\mathbf{v}$ to equations (1.2), (1.4). Finally, lemmas supporting the proof of the existence theorem are provided in Section 3 (which presents a proof of the existence of a solution to the linearized equations used in the iteration scheme) and in Section 4 (which contains other lemmas used in the proof of the theorem).

2. Existence theorem

The main tools utilized in the existence proof are a priori estimates. We will work with the Sobolev space $H^s(\Omega)$ (where $s\geq 0$ is an integer) of real-valued functions in $L^2(\Omega)$ whose distribution derivatives up to order s are in $L^2(\Omega)$, with norm given by $\|u\|_s^2=\sum_{|\alpha|\leq s}\int_\Omega|D^\alpha u|^2d\mathbf{x}$ and inner product $(u,v)_s=\sum_{|\alpha|\leq s}\int_\Omega(D^\alpha u)\cdot(D^\alpha v)d\mathbf{x}$. Here, we adopt the standard multi-index notation. We will also use the notation $\|u\|_s^2=\sum_{0\leq r\leq s}\int_\Omega|D^r u|^2d\mathbf{x}$, where $D^r u$ is the set of all space derivatives $D^\alpha u$ with $|\alpha|=r$, and $|D^r u|^2=\sum_{|\alpha|=r}|D^\alpha u|^2$, where $r\geq 0$ is an integer. Also, we let both ∇u and Du denote the gradient of u. We will use standard function spaces. $L^\infty([0,T],H^s(\Omega))$ is the space of bounded measurable functions from $[0,T]$ into $H^s(\Omega)$, with the norm $\|u\|_{s,T}^2=\sup_{0\leq t\leq T}\|u(t)\|_s^2$. $C([0,T],H^s(\Omega))$ is the space of continuous functions from $[0,T]$ into $H^s(\Omega)$. The purpose of this paper is to prove the following theorem:

Theorem 2.1. *Let $\Omega=\mathbb{T}^N$, the N-dimensional torus, where $N=2,3$. Let ψ be a given positive smooth function of $\mathbf{x}$ and t, and let $\psi(\mathbf{x},t)\geq c_0$ for $\mathbf{x}\in\Omega$ and $0\leq t\leq T$, where c_0 is a positive constant. And let $\frac{d}{dt}\int_\Omega\psi d\mathbf{x}=0$. Let p be a given smooth function of ρ, and let p and $\frac{dp}{d\rho}$ be positive functions. Let b be a given positive smooth function of t, and let $\mathbf{x}_0$ be a given point in Ω. Then for time interval*

$0 \le t \le T$, equations (1.2), (1.4), subject to the condition that $\rho(\mathbf{x}_0, t) = b(t)$ for $0 \le t \le T$ and the initial condition $\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}) \in H^s(\Omega)$ where $s > \frac{N}{2} + 2$, have a unique classical solution ρ, $\mathbf{v}$, where ρ is a positive function, provided that $\|\mathbf{v}_0\|_s$, $\|\psi_t\|_{s,T}$, $\|\psi_{tt}\|_{s-1,T}$, and $\max_{0\le t\le T} b(t)$ are sufficiently small. The regularity of the solution is

$$\begin{gathered}\rho \in C([0,T], C^3(\Omega)) \cap L^\infty([0,T], H^{s+1}(\Omega))\\ \mathbf{v} \in C([0,T], C^2(\Omega)) \cap L^\infty([0,T], H^{s}(\Omega))\\ \frac{\partial \mathbf{v}}{\partial t} \in C([0,T], C^1(\Omega)) \cap L^\infty([0,T], H^{s-1}(\Omega))\end{gathered}$$

Proof. We begin by making a change of variables. First, we let $h = \ln(\rho)$, and we define the composite function $g = \hat{p}' \circ \exp$, where $\exp(h) = e^h$, so that $g(h) = (\hat{p}' \circ \exp)(h) = \hat{p}'(e^h) = \hat{p}'(e^{\ln(\rho)}) = \hat{p}'(\rho)$. Then equation (1.4) can be equivalently written as

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + g(h)\nabla h = 0 \tag{2.1}$$

Note that g is a given positive, smooth function of h. We define the interval $G_1 \subset \mathbb{R}$ by $G_1 = (\frac{3}{2}\min_{0\le t\le T}\ln(b(t)), \frac{1}{2}\max_{0\le t\le T}\ln(b(t)))$, where $\ln(b(t)) < 0$ for $0 \le t \le T$, since $b(t) < 1$ by assumption. Let the positive constant $c_3 = \min_{h_*\in\overline{G}_1} g(h_*)$. We will prove that $h(\mathbf{x}, t) \in \overline{G}_1$ for $\mathbf{x} \in \Omega$, $0 \le t \le T$. Recall that $\psi(\mathbf{x}, t) \ge c_0 > 0$. We now define $f = \frac{1}{c_0 c_3}\psi$, and therefore $f(\mathbf{x}, t)g(h(\mathbf{x}, t)) = \frac{\psi(\mathbf{x},t)g(h(\mathbf{x},t))}{c_0c_3} \ge 1$ for $\mathbf{x} \in \Omega$, $0 \le t \le T$. Under this change of variable, equation (1.2) can be equivalently written as

$$\nabla\cdot\mathbf{v} = -f^{-1}\frac{\partial f}{\partial t} - f^{-1}\mathbf{v}\cdot\nabla f \tag{2.2}$$

Next, we let $\mathbf{u} = f\mathbf{v}$. Then equations (2.1), (2.2) can be equivalently written as

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{f}\mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{f^2}(\nabla f\cdot\mathbf{u})\mathbf{u} + \mathbf{u}\frac{1}{f}\frac{\partial f}{\partial t} - fg(h)\nabla h \tag{2.3}$$

$$\nabla\cdot\mathbf{u} = -\frac{\partial f}{\partial t} \tag{2.4}$$

We now use the Helmholtz decomposition $\mathbf{u} = P\mathbf{u} + Q\mathbf{u} = \mathbf{w} + \nabla\phi$, where $\mathbf{w} = P\mathbf{u}$ and $\nabla\phi = Q\mathbf{u}$ and $\nabla\cdot\mathbf{w} = 0$ (see, e.g., [2], [6]). Here, P and $Q = I - P$ are orthogonal projection operators. Then equations (2.3), (2.4) become

$$\begin{aligned}\frac{\partial \mathbf{w}}{\partial t} &= -\frac{\partial\nabla\phi}{\partial t} - \frac{1}{f}(\mathbf{w}+\nabla\phi)\cdot\nabla(\mathbf{w}+\nabla\phi)\\ &\quad + \frac{1}{f^2}(\nabla f\cdot(\mathbf{w}+\nabla\phi))(\mathbf{w}+\nabla\phi) + (\mathbf{w}+\nabla\phi)\frac{1}{f}\frac{\partial f}{\partial t} - fg(h)\nabla h\end{aligned} \tag{2.5}$$

$$\nabla\cdot\mathbf{w} = 0 \tag{2.6}$$

$$\Delta\phi = -\frac{\partial f}{\partial t} \tag{2.7}$$

At time $t = 0$, we have $\mathbf{u}(\mathbf{x}, 0) = f(\mathbf{x}, 0)\mathbf{v}(\mathbf{x}, 0) \in H^s(\Omega)$, and $\mathbf{w}(\mathbf{x}, 0)$ is the solenoidal component of $\mathbf{u}(\mathbf{x}, 0)$, so that $\mathbf{w}(\mathbf{x}, 0) = P\mathbf{u}(\mathbf{x}, 0)$ and $\nabla\cdot\mathbf{w}(\mathbf{x}, 0) = 0$. The compatibility condition for solving the elliptic equation (2.7) is $0 = \int_\Omega \frac{\partial f}{\partial t}d\mathbf{x} = \frac{d}{dt}\int_\Omega f d\mathbf{x}$, where $\Omega = \mathbf{T}^N$, the N-dimensional torus. Therefore we require that $\frac{d}{dt}\int_\Omega \psi d\mathbf{x} = 0$, where $f = \frac{1}{c_0c_3}\psi$.

We will prove the existence of a unique solution $\mathbf{w}$, h, $\nabla\phi$ to (2.5), (2.6), (2.7). It follows that $\mathbf{u}$, h is a solution to (2.3), (2.4), where $\mathbf{u} = \mathbf{w} + \nabla\phi$. And therefore it follows that $\mathbf{v}$, ρ is a solution to (1.2), (1.4), where $\mathbf{v} = \frac{1}{f}\mathbf{u}$, where $f = \frac{1}{c_0 c_3}\psi$, and where $\rho = e^h$.

We now proceed with the proof. We will construct the solution $\mathbf{w}$, h, $\nabla\phi$ to (2.5), (2.6), (2.7) through an iteration scheme. To define the iteration scheme, we will let the terms of the sequence of approximate solutions consist of the functions $\mathbf{w}^k$ and h^k. Note that $\nabla\phi$ is immediately determined by solving the elliptic equation (2.7), since f is a given function. For $k = 0, 1, 2, \dots$, construct $\mathbf{w}^{k+1}$, h^{k+1} from the previous iterates $\mathbf{w}^k$, h^k by solving the linear system of equations

$$
\begin{aligned}
\frac{\partial \mathbf{w}^{k+1}}{\partial t} &= -\frac{\partial \nabla\phi}{\partial t} - \frac{1}{f}(\mathbf{w}^k + \nabla\phi)\cdot\nabla(\mathbf{w}^{k+1} + \nabla\phi) \\
&\quad + \frac{1}{f^2}(\nabla f\cdot(\mathbf{w}^k + \nabla\phi))(\mathbf{w}^{k+1} + \nabla\phi) \\
&\quad + (\mathbf{w}^{k+1} + \nabla\phi)\frac{1}{f}\frac{\partial f}{\partial t} - f g(h^k)\nabla h^{k+1},
\end{aligned} \tag{2.8}
$$

$$
\nabla\cdot\mathbf{w}^{k+1} = 0, \tag{2.9}
$$

under periodic boundary conditions, and satisfying $h^{k+1}(\mathbf{x}_0, t) = \ln(\rho(\mathbf{x}_0, t)) = \ln(b(t))$, and with the initial data $\mathbf{w}^{k+1}(\mathbf{x}, 0) = \mathbf{w}(\mathbf{x}, 0)$. Set the initial iterate $\mathbf{w}^0(\mathbf{x}, t) = \mathbf{w}(\mathbf{x}, 0)$, the initial data. And set the initial iterate $h^0(\mathbf{x}, t) = \ln(\rho(\mathbf{x}_0, t)) = \ln(b(t))$.

Existence of a sufficiently smooth solution to the system of equations (2.8), (2.9) for fixed k which satisfies $h^{k+1}(\mathbf{x}_0, t) = \ln(b(t))$ follows from the proof given in Lemma 3.1 in Section 3. We proceed now to prove convergence of the iterates as $k \to \infty$ to a unique classical solution of (2.5), (2.6), (2.7). First, we will prove uniform boundedness of the approximating sequence in a high Sobolev space norm. Then, we will prove contraction of the approximating sequence in a low Sobolev space norm. A standard compactness argument completes the proof (see Embid [6], Majda [8]).

Proposition 2.2. *Assume that the hypotheses of Theorem 2.1 are satisfied. Let* $\epsilon = \max_{0\le t\le T} b(t)$ *and let*

$$
\max\{\|\mathbf{v}_0\|_s^2, \|f_t\|_{s,T}^2, \|f_{tt}\|_{s-1,T}^2\} \le \max_{0\le t\le T} b(t) = \epsilon,
$$

where $\epsilon < 1$, *and suppose that* ϵ *is sufficiently small. Let* R *be a given positive constant such that* $R \le \frac{1}{2}|\max_{0\le t\le T}\ln(b(t))|$. *Then there exist positive constants* L_1, L_2, L_3, L_4, L_5 *such that the following hold for* $k = 1, 2, 3\dots$,

- (a) $\|\mathbf{w}^k\|_{s,T}^2 \le \epsilon L_1$,
- (b) $\|\nabla h^k\|_{s,T}^2 \le \epsilon L_2$,
- (c) $\|h^k\|_{s+1,T}^2 \le L_3$,
- (d) $\|\partial\mathbf{w}^k/\partial t\|_{s-1,T}^2 \le L_4$,
- (e) $|h^k - h^0|_{L^\infty, T} \le R$
- (f) $\|g(h^k(\mathbf{x}, t))\|_{s,T}^2 \le L_5$

where $s > \frac{N}{2} + 2$, $N = 2, 3$, *and where* $\epsilon L_1 \le 1$, $\epsilon L_2 \le 1$. *And there exist constants* c_1, c_2, c_3, c_4 *such that* $c_1 \le h^k(\mathbf{x}, t) \le c_2 < 0$ *and* $0 < c_3 \le g(h^k(\mathbf{x}, t)) \le c_4$ *for* $\mathbf{x}\in\Omega$, $0 \le t \le T$, $k = 1, 2, 3\dots$.

Proof. The proof is by induction on k. We show only the inductive step. We will derive estimates for ∇h^{k+1}, h^{k+1}, and $\mathbf{w}^{k+1}$, and then use these estimates to prescribe L_1, L_2, L_3, L_4, L_5 a priori, independent of k, so that if ∇h^k, h^k, $\mathbf{w}^k$ satisfy the estimates in (a)-(f), then ∇h^{k+1}, h^{k+1}, $\mathbf{w}^{k+1}$ also satisfy the same estimates. In the estimates that follow, we use C to denote a generic constant whose value may change from one instance to the next, but is independent of ϵ, L_1, L_2, L_3, L_4, L_5.

We frequently use the well-known Sobolev space inequality $\|uv\|_r^2 \le C\|u\|_r^2\|v\|_r^2$ for $r \ge 2$ (from Lemma 4.1 in Section 4) while making the needed estimates. And in the estimates, $s > \frac{N}{2} + 2$, $s_0 = [\frac{N}{2}] + 1 = 2$ for $N = 2, 3$, and $s \ge 4$, and $s_1 = \max\{s-1, s_0\} = s-1$. Also, we will frequently use the estimate $\|\nabla\phi\|_r^2 \le C\|f_t\|_{r-1}^2$ for $r \ge 1$ from Lemma 4.7 in Section 4, where $\Delta\phi = -\frac{\partial f}{\partial t}$.

Estimate for $\|\nabla h^{k+1}\|_s^2$: Applying the divergence operator to equation (2.8) yields

$$
\begin{aligned}
&\nabla\cdot(fg(h^k)\nabla h^{k+1})\\
&= -\frac{\partial\Delta\phi}{\partial t} - (\nabla(\frac{1}{f}(\mathbf{w}^k + \nabla\phi)))^T : (\nabla(\mathbf{w}^{k+1} + \nabla\phi)) - \frac{1}{f}(\mathbf{w}^k + \nabla\phi)\cdot\nabla\Delta\phi \\
&\quad + \nabla\cdot(\frac{1}{f^2}(\nabla f\cdot(\mathbf{w}^k + \nabla\phi))(\mathbf{w}^{k+1} + \nabla\phi)) + \nabla\cdot((\mathbf{w}^{k+1} + \nabla\phi)\frac{1}{f}\frac{\partial f}{\partial t})
\end{aligned}
\tag{2.10}
$$

Applying estimate (4.20) from Lemma 4.7 in Section 4 to equation (2.10), where we use the inequality $f(\mathbf{x}, t)g(h^k(\mathbf{x}, t)) \ge 1$ by the induction hypothesis, yields

$$
\begin{aligned}
&\|\nabla h^{k+1}\|_s^2\\
&\le C\sum_{j=0}^{s}\|D(fg(h^k))\|_{s_1}^{2j}(\|\frac{\partial\Delta\phi}{\partial t}\|_{s-1}^2\\
&\quad + \|(\nabla(\frac{1}{f}(\mathbf{w}^k + \nabla\phi)))^T : (\nabla(\mathbf{w}^{k+1} + \nabla\phi))\|_{s-1}^2)\\
&\quad + C\sum_{j=0}^{s}\|D(fg(h^k))\|_{s_1}^{2j}(\|\frac{1}{f}(\mathbf{w}^k + \nabla\phi)\cdot\nabla\Delta\phi\|_{s-1}^2\\
&\quad + \|\frac{1}{f^2}(\nabla f\cdot(\mathbf{w}^k + \nabla\phi))(\mathbf{w}^{k+1} + \nabla\phi)\|_s^2)\\
&\quad + C\sum_{j=0}^{s}\|D(fg(h^k))\|_{s_1}^{2j}\|(\mathbf{w}^{k+1} + \nabla\phi)\frac{1}{f}\frac{\partial f}{\partial t}\|_s^2\\
&\le C\sum_{j=0}^{s}\|fg(h^k)\|_s^{2j}(\|\frac{\partial\Delta\phi}{\partial t}\|_{s-1}^2 + \|\nabla(\frac{1}{f}(\mathbf{w}^k + \nabla\phi))\|_{s-1}^2\|\nabla(\mathbf{w}^{k+1} + \nabla\phi)\|_{s-1}^2)\\
&\quad + C\sum_{j=0}^{s}\|fg(h^k)\|_s^{2j}(\|\frac{1}{f}\|_{s-1}^2\|\mathbf{w}^k + \nabla\phi\|_{s-1}^2\|\nabla\Delta\phi\|_{s-1}^2\\
&\quad + \|\frac{1}{f^2}\|_s^2\|\nabla f\|_s^2\|\mathbf{w}^k + \nabla\phi\|_s^2\|\mathbf{w}^{k+1} + \nabla\phi\|_s^2)\\
&\quad + C\sum_{j=0}^{s}\|fg(h^k)\|_s^{2j}\|\mathbf{w}^{k+1} + \nabla\phi\|_s^2\|\frac{1}{f}\frac{\partial f}{\partial t}\|_s^2
\end{aligned}
$$

$$
\begin{aligned}
&\le C\sum_{j=0}^{s}\|f\|_s^{2j}\|g(h^k)\|_s^{2j}(\|\frac{\partial\Delta\phi}{\partial t}\|_{s-1}^2+\|\frac{1}{f}\|_s^2(\|\mathbf{w}^k\|_s^2+\|\nabla\phi\|_s^2)(\|\mathbf{w}^{k+1}\|_s^2+\|\nabla\phi\|_s^2))\\
&\quad+C\sum_{j=0}^{s}\|f\|_s^{2j}\|g(h^k)\|_s^{2j}\|\frac{1}{f}\|_{s-1}^2(\|\mathbf{w}^k\|_{s-1}^2+\|\nabla\phi\|_{s-1}^2)\|\Delta\phi\|_s^2\\
&\quad+C\sum_{j=0}^{s}\|f\|_s^{2j}\|g(h^k)\|_s^{2j}\|\frac{1}{f^2}\|_s^2\|\nabla f\|_s^2(\|\mathbf{w}^k\|_s^2+\|\nabla\phi\|_s^2)(\|\mathbf{w}^{k+1}\|_s^2+\|\nabla\phi\|_s^2)\\
&\quad+C\sum_{j=0}^{s}\|f\|_s^{2j}\|g(h^k)\|_s^{2j}(\|\mathbf{w}^{k+1}\|_s^2+\|\nabla\phi\|_s^2)\|\frac{1}{f}\|_s^2\|f_t\|_s^2\\
&\le C\sum_{j=0}^{s}\|f\|_s^{2j}(L_5)^j(\|f_{tt}\|_{s-1}^2+\|\frac{1}{f}\|_s^2(\epsilon L_1+\|f_t\|_{s-1}^2)(\|\mathbf{w}^{k+1}\|_s^2+\|f_t\|_{s-1}^2))\\
&\quad+C\sum_{j=0}^{s}\|f\|_s^{2j}(L_5)^j(\|\frac{1}{f}\|_{s-1}^2(\epsilon L_1+\|f_t\|_{s-2}^2)\|f_t\|_s^2\\
&\quad+\|\frac{1}{f}\|_s^4\|\nabla f\|_s^2(\epsilon L_1+\|f_t\|_{s-1}^2)(\|\mathbf{w}^{k+1}\|_s^2+\|f_t\|_{s-1}^2))\\
&\quad+C\sum_{j=0}^{s}\|f\|_s^{2j}(L_5)^j(\|\mathbf{w}^{k+1}\|_s^2+\|f_t\|_{s-1}^2)\|\frac{1}{f}\|_s^2\|f_t\|_s^2\\
&\le C\sum_{j=0}^{s}\|f\|_{s,T}^{2j}(L_5)^j\Big(\epsilon+\|\frac{1}{f}\|_{s,T}^2(\epsilon L_1+\epsilon)(\|\mathbf{w}^{k+1}\|_s^2+\epsilon)+\|\frac{1}{f}\|_{s-1,T}^2(\epsilon L_1+\epsilon)(\epsilon)\Big)\\
&\quad+C\sum_{j=0}^{s}\|f\|_{s,T}^{2j}(L_5)^j\Big(\|\frac{1}{f}\|_{s,T}^4\|\nabla f\|_{s,T}^2(\epsilon L_1+\epsilon)(\|\mathbf{w}^{k+1}\|_s^2+\epsilon)\\
&\quad+(\|\mathbf{w}^{k+1}\|_s^2+\epsilon)\|\frac{1}{f}\|_{s,T}^2(\epsilon)\Big)\\
&\le C_1(\epsilon+\|\mathbf{w}^{k+1}\|_s^2)
\end{aligned}
\tag{2.11}
$$

where $\|g(h^k)\|_s^2\le L_5$ by the induction hypothesis. And $L_5=C|g|_{s,\overline{G}_1}^2$, where we define $|g|_{s,\overline{G}_1}=\max\{\big|\frac{d^jg}{dh^j}(h_*)\big| : h_*\in\overline{G}_1, 0\le j\le s\}$. And we used the estimate $\|\mathbf{w}^k\|_s^2\le\epsilon L_1$ by the induction hypothesis, where $\epsilon L_1\le 1$. We used the fact that $\Delta\phi=-f_t$, and we used the estimate $\|\nabla\phi\|_r^2\le C\|f_t\|_{r-1}^2$ for $r\ge 1$ from Lemma 4.7 in Section 4. And we used the assumptions that $\|f_{tt}\|_{s-1}^2\le\epsilon$ and $\|f_t\|_s^2\le\epsilon$. The constant C_1 depends on $\|f\|_{s,T}$, $\|\frac{1}{f}\|_{s,T}$, $\|\nabla f\|_{s,T}$, and $|g|_{s,\overline{G}_1}$.

Estimate for $\|\mathbf{w}^{k+1}\|_s^2$: Recall that the initial condition $\mathbf{w}^{k+1}(\mathbf{x},0)=\mathbf{w}(\mathbf{x},0)=P\mathbf{u}(\mathbf{x},0)=P(f(\mathbf{x},0)\mathbf{v}_0)$, where P is an orthogonal projection operator. Then by applying estimate (4.21) from Lemma 4.8 to equation (2.8) we obtain the estimate

$$
\begin{aligned}
&\|\mathbf{w}^{k+1}\|_s^2\\
&\le e^{\beta t}\|P(f(\mathbf{x},0)\mathbf{v}_0)\|_s^2+e^{\beta t}C\int_0^t(\|\frac{\partial\nabla\phi}{\partial t}\|_s^2+\|\frac{1}{f}(\mathbf{w}^k+\nabla\phi)\cdot\nabla(\nabla\phi)\|_s^2)d\tau\\
&\quad+e^{\beta t}C\int_0^t(\|\frac{1}{f^2}(\nabla f\cdot(\mathbf{w}^k+\nabla\phi))\nabla\phi\|_s^2+\|\nabla\phi\frac{f_t}{f}\|_s^2+\|fg(h^k)\nabla h^{k+1}\|_s^2)d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le e^{\beta t}\|f(\mathbf{x},0)\mathbf{v}_0\|_s^2 + e^{\beta t} C\int_0^t (\|\frac{\partial \nabla\phi}{\partial t}\|_s^2 + \|\frac{1}{f}\|_s^2\|\mathbf{w}^k+\nabla\phi\|_s^2\|\nabla\phi\|_{s+1}^2)d\tau \\
&\quad + e^{\beta t}C\int_0^t (\|\frac{1}{f^2}\|_s^2\|\nabla f\|_s^2\|\mathbf{w}^k+\nabla\phi\|_s^2\|\nabla\phi\|_s^2 + \|\nabla\phi\|_s^2\|\frac{1}{f}\|_s^2\|f_t\|_s^2 \\
&\quad + \|f\|_s^2\|g(h^k)\|_s^2\|\nabla h^{k+1}\|_s^2)d\tau \\
&\le e^{\beta t}C\|f(\mathbf{x},0)\|_s^2\|\mathbf{v}_0\|_s^2 + e^{\beta t}C\int_0^t (\|\frac{\partial\nabla\phi}{\partial t}\|_s^2 + \|\frac{1}{f}\|_s^2(\|\mathbf{w}^k\|_s^2+\|\nabla\phi\|_s^2)\|\nabla\phi\|_{s+1}^2)d\tau \\
&\quad + e^{\beta t}C\int_0^t (\|\frac{1}{f^2}\|_s^2\|\nabla f\|_s^2(\|\mathbf{w}^k\|_s^2+\|\nabla\phi\|_s^2)\|\nabla\phi\|_s^2 + \|\nabla\phi\|_s^2\|\frac{1}{f}\|_s^2\|f_t\|_s^2 \\
&\quad + \|f\|_s^2\|g(h^k)\|_s^2\|\nabla h^{k+1}\|_s^2)d\tau \\
&\le e^{\beta t}C\|f\|_{s,T}^2\|\mathbf{v}_0\|_s^2 + e^{\beta t}C\int_0^t (\|f_{tt}\|_{s-1}^2 + \|\frac{1}{f}\|_s^2(\epsilon L_1+\|f_t\|_{s-1}^2)\|f_t\|_s^2)d\tau \\
&\quad + e^{\beta t}C\int_0^t (\|\frac{1}{f^2}\|_s^2\|\nabla f\|_s^2(\epsilon L_1+\|f_t\|_{s-1}^2)\|f_t\|_{s-1}^2 + \|f_t\|_{s-1}^2\|\frac{1}{f}\|_s^2\|f_t\|_s^2 \\
&\quad + \|f\|_s^2\|g(h^k)\|_s^2 C_1(\epsilon+\|\mathbf{w}^{k+1}\|_s^2))d\tau \\
&\le e^{\beta T}C\epsilon\|f\|_{s,T}^2 + e^{\beta T}CT\Big(\epsilon + \|\frac{1}{f}\|_{s,T}^2(\epsilon L_1+\epsilon)\epsilon + \|\frac{1}{f}\|_{s,T}^4\|\nabla f\|_{s,T}^2(\epsilon L_1+\epsilon)\epsilon\Big) \\
&\quad + e^{\beta T}CT\epsilon^2\|\frac{1}{f}\|_{s,T}^2 + e^{\beta T}CT\|f\|_{s,T}^2L_5C_1\epsilon + e^{\beta T}C\|f\|_{s,T}^2L_5C_1\int_0^t\|\mathbf{w}^{k+1}\|_s^2d\tau \\
&\le e^{\beta T}C_2\epsilon(1+T) + e^{\beta T}C_2\int_0^t\|\mathbf{w}^{k+1}\|_s^2d\tau
\end{aligned} \tag{2.12}
$$

where we used the estimates $\|\mathbf{w}^k\|_s^2 \le \epsilon L_1 \le 1$ and $\|g(h^k)\|_s^2 \le L_5$, by the induction hypothesis, where $L_5 = C|g|_{s,\overline{G}_1}^2$. And we used estimate (2.11) for $\|\nabla h^{k+1}\|_s^2$. We also used the estimates $\|\nabla\phi\|_r^2 \le C\|f_t\|_{r-1}^2$ and $\|\nabla\phi_t\|_r^2 \le C\|f_{tt}\|_{r-1}^2$ for $r \ge 1$ from Lemma 4.7, where $\Delta\phi = -f_t$. And we used the assumptions that $\|f_t\|_{s,T}^2 \le \epsilon$, $\|f_{tt}\|_{s-1,T}^2 \le \epsilon$, and $\|\mathbf{v}_0\|_s^2 \le \epsilon$. Here C_2 depends on $\|f\|_{s,T}$, $\|\frac{1}{f}\|_{s,T}$, $\|\nabla f\|_{s,T}$, and $|g|_{s,\overline{G}_1}$.

From Lemma 4.8 in Section 4 we obtain the estimate:

$$
\begin{aligned}
\beta &\le C(1+\|\frac{1}{f}\|_{s_1+1,T}(\|\mathbf{w}^k\|_{s_1+1,T}+\|\nabla\phi\|_{s_1+1,T})) + C\|\frac{1}{f^2}\|_{s_1+1,T} \\
&\quad \times \|\nabla f\|_{s_1+1,T}(\|\mathbf{w}^k\|_{s_1+1,T}+\|\nabla\phi\|_{s_1+1,T}) + C\|\frac{1}{f}\frac{\partial f}{\partial t}\|_{s_1+1,T} \\
&\le C\big(1+\|\frac{1}{f}\|_{s,T}(\epsilon^{1/2}L_1^{1/2}+\|f_t\|_{s-1,T})\big) \\
&\quad + C\|\frac{1}{f}\|_{s,T}^2\|\nabla f\|_{s,T}(\epsilon^{1/2}L_1^{1/2}+\|f_t\|_{s-1,T}) + C\|\frac{1}{f}\|_{s,T}\|f_t\|_{s,T} \\
&\le C_3
\end{aligned} \tag{2.13}
$$

where $s_1 = s-1$ and $s \ge 4$. We used the estimate $\|\mathbf{w}^k\|_s \le \epsilon^{1/2}L_1^{1/2} \le 1$, by the induction hypothesis. And we used the assumption that $\|f_t\|_{s,T}^2 \le \epsilon < 1$. And C_3 depends on $\|\frac{1}{f}\|_{s,T}$, $\|\nabla f\|_{s,T}$.

By applying Gronwall's inequality to (2.12), we obtain for $\mathbf{w}^{k+1}$ the estimate

$$\|\mathbf{w}^{k+1}\|_s^2 \le \epsilon C_4(1+T)(1+C_4Te^{C_4T}) \tag{2.14}$$

where $C_4 = e^{C_3T}C_2$. We now define $L_1 = C_4(1+T)(1+C_4Te^{C_4T})$. It follows that $\|\mathbf{w}^{k+1}\|_s^2 \le \epsilon L_1$, where we choose ϵ sufficiently small so that $\epsilon L_1 \le 1$. Substituting inequality (2.14) into (2.11) yields

$$\|\nabla h^{k+1}\|_s^2 \le C_1(\epsilon + \|\mathbf{w}^{k+1}\|_s^2) \le \epsilon C_1(1+L_1) \tag{2.15}$$

We now define $L_2 = C_1(1+L_1)$. It follows that $\|\nabla h^{k+1}\|_s^2 \le \epsilon L_2$, where we choose ϵ sufficiently small so that $\epsilon L_2 \le 1$. And L_1, L_2 depend on $\|f\|_{s,T}$, $\|\frac{1}{f}\|_{s,T}$, $\|\nabla f\|_{s,T}$, and $|g|_{s,\overline{G}_1}$. This completes the proof of parts (a) and (b) of Proposition 2.2.

Estimate for $\|h^{k+1}\|_{s+1}^2$: By Lemma 4.3 in Section 4, and using inequality (2.15), we obtain

$$\begin{aligned}
\|h^{k+1}\|_{s+1}^2 &\le C(\|h^{k+1}(\mathbf{x}_0,t)\|_{s+1}^2 + \|\nabla(h^{k+1}(\mathbf{x}_0,t))\|_s^2 + \|\nabla h^{k+1}\|_s^2) \\
&\le C(|\Omega| \max_{0\le t\le T} |h^{k+1}(\mathbf{x}_0,t)|^2 + \epsilon L_2) \\
&\le C(|\Omega| \max_{0\le t\le T} |\ln(b(t))|^2 + 1) = L_3
\end{aligned} \tag{2.16}$$

since $h^{k+1}(\mathbf{x}_0,t) = \ln(b(t))$ and $\epsilon L_2 \le 1$. This completes the proof of part (c) of Proposition 2.2.

Estimate for $\|\mathbf{w}_t^{k+1}\|_{s-1}^2$: We immediately obtain from equation (2.8) the following:

$$\begin{aligned}
&\|\mathbf{w}_t^{k+1}\|_{s-1}^2 \\
&\le C\|\frac{\partial\nabla\phi}{\partial t}\|_{s-1}^2 + C\|\frac{1}{f}\|_{s-1}^2(\|\mathbf{w}^k\|_{s-1}^2 + \|\nabla\phi\|_{s-1}^2)(\|\mathbf{w}^{k+1}\|_s^2 + \|\nabla\phi\|_s^2) \\
&\quad + C\|\frac{1}{f^2}\|_{s-1}^2\|\nabla f\|_{s-1}^2(\|\mathbf{w}^k\|_{s-1}^2 + \|\nabla\phi\|_{s-1}^2)(\|\mathbf{w}^{k+1}\|_{s-1}^2 + \|\nabla\phi\|_{s-1}^2) \\
&\quad + C(\|\mathbf{w}^{k+1}\|_{s-1}^2 + \|\nabla\phi\|_{s-1}^2)\|\frac{1}{f}\frac{\partial f}{\partial t}\|_{s-1}^2 + C\|f\|_{s-1}^2\|g(h^k)\|_{s-1}^2\|\nabla h^{k+1}\|_{s-1}^2 \\
&\le C\|f_{tt}\|_{s-2,T}^2 + C\|\frac{1}{f}\|_{s-1,T}^2(\epsilon L_1 + \|f_t\|_{s-2,T}^2)(\epsilon L_1 + \|f_t\|_{s-1,T}^2) \\
&\quad + C\|\frac{1}{f}\|_{s-1,T}^4\|\nabla f\|_{s-1,T}^2(\epsilon L_1 + \|f_t\|_{s-2,T}^2)(\epsilon L_1 + \|f_t\|_{s-2,T}^2) \\
&\quad + C(\epsilon L_1 + \|f_t\|_{s-2,T}^2)\|\frac{1}{f}\|_{s-1,T}^2\|f_t\|_{s-1,T}^2 + C\|f\|_{s-1,T}^2 L_5(\epsilon L_2) \\
&\le C_5
\end{aligned} \tag{2.17}$$

where we used the estimates $\|\mathbf{w}^{k+1}\|_s^2 \le \epsilon L_1 \le 1$ and $\|\nabla h^{k+1}\|_{s-1}^2 \le \epsilon L_2 \le 1$ from (2.14), (2.15). And we used the estimates $\|\mathbf{w}^k\|_s^2 \le \epsilon L_1 \le 1$, $\|g(h^k)\|_s^2 \le L_5$, by the induction hypothesis, where $L_5 = C|g|_{s,\overline{G}_1}^2$. And we used the estimate $\|\nabla\phi\|_r^2 \le C\|f_t\|_{r-1}^2$ for $r \ge 1$ by Lemma 4.7, where $\Delta\phi = -f_t$. And we used the assumptions that $\|f_t\|_{s,T}^2 \le \epsilon < 1$ and $\|f_{tt}\|_{s-1,T}^2 \le \epsilon < 1$. And C_5 depends on $\|\frac{1}{f}\|_{s-1,T}$, $\|\nabla f\|_{s-1,T}$, $\|f\|_{s-1,T}$, and $|g|_{s,\overline{G}_1}$. We now define $L_4 = C_5$. This completes the proof of part(d) of Proposition 2.2.

Estimate for $|h^{k+1}-h^0|_{L^\infty}$: By Lemma 4.3, we obtain the inequality

$$|h^{k+1}-h^0|_{L^\infty} \le C\|\nabla(h^{k+1}-h^0)\|_1 = C\|\nabla h^{k+1}\|_1 \le C\epsilon^{1/2}L_2^{1/2} \le R \tag{2.18}$$

where we used estimate (2.15), and where we choose ϵ sufficiently small so that $C\epsilon^{1/2}L_2^{1/2} \le R$. Here we used the facts that $h^{k+1}(\mathbf{x}_0,t) = h^0(\mathbf{x}_0,t)$ and that $h^0(\mathbf{x},t) = \ln(b(t))$. This completes the proof of part (e).

Since $R \le \frac{1}{2}|\max_{0\le t\le T}\ln(b(t))|$, $|h^{k+1}-h^0| \le \frac{1}{2}|\max_{0\le t\le T}\ln(b(t))|$. And since $h^0(\mathbf{x},t) = \ln(b(t))$, where $\max_{0\le t\le T} b(t) = \epsilon$ and $\epsilon < 1$, so that $\ln(b(t)) < 0$, we obtain the inequalities

$$\begin{aligned} h^{k+1}(\mathbf{x},t) &\le h^0(\mathbf{x},t) + \frac{1}{2}|\max_{0\le t\le T}\ln(b(t))| = \ln(b(t)) - \frac{1}{2}\max_{0\le t\le T}\ln(b(t)) \\ &\le \frac{1}{2}\max_{0\le t\le T}\ln(b(t)) = \frac{1}{2}\ln(\epsilon) \end{aligned} \tag{2.19}$$

$$\begin{aligned} h^{k+1}(\mathbf{x},t) &\ge h^0(\mathbf{x},t) - \frac{1}{2}|\max_{0\le t\le T}\ln(b(t))| = \ln(b(t)) + \frac{1}{2}\max_{0\le t\le T}\ln(b(t)) \\ &\ge \frac{3}{2}\min_{0\le t\le T}(\ln(b(t))) \end{aligned} \tag{2.20}$$

We have defined the interval $G_1 \subset \mathbb{R}$ by

$$\begin{aligned} G_1 = (c_1,c_2) &= (\frac{3}{2}\min_{0\le t\le T}\ln(b(t)), \frac{1}{2}\max_{0\le t\le T}\ln(b(t))) \\ &= (\frac{3}{2}\min_{0\le t\le T}\ln(b(t)), \frac{1}{2}\ln(\epsilon)), \end{aligned}$$

so that $h^{k+1}(\mathbf{x},t) \in \overline{G}_1$ for $\mathbf{x}\in\Omega$, $0\le t\le T$. Since g is a smooth, positive function of h, there exist constants $c_3 = \min_{h_*\in\overline{G}_1} g(h_*)$, $c_4 = \max_{h_*\in\overline{G}_1} g(h_*)$, so that $0 < c_3 \le g(h_*) \le c_4$ for all $h_*\in\overline{G}_1$. And so $0 < c_3 \le g(h^{k+1}(\mathbf{x},t)) \le c_4$ for $\mathbf{x}\in\Omega$, $0\le t\le T$.

Since $g(h) = (\hat{p}'\circ\exp)(h) = \hat{p}'(e^h)$, where $\hat{p}'$ is a smooth function, it follows that

$$|\frac{dg}{dh}(h_*)| = |\hat{p}''(e^{h_*})e^{h_*}| \le \max_{h_*\in\overline{G}_1}|\hat{p}''(e^{h_*})|e^{\frac{1}{2}\ln\epsilon} \le C\epsilon^{1/2} \tag{2.21}$$

$$\begin{aligned} |\frac{d^2g}{dh^2}(h_*)| &= |\hat{p}''(e^{h_*})e^{h_*} + \hat{p}'''(e^{h_*})(e^{h_*})^2| \\ &\le \max_{h_*\in\overline{G}_1}|\hat{p}''(e^{h_*})|e^{\frac{1}{2}\ln\epsilon} + \max_{h_*\in\overline{G}_1}|\hat{p}'''(e^{h_*})|\big(e^{\frac{1}{2}\ln\epsilon}\big)^2 \le C\epsilon^{1/2} \end{aligned} \tag{2.22}$$

for $h_*\in\overline{G}_1$, where we assume that $\max_{h_*\in\overline{G}_1}|\hat{p}''(e^{h_*})| \le C$ and that

$$\epsilon^{1/2}\max_{h_*\in\overline{G}_1}|\hat{p}'''(e^{h_*})| \le C,$$

where C does not depend on ϵ. From (2.21), (2.22), it follows that

$$\big|\frac{dg}{dh}\big|_{1,\overline{G}_1} \le C\epsilon^{1/2} \tag{2.23}$$

where we define $|f|_{r,\overline{G}_1} = \max\{\big|\frac{d^jf}{dh^j}(h_*)\big| : h_*\in\overline{G}_1, 0\le j\le r\}$.

Estimate for $\|g(h^{k+1})\|_s^2$: By Lemma 4.4 and by (2.15) and by the inequality $g(h^{k+1}(\mathbf{x},t)) \le c_4$ which was just proven, it follows that

$$\begin{aligned}\|g(h^{k+1})\|_s^2 &\le \|g(h^{k+1})\|_0^2 + C\|D(g(h^{k+1}))\|_{s-1}^2 \\ &\le c_4^2\|\Omega| + C\big|\frac{dg}{dh}\big|_{s-1,\overline{G}_1}^2 \sum_{1\le j\le s} \|Dh^{k+1}\|_{s-1}^{2j} \\ &\le c_4^2\|\Omega| + C\big|\frac{dg}{dh}\big|_{s-1,\overline{G}_1}^2 \sum_{1\le j\le s} (\epsilon L_2)^j \\ &\le C|g|_{s,\overline{G}_1}^2 = L_5\end{aligned} \tag{2.24}$$

where we used that $\epsilon L_2 \le 1$, and we used the fact that $c_4^2 = (\max_{h_*\in\overline{G}_1} g(h_*))^2 = |g|_{0,\overline{G}_1}^2$. This completes the proof of Proposition 2.2. □

Next, we give the proof of contraction in a low Sobolev space norm.

Proposition 2.3. *Assume that the hypotheses of Theorem 2.1 hold. Then*
- (a) $\sum_{k=0}^{\infty}(\|\mathbf{w}^{k+1} - \mathbf{w}^k\|_{1,T}^2 + \epsilon\|\nabla(h^{k+1} - h^k)\|_{1,T}^2) < \infty$,
- (b) $\sum_{k=0}^{\infty} \|h^{k+1} - h^k\|_{2,T}^2 < \infty$,
- (c) $\sum_{k=0}^{\infty} \|\mathbf{w}_t^{k+1} - \mathbf{w}_t^k\|_{0,T}^2 < \infty$.

Proof. Subtracting equation (2.8) for $\mathbf{w}^k$, $\nabla\rho^k$ from equation (2.8) for $\mathbf{w}^{k+1}$, $\nabla\rho^{k+1}$ yields

$$\begin{aligned}&\frac{\partial}{\partial t}(\mathbf{w}^{k+1} - \mathbf{w}^k) \\ &= -\frac{1}{f}(\mathbf{w}^k + \nabla\phi)\cdot\nabla(\mathbf{w}^{k+1} - \mathbf{w}^k) - \frac{1}{f}(\mathbf{w}^k - \mathbf{w}^{k-1})\cdot\nabla(\mathbf{w}^k + \nabla\phi) \\ &\quad + \frac{1}{f^2}(\nabla f\cdot(\mathbf{w}^k + \nabla\phi))(\mathbf{w}^{k+1} - \mathbf{w}^k) + \frac{1}{f^2}(\nabla f\cdot(\mathbf{w}^k - \mathbf{w}^{k-1}))(\mathbf{w}^k + \nabla\phi) \\ &\quad + (\mathbf{w}^{k+1} - \mathbf{w}^k)\frac{1}{f}\frac{\partial f}{\partial t} - fg(h^k)\nabla(h^{k+1} - h^k) - f(g(h^k) - g(h^{k-1}))\nabla h^k\end{aligned} \tag{2.25}$$

where $\nabla\cdot(\mathbf{w}^{k+1} - \mathbf{w}^k) = 0$ and where $\mathbf{w}^{k+1}(\mathbf{x},0) - \mathbf{w}^k(\mathbf{x},0) = 0$.

In the estimates that follow, we will frequently use the well-known Sobolev space inequalities $\|uv\|_1^2 \le C\|u\|_3^2\|v\|_1^2$, and $\|uv\|_r^2 \le C\|u\|_r^2\|v\|_r^2$ for $r \ge 2$, and $|u|_{L^\infty}^2 \le C\|u\|_{s_0}^2$, where $s_0 = [\frac{N}{2}] + 1 = 2$ for $N = 2, 3$ (from Lemma 4.1 in Section 4).

Estimate for $\|\nabla(h^{k+1} - h^k)\|_1^2$: Applying the divergence operator to (2.25) yields

$$\begin{aligned}&\nabla\cdot\big(fg(h^k)\nabla(h^{k+1} - h^k)\big) \\ &= -(\nabla(\frac{1}{f}(\mathbf{w}^k + \nabla\phi)))^T : (\nabla(\mathbf{w}^{k+1} - \mathbf{w}^k)) \\ &\quad - \nabla\cdot\Big(\frac{1}{f}(\mathbf{w}^k - \mathbf{w}^{k-1})\cdot\nabla(\mathbf{w}^k \\ &\quad + \nabla\phi)\Big) + \nabla\cdot\Big(\frac{1}{f^2}(\nabla f\cdot(\mathbf{w}^k + \nabla\phi))(\mathbf{w}^{k+1} - \mathbf{w}^k)\Big) \\ &\quad + \nabla\cdot\big(\frac{1}{f^2}(\nabla f\cdot(\mathbf{w}^k - \mathbf{w}^{k-1}))(\mathbf{w}^k + \nabla\phi)\Big) + \nabla\cdot\Big((\mathbf{w}^{k+1} - \mathbf{w}^k)\frac{1}{f}\frac{\partial f}{\partial t}\Big) \\ &\quad - \nabla\cdot\big(f(g(h^k) - g(h^{k-1}))\nabla h^k\big)\end{aligned} \tag{2.26}$$

Applying estimate (4.20) from Lemma 4.7 to equation (2.26), and using the fact that $f(\mathbf{x},t)g(h^k(\mathbf{x},t)) \geq 1$, and using Lemma 4.1, yields

$$
\begin{aligned}
&\|\nabla(h^{k+1}-h^k)\|_1^2 \\
&\leq C\sum_{j=0}^{1}\|D(fg(h^k))\|_2^{2j}\|(\nabla(\frac{1}{f}(\mathbf{w}^k+\nabla\phi)))^T:(\nabla(\mathbf{w}^{k+1}-\mathbf{w}^k))\|_0^2 \\
&\quad + C\sum_{j=0}^{1}\|D(fg(h^k))\|_2^{2j}\Big(\|\frac{1}{f}(\mathbf{w}^k-\mathbf{w}^{k-1})\cdot\nabla(\mathbf{w}^k+\nabla\phi)\|_1^2 \\
&\quad + \|\frac{1}{f^2}(\nabla f\cdot(\mathbf{w}^k+\nabla\phi))(\mathbf{w}^{k+1}-\mathbf{w}^k)\|_1^2\Big) \\
&\quad + C\sum_{j=0}^{1}\|D(fg(h^k))\|_2^{2j}\Big(\|\frac{1}{f^2}(\nabla f\cdot(\mathbf{w}^k-\mathbf{w}^{k-1}))(\mathbf{w}^k+\nabla\phi)\|_1^2 \\
&\quad + \|(\mathbf{w}^{k+1}-\mathbf{w}^k)\frac{1}{f}\frac{\partial f}{\partial t}\|_1^2\Big) + C\sum_{j=0}^{1}\|D(fg(h^k))\|_2^{2j}\|f(g(h^k)-g(h^{k-1}))\nabla h^k\|_1^2 \\
&\leq C\sum_{j=0}^{1}\|fg(h^k)\|_3^{2j}\Big(|\nabla(\frac{1}{f}(\mathbf{w}^k+\nabla\phi))|_{L^\infty}^2\|\nabla(\mathbf{w}^{k+1}-\mathbf{w}^k)\|_0^2 \\
&\quad + \|\frac{1}{f}(\mathbf{w}^k-\mathbf{w}^{k-1})\|_1^2\|\nabla(\mathbf{w}^k+\nabla\phi)\|_3^2\Big) \\
&\quad + C\sum_{j=0}^{1}\|fg(h^k)\|_3^{2j}\Big(\|\frac{1}{f^2}(\nabla f\cdot(\mathbf{w}^k+\nabla\phi)\|_3^2\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2 \\
&\quad + \|\frac{1}{f^2}(\nabla f\cdot(\mathbf{w}^k-\mathbf{w}^{k-1}))\|_1^2\|\mathbf{w}^k+\nabla\phi\|_3^2\Big) \\
&\quad + C\sum_{j=0}^{1}\|fg(h^k)\|_3^{2j}\Big(\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2\|\frac{1}{f}\frac{\partial f}{\partial t}\|_3^2 + \|g(h^k)-g(h^{k-1})\|_1^2\|f\nabla h^k\|_3^2\Big) \\
&\leq C\sum_{j=0}^{1}\|f\|_3^{2j}\|g(h^k)\|_3^{2j}\Big(\|\nabla(\frac{1}{f}(\mathbf{w}^k+\nabla\phi))\|_2^2\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2 \\
&\quad + \|\frac{1}{f}\|_3^2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_1^2\|\mathbf{w}^k+\nabla\phi\|_4^2\Big) \\
&\quad + C\sum_{j=0}^{1}\|f\|_3^{2j}\|g(h^k)\|_3^{2j}\Big(\|\frac{1}{f^2}\nabla f\|_3^2\|\mathbf{w}^k+\nabla\phi\|_3^2\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2 \\
&\quad + \|\frac{1}{f^2}\nabla f\|_3^2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_1^2\|\mathbf{w}^k+\nabla\phi\|_3^2\Big) \\
&\quad + C\sum_{j=0}^{1}\|f\|_3^{2j}\|g(h^k)\|_3^{2j}\Big(\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2\|\frac{1}{f}\|_3^2\|f_t\|_3^2 + \|g(h^k) \\
&\quad - g(h^{k-1})\|_1^2\|f\|_3^2\|\nabla h^k\|_3^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\le C\sum_{j=0}^{1}\|f\|_3^{2j}(L_5)^j\Big(\|\frac{1}{f}\|_3^2(\|\mathbf{w}^k\|_3^2+\|\nabla\phi\|_3^2)\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2\\
&\quad+\|\frac{1}{f}\|_3^2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_1^2(\|\mathbf{w}^k\|_4^2+\|\nabla\phi\|_4^2)\Big)\\
&\quad+C\sum_{j=0}^{1}\|f\|_3^{2j}(L_5)^j\Big(\|\frac{1}{f^2}\nabla f\|_3^2(\|\mathbf{w}^k\|_3^2+\|\nabla\phi\|_3^2)\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2\\
&\quad+\|\frac{1}{f^2}\nabla f\|_3^2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_1^2(\|\mathbf{w}^k\|_3^2+\|\nabla\phi\|_3^2)\Big)\\
&\quad+C\sum_{j=0}^{1}\|f\|_3^{2j}(L_5)^j\Big(\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2\|\frac{1}{f}\|_3^2\|f_t\|_3^2\\
&\quad+\|g(h^k)-g(h^{k-1})\|_1^2\|f\|_3^2\|\nabla h^k\|_3^2\Big)\\
&\le C\sum_{j=0}^{1}\|f\|_{3,T}^{2j}(L_5)^j\Big(\|\frac{1}{f}\|_{3,T}^2(\epsilon L_1+\|f_t\|_{2,T}^2)\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2\\
&\quad+\|\frac{1}{f}\|_{3,T}^2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_1^2(\epsilon L_1+\|f_t\|_{3,T}^2)\Big)\\
&\quad+C\sum_{j=0}^{1}\|f\|_{3,T}^{2j}(L_5)^j\Big\|\frac{1}{f}\Big\|_{3,T}^4\|\nabla f\|_{3,T}^2(\epsilon L_1+\|f_t\|_{2,T}^2)\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2\\
&\quad+C\sum_{j=0}^{1}\|f\|_{3,T}^{2j}(L_5)^j\Big\|\frac{1}{f}\Big\|_{3,T}^4\|\nabla f\|_{3,T}^2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_1^2(\epsilon L_1+\|f_t\|_{2,T}^2)\\
&\quad+C\sum_{j=0}^{1}\|f\|_{3,T}^{2j}(L_5)^j\Big(\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2\|\frac{1}{f}\|_{3,T}^2\|f_t\|_{3,T}^2+\|g(h^k)\\
&\quad-g(h^{k-1})\|_1^2\|f\|_{3,T}^2(\epsilon L_2)\Big)\\
&\le C_6\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2+C_6\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_1^2+C_6\|g(h^k)-g(h^{k-1})\|_1^2
\end{aligned}
\tag{2.27}
$$

where we used the estimates $\|\mathbf{w}^k\|_4^2 \le \epsilon L_1 \le 1$, $\|\nabla h^k\|_3^2 \le \epsilon L_2 \le 1$, $\|g(h^k)\|_3^2 \le L_5$, where $L_5 = C|g|^2_{s,\overline{G}_1}$, from Proposition 2.2. We used the assumption that $\|f_t\|_{3,T}^2 \le \epsilon < 1$. And we used the estimate $\|\nabla\phi\|_r^2 \le C\|f_t\|_{r-1}^2$ for $r \ge 1$ from Lemma 4.7. And C_6 depends on $\|\frac{1}{f}\|_{3,T}$,$\|\nabla f\|_{3,T}$,$\|f\|_{3,T}$, $|g|_{s,\overline{G}_1}$.

By Lemma 4.5 and Lemma 4.3, we obtain the estimate

$$
\begin{aligned}
\|g(h^k)-g(h^{k-1})\|_1^2 &\le C\Big|\frac{dg}{dh}\Big|_{1,\overline{G}_1}^2(1+\|\nabla h^k\|_0^2+\|\nabla h^{k-1}\|_0^2)\|h^k-h^{k-1}\|_2^2\\
&\le C\Big|\frac{dg}{dh}\Big|_{1,\overline{G}_1}^2(1+2\epsilon L_2)\|\nabla(h^k-h^{k-1})\|_1^2\\
&\le C\epsilon\|\nabla(h^k-h^{k-1})\|_1^2
\end{aligned}
\tag{2.28}
$$

where we used the estimate $\big|\frac{dg}{dh}\big|_{1,\overline{G}_1}^2 \le C\epsilon$ by inequality (2.23), and we used the estimates $\|\nabla h^k\|_0^2 \le \epsilon L_2 \le 1$, $\|\nabla h^{k-1}\|_0^2 \le \epsilon L_2 \le 1$ from Proposition 2.2. We used Lemma 4.3 from Section 4, as well as the fact that $h^k(\mathbf{x}_0,t) = h^{k-1}(\mathbf{x}_0,t)$, to obtain the estimate $\|h^k - h^{k-1}\|_2^2 \le C\|\nabla(h^k - h^{k-1})\|_1^2$.

Substituting (2.28) into (2.27) yields

$$\begin{aligned} &\|\nabla(h^{k+1}-h^k)\|_1^2 \\ &\le C_7\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2 + C_7\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_1^2 + \epsilon C_7\|\nabla(h^k-h^{k-1})\|_1^2 \end{aligned} \tag{2.29}$$

where $C_7 = C_6(1+C)$.

Estimate for $\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2$: Applying estimate (4.21) from Lemma 4.8 to equation (2.25), and using Lemma 4.1, yields

$$\begin{aligned}
&\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2 \\
&\le Ce^{\beta t}\int_0^t \|\frac{1}{f}(\mathbf{w}^k-\mathbf{w}^{k-1})\cdot\nabla(\mathbf{w}^k+\nabla\phi)\|_1^2 d\tau \\
&\quad + Ce^{\beta t}\int_0^t (\|\frac{1}{f^2}(\nabla f\cdot(\mathbf{w}^k-\mathbf{w}^{k-1}))(\mathbf{w}^k+\nabla\phi)\|_1^2 + \|fg(h^k)\nabla(h^{k+1}-h^k)\|_1^2)d\tau \\
&\quad + Ce^{\beta t}\int_0^t \|f(g(h^k)-g(h^{k-1}))\nabla h^k\|_1^2 d\tau \\
&\le Ce^{\beta t}\int_0^t \Big(\|\frac{1}{f}(\mathbf{w}^k-\mathbf{w}^{k-1})\|_1^2\|\nabla(\mathbf{w}^k+\nabla\phi)\|_3^2 \\
&\quad + \|\frac{1}{f^2}\nabla f\cdot(\mathbf{w}^k-\mathbf{w}^{k-1})\|_1^2\|\mathbf{w}^k+\nabla\phi\|_3^2\Big)d\tau \\
&\quad + Ce^{\beta t}\int_0^t (\|fg(h^k)\|_3^2\|\nabla(h^{k+1}-h^k)\|_1^2 + \|g(h^k)-g(h^{k-1})\|_1^2\|f\nabla h^k\|_3^2)d\tau \\
&\le Ce^{\beta t}\int_0^t (\|\frac{1}{f}\|_3^2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_1^2(\|\mathbf{w}^k\|_4^2+\|\nabla\phi\|_4^2) \\
&\quad + \|\frac{1}{f^2}\nabla f\|_3^2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_1^2(\|\mathbf{w}^k\|_3^2+\|\nabla\phi\|_3^2))d\tau \\
&\quad + Ce^{\beta t}\int_0^t (\|f\|_3^2\|g(h^k)\|_3^2\|\nabla(h^{k+1}-h^k)\|_1^2 + \|g(h^k)-g(h^{k-1})\|_1^2\|f\|_3^2\|\nabla h^k\|_3^2)d\tau \\
&\le Ce^{\beta t}\int_0^t \Big(\|\frac{1}{f}\|_3^2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_1^2(\epsilon L_1+\|f_t\|_3^2) \\
&\quad + \|\frac{1}{f}\|_3^4\|\nabla f\|_3^2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_1^2(\epsilon L_1+\|f_t\|_2^2)\Big)d\tau \\
&\quad + Ce^{\beta t}\int_0^t (\|f\|_3^2(L_5)\Big(C_7\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2 + C_7\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_1^2 \\
&\quad + \epsilon C_7\|\nabla(h^k-h^{k-1})\|_1^2\Big)d\tau \\
&\quad + Ce^{\beta t}\int_0^t \epsilon\|\nabla(h^k-h^{k-1})\|_1^2\|f\|_3^2(\epsilon L_2)d\tau \\
&\le C_8\int_0^t (\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_1^2 + \epsilon\|\nabla(h^k-h^{k-1})\|_1^2)d\tau + C_8\int_0^t \|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2 d\tau
\end{aligned} \tag{2.30}$$

where

$$\beta \le C\Big(1+\|\frac{1}{f}\|_{3,T}(\|\mathbf{w}^k\|_{3,T}+\|\nabla\phi\|_{3,T}) + \|\frac{1}{f^2}\|_{3,T}\|\nabla f\|_{3,T}(\|\mathbf{w}^k\|_{3,T}$$

$$+ \|\nabla\phi\|_{3,T}) + \|\frac{1}{f}\frac{\partial f}{\partial t}\|_{3,T}\Big) \le C_3$$

from estimate (2.13). Here we used estimate (2.29) for $\|\nabla(h^{k+1} - h^k)\|_1^2$, and we used estimate (2.28) for $\|g(h^k) - g(h^{k-1})\|_1^2$. And we used the estimates $\|\mathbf{w}^k\|_4^2 \le \epsilon L_1 \le 1$, $\|\nabla h^k\|_3^2 \le \epsilon L_2 \le 1$, and $\|g(h^k)\|_3^2 \le L_5$, where $L_5 = C|g|^2_{s,\overline{G}_1}$, from Proposition 2.2. And we used the assumption that $\|f_t\|_3^2 \le \epsilon < 1$. And C_8 depends on $\|\frac{1}{f}\|_{3,T}$, $\|\nabla f\|_{3,T}$, $\|f\|_{3,T}$, and $|g|_{s,\overline{G}_1}$.

Applying Gronwall's inequality to (2.30) yields

$$\|\mathbf{w}^{k+1} - \mathbf{w}^k\|_1^2 \le C_9 \int_0^t (\|\mathbf{w}^k - \mathbf{w}^{k-1}\|_1^2 + \epsilon\|\nabla(h^k - h^{k-1})\|_1^2)d\tau \tag{2.31}$$

where $C_9 = C_8(1 + C_8 T e^{C_8 T})$. Substituting (2.31) into (2.29) yields

$$\begin{aligned} \|\nabla(h^{k+1} - h^k)\|_1^2 &\le C_{10}(\|\mathbf{w}^k - \mathbf{w}^{k-1}\|_1^2 + \epsilon\|\nabla(h^k - h^{k-1})\|_1^2) \\ &\quad + C_{10}\int_0^t (\|\mathbf{w}^k - \mathbf{w}^{k-1}\|_1^2 + \epsilon\|\nabla(h^k - h^{k-1})\|_1^2)d\tau \end{aligned} \tag{2.32}$$

where $C_{10} = C_7(1 + C_9)$.

Multiplying (2.32) by ϵ and then adding the resulting inequality to (2.31), yields

$$\begin{aligned} &\|\mathbf{w}^{k+1} - \mathbf{w}^k\|_1^2 + \epsilon\|\nabla(h^{k+1} - h^k)\|_1^2 \\ &\le \epsilon C_{11}\Big(\|\mathbf{w}^k - \mathbf{w}^{k-1}\|_1^2 + \epsilon\|\nabla(h^k - h^{k-1})\|_1^2\Big) \\ &\quad + C_{11}\int_0^t (\|\mathbf{w}^k - \mathbf{w}^{k-1}\|_1^2 + \epsilon\|\nabla(h^k - h^{k-1})\|_1^2)d\tau \end{aligned} \tag{2.33}$$

where $C_{11} = C_9 + C_{10}$.

Applying Lemma 4.6 to (2.33), where we define $Z^k(t) = \|\mathbf{w}^k - \mathbf{w}^{k-1}\|_1^2 + \epsilon\|\nabla(h^k - h^{k-1})\|_1^2$, yields the inequality

$$\begin{aligned} &\|\mathbf{w}^{k+1} - \mathbf{w}^k\|_{1,T}^2 + \epsilon\|\nabla(h^{k+1} - h^k)\|_{1,T}^2 \\ &\le (2\epsilon C_{11})^k e^{\frac{T}{\epsilon}}\left(\|\mathbf{w}^1 - \mathbf{w}^0\|_{1,T}^2 + \epsilon\|\nabla(h^1 - h^0)\|_{1,T}^2\right) \end{aligned} \tag{2.34}$$

where we choose ϵ sufficiently small so that $2\epsilon C_{11} < 1$. Then it follows from (2.34) that

$$\sum_{k=0}^{\infty}(\|\mathbf{w}^{k+1} - \mathbf{w}^k\|_{1,T}^2 + \epsilon\|\nabla(h^{k+1} - h^k)\|_{1,T}^2) < \infty. \tag{2.35}$$

This completes the proof of part (a) of Proposition 2.3.

By Lemma 4.3 and the fact that $h^{k+1}(\mathbf{x}_0, t) = h^k(\mathbf{x}_0, t)$, we have the inequality $\|h^{k+1} - h^k\|_2^2 \le C\|\nabla(h^{k+1} - h^k)\|_1^2$. Then from (2.35), it follows that

$$\sum_{k=0}^{\infty}\|h^{k+1} - h^k\|_{2,T}^2 \le C\sum_{k=0}^{\infty}\|\nabla(h^{k+1} - h^k)\|_{1,T}^2 < \infty. \tag{2.36}$$

This completes the proof of part (b) of Proposition 2.3.

Estimate for $\|\mathbf{w}_t^{k+1} - \mathbf{w}_t^k\|_0^2$: From (2.25) we obtain the inequality

$$\begin{aligned} &\|\frac{\partial}{\partial t}(\mathbf{w}^{k+1} - \mathbf{w}^k)\|_0^2 \\ &\le C\|\frac{1}{f}(\mathbf{w}^k + \nabla\phi)\cdot\nabla(\mathbf{w}^{k+1} - \mathbf{w}^k)\|_0^2 + C\|\frac{1}{f}(\mathbf{w}^k - \mathbf{w}^{k-1})\cdot\nabla(\mathbf{w}^k + \nabla\phi)\|_0^2 \end{aligned}$$

$$
\begin{aligned}
&+ C\|\frac{1}{f^2}(\nabla f\cdot(\mathbf{w}^k+\nabla\phi))(\mathbf{w}^{k+1}-\mathbf{w}^k)\|_0^2\\
&+ C\|\frac{1}{f^2}(\nabla f\cdot(\mathbf{w}^k-\mathbf{w}^{k-1}))(\mathbf{w}^k+\nabla\phi)\|_0^2 + C\|(\mathbf{w}^{k+1}-\mathbf{w}^k)\frac{1}{f}\frac{\partial f}{\partial t}\|_0^2\\
&+ C\|fg(h^k)\nabla(h^{k+1}-h^k)\|_0^2 + C\|f(g(h^k)-g(h^{k-1}))\nabla h^k\|_0^2\\
\le{}& C\big|\frac{1}{f}\big|_{L^\infty}^2(|\mathbf{w}^k|_{L^\infty}^2+|\nabla\phi|_{L^\infty}^2)\|\nabla(\mathbf{w}^{k+1}-\mathbf{w}^k)\|_0^2\\
&+ C\big|\frac{1}{f}\big|_{L^\infty}^2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_0^2(|D\mathbf{w}^k|_{L^\infty}^2+|D(\nabla\phi)|_{L^\infty}^2)\\
&+ C\big|\frac{1}{f^2}\big|_{L^\infty}^2|\nabla f|_{L^\infty}^2(|\mathbf{w}^k|_{L^\infty}^2+|\nabla\phi|_{L^\infty}^2)\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_0^2\\
&+ C\big|\frac{1}{f^2}\big|_{L^\infty}^2|\nabla f|_{L^\infty}^2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_0^2(|\mathbf{w}^k|_{L^\infty}^2+|\nabla\phi|_{L^\infty}^2)\\
&+ C\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_0^2\big|\frac{1}{f}\frac{\partial f}{\partial t}\big|_{L^\infty}^2 + C|f|_{L^\infty}^2|g(h^k)|_{L^\infty}^2\|\nabla(h^{k+1}-h^k)\|_0^2\\
&+ C|f|_{L^\infty}^2\|g(h^k)-g(h^{k-1})\|_0^2|\nabla h^k|_{L^\infty}^2\\
\le{}& C\big\|\frac{1}{f}\big\|_2^2(\epsilon L_1+\|f_t\|_1^2)\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2 + C\big\|\frac{1}{f}\big\|_2^2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_1^2(\epsilon L_1+\|f_t\|_2^2)\\
&+ C\big\|\frac{1}{f}\big\|_2^4\|\nabla f\|_2^2(\epsilon L_1+\|f_t\|_1^2)\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2\\
&+ C\Big\|\frac{1}{f}\Big\|_2^4\|\nabla f\|_2^2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_1^2(\epsilon L_1+\|f_t\|_1^2)\\
&+ C\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_1^2\big\|\frac{1}{f}\big\|_2^2\|f_t\|_2^2 + C\|f\|_2^2(L_5)\|\nabla(h^{k+1}-h^k)\|_1^2\\
&+ C\|f\|_2^2\|g(h^k)-g(h^{k-1})\|_0^2(\epsilon L_2)\\
\le{}& C_{12}\Big(\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_{1,T}^2 + \|\mathbf{w}^k-\mathbf{w}^{k-1}\|_{1,T}^2 + \|\nabla(h^{k+1}-h^k)\|_{1,T}^2\\
&+ \epsilon\|h^k-h^{k-1}\|_{2,T}^2\Big)
\end{aligned}\tag{2.37}
$$

where we used the estimates $|g(h^k)|_{L^\infty}^2 \le C\|g(h^k)\|_2^2 \le CL_5$, where $L_5 = C|g|_{s,\overline{G}_1}^2$, and $|\mathbf{w}^k|_{L^\infty}^2 \le C\|\mathbf{w}^k\|_2^2 \le C\epsilon L_1$, $|D\mathbf{w}^k|_{L^\infty}^2 \le C\|\mathbf{w}^k\|_3^2 \le C\epsilon L_1$, and $|\nabla h^k|_{L^\infty}^2 \le C\|\nabla h^k\|_2^2 \le C\epsilon L_2$ from Lemma 4.1 and Proposition 2.2, where $\epsilon L_1 \le 1$ and $\epsilon L_2 \le 1$. And we used the estimate $\|g(h^k)-g(h^{k-1})\|_0^2 \le C\big|\frac{dg}{dh}\big|_{0,\overline{G}_1}^2\|h^k-h^{k-1}\|_0^2 \le C\epsilon\|h^k - h^{k-1}\|_2^2$ by Lemma 4.5 and estimate (2.23), where we use the fact that $\big|\frac{dg}{dh}\big|_{0,\overline{G}_1}^2 \le \big|\frac{dg}{dh}\big|_{1,\overline{G}_1}^2 \le C\epsilon$. And we used the estimates $|\nabla\phi|_{L^\infty}^2 \le C\|\nabla\phi\|_2^2 \le C\|f_t\|_1^2$ and $|D(\nabla\phi)|_{L^\infty}^2 \le C\|\nabla\phi\|_3^2 \le C\|f_t\|_2^2$ from Lemma 4.7. And we used the assumption that $\|f_t\|_2^2 \le \epsilon < 1$. And C_{12} depends on $\|\frac{1}{f}\|_{2,T}$, $\|\nabla f\|_{2,T}$, $\|f\|_{2,T}$, and $|g|_{s,\overline{G}_1}$.

Then by (2.35), (2.36), (2.37), it follows that

$$
\sum_{k=0}^{\infty}\|\frac{\partial}{\partial t}(\mathbf{w}^{k+1}-\mathbf{w}^k)\|_{0,T}^2 < \infty. \tag{2.38}
$$

This completes the proof of Proposition 2.3. □

Using Proposition 2.2 and Proposition 2.3, we now complete the proof of Theorem 2.1. From Proposition 2.3, $\|\mathbf{w}^{k+1} - \mathbf{w}^k\|_{1,T}^2 \to 0$ and $\|h^{k+1} - h^k\|_{2,T}^2 \to 0$ as $k \to \infty$. Therefore, we conclude that there exist $\mathbf{w} \in C([0,T], H^1(\Omega))$ and $h \in C([0,T], H^2(\Omega))$ such that $\|\mathbf{w}^k - \mathbf{w}\|_{1,T} \to 0$, and $\|h^k - h\|_{2,T} \to 0$, as $k \to \infty$. Using the interpolation inequalities $\|g\|_{s'} \le C\|g\|_1^{\gamma}\|g\|_s^{1-\gamma}$ and $\|g\|_{s'+1} \le C\|g\|_2^{\gamma}\|g\|_{s+1}^{1-\gamma}$, where $\gamma = \frac{s-s'}{s-1}$, and $1 < s' < s$, and $s \ge 4$ (see Lemma 4.1), and using Proposition 2.2, we can conclude that $\|\mathbf{w}^k - \mathbf{w}\|_{s',T} \to 0$ and $\|h^k - h\|_{s'+1,T} \to 0$ as $k \to \infty$ for $1 < s' < s$. For $s' > \frac{N}{2} + 2$, Sobolev's lemma implies that $\mathbf{w}^k \to \mathbf{w}$ in $C([0,T], C^2(\Omega))$ and $h^k \to h$ in $C([0,T], C^3(\Omega))$. From the linear system of equations (2.8), (2.9) it follows that $\|\mathbf{w}_t^k - \mathbf{w}_t\|_{s'-1,T} \to 0$, as $k \to \infty$, so that $\mathbf{w}_t^k \to \mathbf{w}_t$ in $C([0,T], C^1(\Omega))$, and $\mathbf{w}$, h is a classical solution of the system of equations (2.5), (2.6). The additional facts that $\mathbf{w} \in L^\infty([0,T], H^s(\Omega))$, and $h \in L^\infty([0,T], H^{s+1}(\Omega))$ can be deduced using boundedness in high norm and a standard compactness argument (see, for example, Embid [6], Majda [8]). The uniqueness of the solution follows from the proof of Proposition 2.3 by a standard argument using estimates similar to the estimates used in the proof of Proposition 2.3. And therefore $\mathbf{v}$, ρ is a unique classical solution of the system of equations (1.2), (1.4), where $\mathbf{v} = \frac{\mathbf{w}+\nabla\phi}{f}$, $f = \frac{1}{c_0 c_3}\psi$, and $\rho = e^h$. This completes the proof of the theorem. □

A final remark We now present a proof that the standard conservation of mass equation $\frac{\partial\rho}{\partial t} + \mathbf{v}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{v} = 0$ is approximately satisfied by the solution ρ, $\mathbf{v}$ to the system of equations (1.2), (1.4), provided that the hypotheses of Theorem 2.1 are satisfied and that $\max_{0\le t\le T}\big|\frac{b'(t)}{b(t)}\big|^2 \le C$, where C is a generic constant which does not depend on ϵ.

We use the following inequality:

$$
\begin{aligned}
&|\frac{\partial\rho}{\partial t} + \mathbf{v}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{v}|_{L^\infty}^2 \\
&\le C|\rho_t|_{L^\infty}^2 + C|\mathbf{v}|_{L^\infty}^2|\nabla\rho|_{L^\infty}^2 + C|\rho|_{L^\infty}^2|\nabla\cdot\mathbf{v}|_{L^\infty}^2 \\
&\le C|\rho_t|_{L^\infty}^2 + C_{13}|\nabla\rho|_{L^\infty}^2 + C_{13}|\rho|_{L^\infty}^2
\end{aligned} \tag{2.39}
$$

where we used the estimate

$$
\begin{aligned}
|\mathbf{v}|_{L^\infty}^2 &\le C\|\mathbf{v}\|_2^2 = C\|\frac{(\mathbf{w}+\nabla\phi)}{f}\|_2^2 \\
&\le C\|\frac{1}{f}\|_2^2(\|\mathbf{w}\|_2^2 + \|\nabla\phi\|_2^2) \\
&\le C\|\frac{1}{f}\|_2^2(\epsilon L_1 + \|f_t\|_1^2) \le C\|\frac{1}{f}\|_{2,T}^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|\nabla\cdot\mathbf{v}|_{L^\infty}^2 &\le C\|\nabla\cdot\mathbf{v}\|_2^2 \le C\|\mathbf{v}\|_3^2 \\
&= C\|\frac{(\mathbf{w}+\nabla\phi)}{f}\|_3^2 \le C\|\frac{1}{f}\|_3^2(\|\mathbf{w}\|_3^2 + \|\nabla\phi\|_3^2) \\
&\le C\|\frac{1}{f}\|_3^2(\epsilon L_1 + \|f_t\|_2^2) \le C\|\frac{1}{f}\|_{3,T}^2.
\end{aligned}
$$

We used the estimate $\|\mathbf{w}\|_s^2 \le \epsilon L_1 \le 1$, and the assumption that $\|f_t\|_2^2 \le \epsilon < 1$. Also we use $C_{13} = C\|\frac{1}{f}\|_{3,T}^2$.

We next obtain estimates for $|\rho|^2_{L^\infty}$, $|\nabla\rho|^2_{L^\infty}$, and $|\rho_t|^2_{L^\infty}$ to use in inequality (2.39).

Estimate for $|\rho|^2_{L^\infty}$: By inequality (2.19), and using the fact that $\rho = e^{\ln(\rho)} = e^h$, we have:

$$|\rho|^2_{L^\infty} = |e^h|^2_{L^\infty} \le (e^{\frac{1}{2}\ln(\epsilon)})^2 = \epsilon \tag{2.40}$$

Estimate for $|\nabla\rho|^2_{L^\infty}$: Using the fact that $\nabla\rho = \rho\nabla(\ln(\rho)) = \rho\nabla h$, we obtain the estimate

$$|\nabla\rho|^2_{L^\infty} = |\rho\nabla h|^2_{L^\infty} \le |\rho|^2_{L^\infty}|\nabla h|^2_{L^\infty} \le \epsilon(C\epsilon L_2) \le \epsilon C \tag{2.41}$$

where we used (2.40) and we used the estimate $|\nabla h|^2_{L^\infty} \le C\|\nabla h\|^2_2 \le C(\epsilon L_2)$, where $\epsilon L_2 \le 1$.

Estimate for $|\rho_t|^2_{L^\infty}$: Using the fact that $\rho_t = \rho(\ln(\rho))_t = \rho h_t$, and using (2.40), we obtain

$$|\rho_t|^2_{L^\infty} = |\rho h_t|^2_{L^\infty} \le |\rho|^2_{L^\infty}|h_t|^2_{L^\infty} \le \epsilon|h_t|^2_{L^\infty} \le \epsilon C\|h_t\|^2_2 \tag{2.42}$$

By Lemma 4.3 in Section 4 and by the fact that $h_t(\mathbf{x}_0,t) = \frac{\rho_t(\mathbf{x}_0,t)}{\rho(\mathbf{x}_0,t)} = \frac{b'(t)}{b(t)}$, we obtain the estimate

$$\begin{aligned}
\|h_t\|^2_2 &\le C(\|h_t(\mathbf{x}_0,t)\|^2_2 + \|\nabla(h_t(\mathbf{x}_0,t))\|^2_1 + \|\nabla h_t\|^2_1) \\
&\le C(|\Omega| \max_{0\le t\le T}|h_t(\mathbf{x}_0,t)|^2 + \|\nabla h_t\|^2_1) \\
&\le C(|\Omega| \max_{0\le t\le T}\Big|\frac{b'(t)}{b(t)}\Big|^2 + \|\nabla h_t\|^2_1) \\
&\le C(|\Omega| + \|\nabla h_t\|^2_1)
\end{aligned} \tag{2.43}$$

where we used the assumption that $\max_{0\le t\le T}\big|\frac{b'(t)}{b(t)}\big|^2 \le C$.

Estimate for $\|\nabla h_t\|^2_1$: Applying $\frac{\partial}{\partial t}$ to equation (2.5) yields

$$\begin{aligned}
&\frac{\partial^2\mathbf{w}}{\partial t^2} + fg(h)\nabla h_t \\
&= -\frac{\partial}{\partial t}(\frac{1}{f}(\mathbf{w}+\nabla\phi)\cdot\nabla(\mathbf{w}+\nabla\phi)) + \frac{\partial}{\partial t}(\frac{1}{f^2}(\nabla f\cdot(\mathbf{w}+\nabla\phi))(\mathbf{w}+\nabla\phi)) \\
&\quad + \frac{\partial}{\partial t}((\mathbf{w}+\nabla\phi)\frac{1}{f}\frac{\partial f}{\partial t}) - f_t g(h)\nabla h - f\frac{dg}{dh}\frac{\partial h}{\partial t}\nabla h - \frac{\partial^2\nabla\phi}{\partial t^2}
\end{aligned} \tag{2.44}$$

Next, applying the divergence operator to (2.44) yields

$$\begin{aligned}
&\nabla\cdot(fg(h)\nabla h_t) \\
&= -\nabla\cdot\Big((\frac{\partial}{\partial t}(\frac{1}{f}))(\mathbf{w}+\nabla\phi)\cdot\nabla(\mathbf{w}+\nabla\phi)\Big) - \nabla\cdot(\frac{1}{f}(\mathbf{w}_t+\nabla\phi_t)\cdot\nabla(\mathbf{w}+\nabla\phi)) \\
&\quad - \nabla\cdot(\frac{1}{f}(\mathbf{w}+\nabla\phi)\cdot\nabla(\mathbf{w}_t+\nabla\phi_t)) + \nabla\cdot(\Big(\frac{\partial}{\partial t}(\frac{1}{f^2}\nabla f)\cdot(\mathbf{w}+\nabla\phi)\Big)(\mathbf{w}+\nabla\phi)) \\
&\quad + \nabla\cdot(\frac{1}{f^2}(\nabla f\cdot(\mathbf{w}_t+\nabla\phi_t))(\mathbf{w}+\nabla\phi)) + \nabla\cdot(\frac{1}{f^2}(\nabla f\cdot(\mathbf{w}+\nabla\phi))(\mathbf{w}_t+\nabla\phi_t)) \\
&\quad + \nabla\cdot((\mathbf{w}_t+\nabla\phi_t)\frac{f_t}{f}) + \nabla\cdot((\mathbf{w}+\nabla\phi)\frac{\partial}{\partial t}\Big(\frac{f_t}{f}\Big)) - \nabla\cdot(f_t g(h)\nabla h) \\
&\quad - \nabla f\cdot(\frac{dg}{dh}h_t\nabla h) - \nabla(\frac{dg}{dh})\cdot(fh_t\nabla h) - f\frac{dg}{dh}(\nabla h_t\cdot\nabla h) - f\frac{dg}{dh}h_t\Delta h - \frac{\partial^2\Delta\phi}{\partial t^2}
\end{aligned} \tag{2.45}$$

We will use the well-known inequality $\|uv\|_1^2 \le \|uv\|_2^2 \le C\|u\|_2^2\|v\|_2^2$ (see Lemma 4.1). Then applying estimate (4.20) from Lemma 4.7 to (2.45), and using the fact that $f(\mathbf{x},t)g(h(\mathbf{x},t)) \ge 1$, yields

$$
\begin{aligned}
&\|\nabla h_t\|_1^2 \\
&\le C\sum_{j=0}^{1}\|D(fg(h))\|_2^{2j}(\|\big(\frac{\partial}{\partial t}\big(\frac{1}{f}\big)\big)(\mathbf{w}+\nabla\phi)\cdot\nabla(\mathbf{w}+\nabla\phi)\|_1^2 \\
&\quad +\|\frac{1}{f}(\mathbf{w}_t+\nabla\phi_t)\cdot\nabla(\mathbf{w}+\nabla\phi)\|_1^2) \\
&\quad + C\sum_{j=0}^{1}\|D(fg(h))\|_2^{2j}(\|\frac{1}{f}(\mathbf{w}+\nabla\phi)\cdot\nabla(\mathbf{w}_t+\nabla\phi_t)\|_1^2 \\
&\quad +\|\left(\frac{\partial}{\partial t}(\frac{1}{f^2}\nabla f)\cdot(\mathbf{w}+\nabla\phi)\right)(\mathbf{w}+\nabla\phi)\|_1^2) \\
&\quad + C\sum_{j=0}^{1}\|D(fg(h))\|_2^{2j}(\|\frac{1}{f^2}(\nabla f\cdot(\mathbf{w}_t+\nabla\phi_t))(\mathbf{w}+\nabla\phi)\|_1^2 \\
&\quad +\|\frac{1}{f^2}(\nabla f\cdot(\mathbf{w}+\nabla\phi))(\mathbf{w}_t+\nabla\phi_t)\|_1^2) \\
&\quad + C\sum_{j=0}^{1}\|D(fg(h))\|_2^{2j}(\|(\mathbf{w}_t+\nabla\phi_t)\frac{f_t}{f}\|_1^2+\|(\mathbf{w}+\nabla\phi)\frac{\partial}{\partial t}\big(\frac{f_t}{f}\big)\|_1^2 \\
&\quad +\|f_t g(h)\nabla h\|_1^2+\|\nabla f\cdot(\frac{dg}{dh}h_t\nabla h)\|_0^2) \\
&\quad + C\sum_{j=0}^{1}\|D(fg(h))\|_2^{2j}(\|\nabla(\frac{dg}{dh})\cdot(fh_t\nabla h)\|_0^2+\|f\frac{dg}{dh}(\nabla h_t\cdot\nabla h)\|_0^2 \\
&\quad +\|f\frac{dg}{dh}h_t\Delta h\|_0^2+\|\frac{\partial^2\Delta\phi}{\partial t^2}\|_0^2) \\
&\le C\sum_{j=0}^{1}\|f\|_3^{2j}\|g(h)\|_3^{2j}(\|\frac{\partial}{\partial t}\big(\frac{1}{f}\big)\|_2^2\|\mathbf{w}+\nabla\phi\|_2^2\|\nabla(\mathbf{w}+\nabla\phi)\|_2^2 \\
&\quad +\|\frac{1}{f}\|_2^2\|\mathbf{w}_t+\nabla\phi_t\|_2^2\|\nabla(\mathbf{w}+\nabla\phi)\|_2^2) \\
&\quad + C\sum_{j=0}^{1}\|f\|_3^{2j}\|g(h)\|_3^{2j}(\|\frac{1}{f}\|_2^2\|\mathbf{w}+\nabla\phi\|_2^2\|\nabla(\mathbf{w}_t+\nabla\phi_t)\|_2^2 \\
&\quad +\|\frac{\partial}{\partial t}\big(\frac{1}{f^2}\nabla f\big)\|_2^2\|\mathbf{w}+\nabla\phi\|_2^2\|\mathbf{w}+\nabla\phi\|_2^2) \\
&\quad + C\sum_{j=0}^{1}\|f\|_3^{2j}\|g(h)\|_3^{2j}(\|\frac{1}{f^2}\nabla f\|_2^2\|\mathbf{w}_t+\nabla\phi_t\|_2^2\|\mathbf{w}+\nabla\phi\|_2^2 \\
&\quad +\|\frac{1}{f^2}\nabla f\|_2^2\|\mathbf{w}+\nabla\phi\|_2^2\|\mathbf{w}_t+\nabla\phi_t\|_2^2)
\end{aligned}
$$

$$
\begin{aligned}
&+ C\sum_{j=0}^{1}\|f\|_3^{2j}\|g(h)\|_3^{2j}(\|\mathbf{w}_t+\nabla\phi_t\|_2^2\|\frac{f_t}{f}\|_2^2+\|\mathbf{w}+\nabla\phi\|_2^2\|\frac{\partial}{\partial t}(\frac{f_t}{f})\|_2^2\\
&+\|f_t\|_2^2\|g(h)\|_2^2\|\nabla h\|_2^2)\\
&+ C\sum_{j=0}^{1}\|f\|_3^{2j}\|g(h)\|_3^{2j}(|\nabla f|_{L^\infty}^2|\frac{dg}{dh}|_{L^\infty}^2|h_t|_{L^\infty}^2\|\nabla h\|_0^2\\
&+\|\nabla(\frac{dg}{dh})\|_0^2|f|_{L^\infty}^2|h_t|_{L^\infty}^2|\nabla h|_{L^\infty}^2)\\
&+ C\sum_{j=0}^{1}\|f\|_3^{2j}\|g(h)\|_3^{2j}(|f|_{L^\infty}^2|\frac{dg}{dh}|_{L^\infty}^2\|\nabla h_t\|_0^2|\nabla h|_{L^\infty}^2\\
&+|f|_{L^\infty}^2|\frac{dg}{dh}|_{L^\infty}^2|h_t|_{L^\infty}^2\|\Delta h\|_0^2+\|\frac{\partial^2\Delta\phi}{\partial t^2}\|_0^2)\\
\le{}& C\sum_{j=0}^{1}\|f\|_3^{2j}(L_5)^j(\|\frac{\partial}{\partial t}(\frac{1}{f})\|_2^2(\epsilon L_1+\|f_t\|_1^2)(\epsilon L_1\\
&+\|f_t\|_2^2)+\|\frac{1}{f}\|_2^2(L_4+\|f_{tt}\|_1^2)(\epsilon L_1+\|f_t\|_2^2))\\
&+ C\sum_{j=0}^{1}\|f\|_3^{2j}(L_5)^j(\|\frac{1}{f}\|_2^2(\epsilon L_1+\|f_t\|_1^2)(L_4+\|f_{tt}\|_2^2)\\
&+\|\frac{\partial}{\partial t}(\frac{1}{f^2}\nabla f)\|_2^2(\epsilon L_1+\|f_t\|_1^2)(\epsilon L_1+\|f_t\|_1^2))\\
&+ C\sum_{j=0}^{1}\|f\|_3^{2j}(L_5)^j(\|\frac{1}{f^2}\nabla f\|_2^2(L_4+\|f_{tt}\|_1^2)(\epsilon L_1+\|f_t\|_1^2)\\
&+\|\frac{1}{f^2}\nabla f\|_2^2(\epsilon L_1+\|f_t\|_1^2)(L_4+\|f_{tt}\|_1^2))\\
&+ C\sum_{j=0}^{1}\|f\|_3^{2j}(L_5)^j((L_4+\|f_{tt}\|_1^2)\|\frac{f_t}{f}\|_2^2+(\epsilon L_1+\|f_t\|_1^2)\|\frac{\partial}{\partial t}(\frac{f_t}{f})\|_2^2\\
&+\|f_t\|_2^2L_5(\epsilon L_2)+\|\nabla f\|_2^2(\epsilon)\|h_t\|_2^2(\epsilon L_2))\\
&+ C\sum_{j=0}^{1}\|f\|_3^{2j}(L_5)^j\Big((\epsilon)(\epsilon L_2)\|f\|_2^2\|h_t\|_2^2(\epsilon L_2)+\|f\|_2^2(\epsilon)\|h_t\|_1^2(\epsilon L_2)\\
&+\|f\|_2^2(\epsilon)\|h_t\|_2^2(\epsilon L_2)+\|f_{ttt}\|_0^2\Big)\\
\le{}& C_{14}(1+\epsilon\|h_t\|_2^2) \qquad (2.46)
\end{aligned}
$$

Here we used the estimates $\|\nabla(\mathbf{w}_t+\nabla\phi_t)\|_2^2 \le C\|\mathbf{w}_t+\nabla\phi_t\|_3^2 \le C(\|\mathbf{w}_t\|_3^2+\|\nabla\phi_t\|_3^2) \le C(L_4+\|f_{tt}\|_2^2)$, and $\|\nabla(\mathbf{w}+\nabla\phi)\|_2^2 \le C\|\mathbf{w}+\nabla\phi\|_3^2 \le C(\|\mathbf{w}\|_3^2+\|\nabla\phi\|_3^2) \le C(\epsilon L_1+\|f_t\|_2^2)$, where $\epsilon L_1 \le 1$. We used the fact that $\Delta\phi = -f_t$, and we used the estimate $\|\nabla\phi_t\|_3^2 \le C\|f_{tt}\|_2^2$ from Lemma 4.7 in Section 4. And we used the estimate $\|\Delta h\|_0^2 \le C\|\nabla h\|_1^2 \le C(\epsilon L_2)$, where $\epsilon L_2 \le 1$. And we used the inequalities $\|f_t\|_3^2 \le \epsilon < 1$, $\|f_{tt}\|_2^2 \le \epsilon < 1$, $\|g(h)\|_3^2 \le L_5$. We used the estimate $|\frac{dg}{dh}|_{L^\infty}^2 \le \big|\frac{dg}{dh}\big|_{0,\overline{G}_1}^2 \le \big|\frac{dg}{dh}\big|_{1,\overline{G}_1}^2 \le C\epsilon$ by inequality (2.23). And we used Lemma 4.4

to obtain the estimate $\|\nabla(\frac{dg}{dh})\|_0^2 \le \big|\frac{d^2g}{dh^2}\big|^2_{0,\overline{G}_1}\|\nabla h\|_0^2 \le \big|\frac{dg}{dh}\big|^2_{1,\overline{G}_1}\|\nabla h\|_0^2 \le (C\epsilon)(\epsilon L_2)$. Here C_{14} depends on $\|f\|_{s,T}$, $\|\frac{1}{f}\|_{s,T}$, $\|\nabla f\|_{s,T}$, $|g|_{s,\overline{G}_1}$, and $\|f_{ttt}\|_{0,T}$.

By substituting estimate (2.46) into (2.43), it follows that

$$\|h_t\|_2^2 \le C(|\Omega| + \|\nabla h_t\|_1^2) \le C(|\Omega| + C_{14}(1+\epsilon\|h_t\|_2^2)) \tag{2.47}$$

Choosing ϵ sufficiently small so that $CC_{14}\epsilon \le \frac{1}{2}$, and re-arranging terms in (2.47), yields

$$\|h_t\|_2^2 \le C(|\Omega| + C_{14}) \tag{2.48}$$

By substituting estimate (2.48) into estimate (2.42), it follows that

$$|\rho_t|^2_{L^\infty} \le \epsilon C\|h_t\|_2^2 \le \epsilon C(|\Omega| + C_{14}) = \epsilon C_{15} \tag{2.49}$$

where C_{15} depends on $\|f\|_{s,T}$, $\|\frac{1}{f}\|_{s,T}$, $\|\nabla f\|_{s,T}$, $|g|_{s,\overline{G}_1}$, and $\|f_{ttt}\|_{0,T}$.

From substituting estimates (2.40), (2.41), (2.49) into (2.39), we obtain

$$|\frac{\partial\rho}{\partial t} + \mathbf{v}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{v}|^2_{L^\infty} \le C|\rho_t|^2_{L^\infty} + C_{13}|\nabla\rho|^2_{L^\infty} + C_{13}|\rho|^2_{L^\infty} \le \epsilon C_{16} \tag{2.50}$$

where C_{16} depends on $\|f\|_{s,T}$, $\|\frac{1}{f}\|_{s,T}$, $\|\nabla f\|_{s,T}$, $|g|_{s,\overline{G}_1}$, and $\|f_{ttt}\|_{0,T}$.

It follows from (2.50) that the standard conservation of mass equation $\frac{\partial\rho}{\partial t} + \mathbf{v}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{v} = 0$ is approximately satisfied by the solution ρ, $\mathbf{v}$ to the system of equations (1.2), (1.4).

3. Existence for the linear system of equations

In this section we present the proof of the existence of a solution to the linear equations (2.8), (2.9) used in the iteration scheme in Section 2. Lemmas supporting the proof appear in Section 4.

Lemma 3.1. *Let $\mathbf{w}_0 \in H^s(\Omega)$ where $\nabla\cdot\mathbf{w}_0 = 0$, and let $\mathbf{b}_1 \in C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^s(\Omega))$, $b_2 \in C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^s(\Omega))$, $b_3 \in C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^s(\Omega))$, $\mathbf{g} \in C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^s(\Omega))$, where $s > \frac{N}{2} + 2$, for $N = 2,3$, and where $b_3(\mathbf{x},t) \ge 1$ for $\mathbf{x} \in \Omega = \mathbb{T}^N$, $0 \le t \le T$. Let b be a given positive smooth function of t for $0 \le t \le T$. Let $\mathbf{x}_0 \in \Omega$ be a given point. Then there exists a unique classical solution $\mathbf{w}$, h for $\mathbf{x} \in \Omega$ and $0 \le t \le T$ of*

$$\frac{\partial\mathbf{w}}{\partial t} + \mathbf{b}_1\cdot\nabla\mathbf{w} + b_2\mathbf{w} + b_3\nabla h = \mathbf{g}, \tag{3.1}$$

$$\nabla\cdot\mathbf{w} = 0, \tag{3.2}$$

$$h(\mathbf{x}_0,t) = \ln(b(t)), \quad \mathbf{w}(\mathbf{x},0) = \mathbf{w}_0(\mathbf{x}), \quad \nabla\cdot\mathbf{w}_0 = 0, \tag{3.3}$$

and

$$\mathbf{w} \in C([0,T], C^2(\Omega)) \cap L^\infty([0,T], H^s(\Omega)),$$
$$h \in C([0,T], C^3(\Omega)) \cap L^\infty([0,T], H^{s+1}(\Omega)).$$

Proof. First, we change the equations to an equivalent system by employing the projections P and $Q = I - P$, where P is the orthogonal projection of $L^2(\Omega)$ onto the solenoidal vector field and Q is the orthogonal projection of $L^2(\Omega)$ onto the gradient vector field. Applying the operator P to (3.1), and using the fact that $P\mathbf{w} = \mathbf{w}$, we obtain the equation

$$\frac{\partial\mathbf{w}}{\partial t} + \mathbf{b}_1\cdot\nabla\mathbf{w} + b_2\mathbf{w} = J := Q(\mathbf{b}_1\cdot\nabla P\mathbf{w}) + Q(b_2P\mathbf{w}) - P(b_3\nabla h) + P\mathbf{g} \tag{3.4}$$

Applying the operator Q to (3.1), and using the fact that $P\mathbf{w} = \mathbf{w}$, we obtain the equation

$$Q\left(\mathbf{b}_1 \cdot \nabla P\mathbf{w} + b_2 P\mathbf{w} + b_3 \nabla h - \mathbf{g}\right) = 0. \tag{3.5}$$

With the given initial data $\mathbf{w}_0$, where $\nabla \cdot \mathbf{w}_0 = 0$, the system of equations (3.1), (3.2) and the system of equations (3.4), (3.5) are equivalent. (The proof is standard; see, for example, Embid [6]).

From the definition of Q, it follows that equation (3.5) is equivalent to

$$\nabla \cdot (\mathbf{b}_1 \cdot \nabla P\mathbf{w} + b_2 P\mathbf{w} + b_3 \nabla h - \mathbf{g}) = 0. \tag{3.6}$$

Re-arranging terms yields

$$\begin{aligned} \nabla \cdot (b_3 \nabla h) &= -\nabla \cdot (\mathbf{b}_1 \cdot \nabla P\mathbf{w}) - \nabla \cdot (b_2 P\mathbf{w}) + \nabla \cdot \mathbf{g} \\ &= -(\nabla \mathbf{b}_1)^T : \nabla(P\mathbf{w}) - \nabla b_2 \cdot (P\mathbf{w}) + \nabla \cdot \mathbf{g} \end{aligned} \tag{3.7}$$

Next, we construct the solution $\mathbf{w}$, h of the system of equations (3.4), (3.7) using the method of successive approximation as follows: Set the initial iterate $\mathbf{w}^0(\mathbf{x},t) = \mathbf{w}_0(\mathbf{x})$, the initial data, and set the initial iterate $h^0(\mathbf{x},t) = \ln(b(t))$. For $k = 0, 1, 2, \dots$ define $\mathbf{w}^{k+1}$, h^{k+1}, as the solution of the equations

$$\frac{\partial \mathbf{w}^{k+1}}{\partial t} + \mathbf{b}_1 \cdot \nabla \mathbf{w}^{k+1} + b_2 \mathbf{w}^{k+1} = J^k, \tag{3.8}$$

$$\nabla \cdot (b_3 \nabla h^{k+1}) = -(\nabla \mathbf{b}_1)^T : \nabla(P\mathbf{w}^k) - \nabla b_2 \cdot (P\mathbf{w}^k) + \nabla \cdot \mathbf{g}, \tag{3.9}$$

where $J^k = Q(\mathbf{b}_1 \cdot \nabla P\mathbf{w}^k) + Q(b_2 P\mathbf{w}^k) - P(b_3 \nabla h^{k+1}) + P\mathbf{g}$, and $\mathbf{w}^{k+1}(\mathbf{x},0) = \mathbf{w}_0(\mathbf{x})$.

The first step is to solve (3.9) for h^{k+1}. By the induction hypothesis, $(\nabla \mathbf{b}_1)^T : \nabla(P\mathbf{w}^k) \in C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^{s-1}(\Omega))$ and $\nabla b_2 \cdot (P\mathbf{w}^k)$ belongs to $C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^{s-1}(\Omega))$.

Also $\nabla \cdot \mathbf{g} \in C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^{s-1}(\Omega))$ and $b_3 \in C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^s(\Omega))$, and $b_3 \geq 1$. Therefore, by Lemma 4.7 there exists a unique zero-mean solution $\overline{h^{k+1}} \in C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^{s+1}(\Omega))$ to equation (3.9). It follows that $h^{k+1}(\mathbf{x},t) = \overline{h^{k+1}}(\mathbf{x},t) + \ln(b(t)) - \overline{h^{k+1}}(\mathbf{x}_0,t)$ is a unique solution to equation (3.9) which satisfies the condition $h^{k+1}(\mathbf{x}_0,t) = \ln(b(t))$.

The next step is to solve (3.8) for $\mathbf{w}^{k+1}$. By Lemma 4.2 and by the induction hypothesis, $Q(\mathbf{b}_1 \cdot \nabla P\mathbf{w}^k) \in C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^s(\Omega))$. And by the induction hypothesis, $Q(b_2 P\mathbf{w}^k) \in C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^s(\Omega))$. And from the previous step, $P(b_3 \nabla h^{k+1}) \in C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^s(\Omega))$. And $P\mathbf{g} \in C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^s(\Omega))$. Therefore, by Lemma 4.8 there exists a unique solution $\mathbf{w}^{k+1} \in C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^s(\Omega))$ to equation (3.8).

Next, we obtain estimates for $\|\nabla(h^{k+1} - h^k)\|_{s,T}^2$, $\|h^{k+1} - h^k\|_{s+1,T}^2$, and $\|\mathbf{w}^{k+1} - \mathbf{w}^k\|_{s,T}^2$. From equations (3.8), (3.9) we obtain the following system of equations for $h^{k+1} - h^k$ and $\mathbf{w}^{k+1} - \mathbf{w}^k$:

$$\frac{\partial(\mathbf{w}^{k+1} - \mathbf{w}^k)}{\partial t} + \mathbf{b}_1 \cdot \nabla(\mathbf{w}^{k+1} - \mathbf{w}^k) + b_2(\mathbf{w}^{k+1} - \mathbf{w}^k) = J^k - J^{k-1} \tag{3.10}$$

$$\nabla \cdot (b_3 \nabla(h^{k+1} - h^k)) = -(\nabla b_1)^T : \nabla(P(\mathbf{w}^k - \mathbf{w}^{k-1})) - \nabla b_2 \cdot P(\mathbf{w}^k - \mathbf{w}^{k-1}), \tag{3.11}$$

where $J^k - J^{k-1} = Q(\mathbf{b}_1 \cdot \nabla P(\mathbf{w}^k - \mathbf{w}^{k-1})) + Q(b_2 P(\mathbf{w}^k - \mathbf{w}^{k-1})) - P(b_3 \nabla(h^{k+1} - h^k))$. Initially we have $(\mathbf{w}^{k+1} - \mathbf{w}^k)(\mathbf{x},0) = 0$. And $h^{k+1}(\mathbf{x}_0,t) = h^k(\mathbf{x}_0,t) = \ln(b(t))$.

Next, applying Lemma 4.7 to equation (3.11), and using the fact that $b_3(\mathbf{x},t) \geq 1$, yields

$$\begin{aligned}\|\nabla(h^{k+1}-h^k)\|_s^2 &\leq C\sum_{j=0}^{s}\|Db_3\|_{s_1}^{2j}\Big(\|(\nabla \mathbf{b}_1)^T:\nabla(P(\mathbf{w}^k-\mathbf{w}^{k-1}))\|_{s-1}^2\\ &\quad+\|\nabla b_2\cdot P(\mathbf{w}^k-\mathbf{w}^{k-1})\|_{s-1}^2\Big)\\ &\leq C\sum_{j=0}^{s}\|Db_3\|_{s-1}^{2j}(\|D\mathbf{b}_1\|_{s-1}^2+\|Db_2\|_{s-1}^2)\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_s^2\\ &\leq C_1\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_s^2\end{aligned} \tag{3.12}$$

where $s > \frac{N}{2}+2$ and $s_1 = s-1$. And C_1 depends on $\|D\mathbf{b}_1\|_{s-1,T}$, $\|Db_2\|_{s-1,T}$, $\|Db_3\|_{s-1,T}$.

By Lemma 4.3, and by the fact that $h^{k+1}(\mathbf{x}_0,t) = h^k(\mathbf{x}_0,t)$, we have the inequality

$$\|h^{k+1}-h^k\|_{s+1}^2 \leq C\|\nabla(h^{k+1}-h^k)\|_s^2 \tag{3.13}$$

From (3.12), (3.13) , it follows that

$$\|h^{k+1}-h^k\|_{s+1}^2 \leq C\|\nabla(h^{k+1}-h^k)\|_s^2 \leq C_2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_s^2 \tag{3.14}$$

where C_2 depends on $\|D\mathbf{b}_1\|_{s-1,T}$, $\|Db_2\|_{s-1,T}$, $\|Db_3\|_{s-1,T}$.

Applying Lemma 4.8 to equation (3.10), and using estimate (3.14) for $\|\nabla(h^{k+1}-h^k)\|_s^2$, yields

$$\begin{aligned}\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_s^2 &\leq Ce^{\beta T}\int_0^t \|Q(\mathbf{b}_1\cdot\nabla P(\mathbf{w}^k-\mathbf{w}^{k-1}))\|_s^2\\ &\quad+\|Q(b_2P(\mathbf{w}^k-\mathbf{w}^{k-1}))\|_s^2+\|P(b_3\nabla(h^{k+1}-h^k))\|_s^2 d\tau\\ &\leq Ce^{\beta T}\int_0^t \|\mathbf{b}_1\|_s^2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_s^2+\|b_2\|_s^2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_s^2\\ &\quad+\|b_3\|_s^2\|\nabla(h^{k+1}-h^k)\|_s^2 d\tau\\ &\leq C_3\int_0^t \|\mathbf{w}^k-\mathbf{w}^{k-1}\|_s^2 d\tau\end{aligned} \tag{3.15}$$

where we used Lemma 4.2 to obtain the estimate $\|Q(\mathbf{b}_1\cdot\nabla P(\mathbf{w}^k-\mathbf{w}^{k-1}))\|_s^2 \leq C\|\mathbf{b}_1\|_s^2\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_s^2$. Here $\beta = C(1+\|\mathbf{b}_1\|_{s_1+1,T}+\|b_2\|_{s_1+1,T})$, and C_3 depends on $\|\mathbf{b}_1\|_{s,T}$, $\|b_2\|_{s,T}$, $\|b_3\|_{s,T}$.

Repeated application of (3.15) yields $\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_{s,T}^2 \leq \frac{(C_3T)^k}{k!}\|\mathbf{w}^1-\mathbf{w}^0\|_{s,T}^2$, from which it follows that $\sum_{k=0}^\infty \|\mathbf{w}^{k+1}-\mathbf{w}^k\|_{s,T}^2 < \infty$, and therefore $\sum_{k=0}^\infty \|h^{k+1}-h^k\|_{s+1,T}^2 < \infty$ by (3.14).

Therefore, there exists $\mathbf{w} \in C([0,T],H^0(\Omega))\cap L^\infty([0,T],H^s(\Omega))$ such that $\mathbf{w}^k \to \mathbf{w}$ as $k\to\infty$ strongly in $C([0,T],H^0(\Omega))\cap L^\infty([0,T],H^s(\Omega))$, and there exists $h \in C([0,T],H^0(\Omega))\cap L^\infty([0,T],H^{s+1}(\Omega))$ such that $h^k \to h$ as $k\to\infty$ strongly in $C([0,T],H^0(\Omega))\cap L^\infty([0,T],H^{s+1}(\Omega))$. And the fact that $\mathbf{w}$ and h is a solution to (3.1), (3.2), (3.3) follows by a standard argument (see, e.g., Majda [8], Embid [6]). □

4. Lemmas supporting the proof

Lemma 4.1 (Standard Calculus Inequalities). *(a) If $f \in H^{s_1}(\Omega)$, $g \in H^{s_2}(\Omega)$ and $s_3 = \min\{s_1, s_2, s_1 + s_2 - s_0\} \geq 0$, where $s_0 = [\frac{N}{2}] + 1$, then $fg \in H^{s_3}(\Omega)$, and $\|fg\|_{s_3} \leq C\|f\|_{s_1}\|g\|_{s_2}$. We note that $s_0 = 2$ for $N = 2$ or $N = 3$. Here, the constant C depends on s_1, s_2, Ω.*

(b) If $f \in H^{s_0}(\Omega) \cap L^\infty(\Omega)$, where $s_0 = [\frac{N}{2}] + 1$, then $|f|_{L^\infty} \leq C\|f\|_{s_0}$.

(c) If $f \in H^{r_2}(\Omega)$ and $r = \theta r_1 + (1-\theta) r_2$, with $0 \leq \theta \leq 1$ and $r_1 < r_2$, then $\|f\|_r \leq C\|f\|_{r_1}^\theta \|f\|_{r_2}^{1-\theta}$. Here C is a constant which depends on r_1, r_2, Ω.

The above inequalities are well known. They appear, for example, in Embid [6].

Lemma 4.2. *If $\mathbf{w}$, $\mathbf{v} \in H^r(\Omega)$, $r > \frac{N}{2} + 1$, $\Omega = \mathbb{T}^N$, then $Q(\mathbf{v}\cdot\nabla P\mathbf{w}) \in H^r(\Omega)$ and $\|Q(\mathbf{v}\cdot\nabla P\mathbf{w})\|_r \leq C\|\mathbf{v}\|_r\|\mathbf{w}\|_r$. Here P, Q are the projection operators from the Helmholtz Decomposition $\mathbf{u} = P\mathbf{u} + Q\mathbf{u}$, where $\nabla \cdot P\mathbf{u} = 0$ and $Q\mathbf{u}$ is a gradient vector field.*

A proof of the above lemma appears in Embid [7].

Lemma 4.3. *Let f, g be $H^r(\Omega)$ functions on a bounded domain Ω, where $r \geq 2$. And let $f(\mathbf{x}_0) = g(\mathbf{x}_0)$ at a point $\mathbf{x}_0 \in \Omega$. Then $f - g$ and f satisfy the estimates*

$$\|f - g\|_0^2 \leq C\|\nabla(f - g)\|_1^2, \tag{4.1}$$

$$\|f - g\|_r^2 \leq C\|\nabla(f - g)\|_{r-1}^2, \tag{4.2}$$

$$|f - g|_{L^\infty}^2 \leq C\|\nabla(f - g)\|_1^2 \tag{4.3}$$

$$\|f\|_r^2 \leq C\|g\|_r^2 + C\|\nabla g\|_{r-1}^2 + C\|\nabla f\|_{r-1}^2 \tag{4.4}$$

where C is a constant which depends on Ω and on r.

Proof. A proof of inequality (4.1) appears in Denny [4]. From (4.1) we obtain the inequality

$$\begin{aligned} \|f - g\|_r^2 &\leq \|f - g\|_0^2 + C\|\nabla(f - g)\|_{r-1}^2 \\ &\leq C\|\nabla(f - g)\|_1^2 + C\|\nabla(f - g)\|_{r-1}^2 \leq C\|\nabla(f - g)\|_{r-1}^2 \end{aligned}$$

for $r \geq 2$. This completes the proof of (4.2). From (4.2) with $r = 2$, and from Lemma 4.1, we obtain

$$|f - g|_{L^\infty}^2 \leq C\|f - g\|_2^2 \leq C\|\nabla(f - g)\|_1^2$$

This completes the proof of (4.3). From (4.2) we obtain the inequality

$$\begin{aligned} \|f\|_r^2 &= \|f - g + g\|_r^2 \leq C\|f - g\|_r^2 + C\|g\|_r^2 \leq C\|\nabla(f - g)\|_{r-1}^2 + C\|g\|_r^2 \\ &\leq C\|\nabla f\|_{r-1}^2 + C\|\nabla g\|_{r-1}^2 + C\|g\|_r^2 \end{aligned}$$

for $r \geq 2$. This completes the proof of the lemma. □

Lemma 4.4. *Let f be a smooth function on an interval $G \subset \mathbb{R}$, and let u be a continuous function such that $u(\mathbf{x}) \in \overline{G}_1$ for $\mathbf{x} \in \Omega$, where $\overline{G}_1 \subset G$ and $u \in H^{r+1}(\Omega)$, where $r \geq 2$. Then*

$$\|D(f(u))\|_0^2 \leq \Big|\frac{df}{du}\Big|_{0,\overline{G}_1}^2 \|Du\|_0^2, \tag{4.5}$$

$$\|D(f(u))\|_1^2 \leq C\Big|\frac{df}{du}\Big|_{1,\overline{G}_1}^2 (\|Du\|_1^2 + \|Du\|_2^2\|Du\|_0^2), \tag{4.6}$$

$$\|D(f(u))\|_r^2 \le C\big|\frac{df}{du}\big|^2_{r,\overline{G}_1} \sum_{1\le j\le r+1} \|Du\|_r^{2j}, \tag{4.7}$$

where $r \ge 2$ and C depends on r, Ω. And we define $|g|_{r,\overline{G}_1} = \max\{\big|\frac{d^j g}{du^j}(u_*)\big| : u_* \in \overline{G}_1, 0 \le j \le r\}$.

Proof. We immediately obtain the estimate

$$\|D(f(u))\|_0^2 = \|\frac{df}{du}Du\|_0^2 \le \big|\frac{df}{du}\big|^2_{0,\overline{G}_1}\|Du\|_0^2$$

Next, we obtain the inequality

$$\begin{aligned}
&\|D(f(u))\|_1^2 = \|\frac{df}{du}Du\|_1^2 \\
&= \|\frac{df}{du}Du\|_0^2 + \|D\big(\frac{df}{du}Du\big)\|_0^2 \\
&\le \|\frac{df}{du}Du\|_0^2 + C|D\big(\frac{df}{du}\big)|^2_{L^\infty}\|Du\|_0^2 + C\big|\frac{df}{du}\big|^2_{0,\overline{G}_1}\|D^2u\|_0^2 \\
&= \|\frac{df}{du}Du\|_0^2 + C|\frac{d^2f}{du^2}Du|^2_{L^\infty}\|Du\|_0^2 + C\big|\frac{df}{du}\big|^2_{0,\overline{G}_1}\|D^2u\|_0^2 \\
&\le \big|\frac{df}{du}\big|^2_{0,\overline{G}_1}\|Du\|_0^2 + C\big|\frac{d^2f}{du^2}\big|^2_{0,\overline{G}_1}|Du|^2_{L^\infty}\|Du\|_0^2 + C\big|\frac{df}{du}\big|^2_{0,\overline{G}_1}\|D^2u\|_0^2 \\
&\le C\big|\frac{df}{du}\big|^2_{1,\overline{G}_1}(\|Du\|_1^2 + |Du|^2_{L^\infty}\|Du\|_0^2) \\
&\le C\big|\frac{df}{du}\big|^2_{1,\overline{G}_1}(\|Du\|_1^2 + \|Du\|_2^2\|Du\|_0^2)
\end{aligned} \tag{4.8}$$

where we used Lemma 4.1 to obtain the estimate $|Du|^2_{L^\infty} \le C\|Du\|_2^2$.

If $r \ge 2$, then we obtain the inequality

$$\begin{aligned}
\|D(f(u))\|_r^2 = \|\frac{df}{du}Du\|_r^2 &\le C\|\frac{df}{du}\|_r^2\|Du\|_r^2 \\
&\le C(\|\frac{df}{du}\|_0^2 + \|D\big(\frac{df}{du}\big)\|_{r-1}^2)\|Du\|_r^2
\end{aligned} \tag{4.9}$$

By (4.9) and by repeating the above argument applied to the terms $\|D\big(\frac{d^jf}{du^j}\big)\|^2_{r-j}$, for $j = 1, 2, \dots, r-2$, that appear on the right-hand side of the inequality, we obtain

$$\begin{aligned}
\|D(f(u))\|_r^2 &\le C(\|\frac{df}{du}\|_0^2 + \|D\big(\frac{df}{du}\big)\|_{r-1}^2)\|Du\|_r^2 \\
&= C(\|\frac{df}{du}\|_0^2 + \|\frac{d^2f}{du^2}Du\|_{r-1}^2)\|Du\|_r^2 \\
&\le C(\|\frac{df}{du}\|_0^2 + \|\frac{d^2f}{du^2}\|_{r-1}^2\|Du\|_{r-1}^2)\|Du\|_r^2 \\
&\le C(\|\frac{df}{du}\|_0^2 + \|\frac{d^2f}{du^2}\|_{r-1}^2\|Du\|_r^2)\|Du\|_r^2 \\
&\le C(\|\frac{df}{du}\|_0^2 + (\|\frac{d^2f}{du^2}\|_0^2 + \|D\big(\frac{d^2f}{du^2}\big)\|_{r-2}^2)\|Du\|_r^2)\|Du\|_r^2 \\
&\le C\sum_{1\le j\le r-1}\|\frac{d^jf}{du^j}\|_0^2\|Du\|_r^{2j} + C\|D\big(\frac{d^{r-1}f}{du^{r-1}}\big)\|_1^2\|Du\|_r^{2(r-1)}
\end{aligned} \tag{4.10}$$

where C is a generic constant which changes from one instance to the next.

Substituting estimate (4.8) for the term $\|D\big(\frac{d^{r-1}f}{du^{r-1}}\big)\|_1^2$ into (4.10) yields

$$\begin{aligned}
&\|D(f(u))\|_r^2\\
&\le C\sum_{1\le j\le r-1}\|\frac{d^jf}{du^j}\|_0^2\|Du\|_r^{2j}+C\|D\big(\frac{d^{r-1}f}{du^{r-1}}\big)\|_1^2\|Du\|_r^{2(r-1)}\\
&\le C\sum_{1\le j\le r-1}\|\frac{d^jf}{du^j}\|_0^2\|Du\|_r^{2j}+C\big|\frac{d^rf}{du^r}\big|_{1,\overline{G}_1}^2(\|Du\|_1^2+\|Du\|_2^2\|Du\|_0^2)\|Du\|_r^{2(r-1)}\\
&\le C\big|\frac{df}{du}\big|_{r-2,\overline{G}_1}^2\sum_{1\le j\le r-1}\|Du\|_r^{2j}+C\big|\frac{df}{du}\big|_{r,\overline{G}_1}^2(\|Du\|_r^{2r}+\|Du\|_r^{2(r+1)})\\
&\le C\big|\frac{df}{du}\big|_{r,\overline{G}_1}^2\sum_{1\le j\le r+1}\|Du\|_r^{2j}
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 4.5. *Let f be a smooth function on an interval $G\subset\mathbb{R}$, and let u_1, u_2 be continuous functions such that $u_1(\mathbf{x})\in\overline{G}_1$, $u_2(\mathbf{x})\in\overline{G}_1$ for $\mathbf{x}\in\Omega$, where $\overline{G}_1\subset G$, and $u_1\in H^r(\Omega)$, $u_2\in H^r(\Omega)$, where $r\ge 2$. Then*

$$\|f(u_1)-f(u_2)\|_0^2\le\big|\frac{df}{du}\big|_{0,\overline{G}_1}^2\|u_1-u_2\|_0^2, \tag{4.11}$$

$$\|f(u_1)-f(u_2)\|_1^2\le C\big|\frac{df}{du}\big|_{1,\overline{G}_1}^2(1+\|Du_1\|_0^2+\|Du_2\|_0^2)\|u_1-u_2\|_2^2, \tag{4.12}$$

where C depends on Ω. Here we define $\big|g\big|_{r,\overline{G}_1}=\max\{\big|\frac{d^jg}{du^j}(u_*)\big|:u_*\in\overline{G}_1,0\le j\le r\}$.

Proof. We begin by writing the calculus identity

$$f(u_1)-f(u_2)=I_f(u_1,u_2)(u_1-u_2) \tag{4.13}$$

where $I_f(u_1,u_2)$ is defined as follows:

$$I_f(u_1,u_2)=\int_0^1\frac{df}{du}(\tau u_1+(1-\tau)u_2)d\tau \tag{4.14}$$

and where the following estimates hold for $I_f(u_1,u_2)$:

$$|I_f(u_1,u_2)|_{L^\infty}^2\le\big|\frac{df}{du}\big|_{0,\overline{G}_1}^2 \tag{4.15}$$

$$\|D(I_f(u_1,u_2))\|_0^2\le C\big|\frac{df}{du}\big|_{1,\overline{G}_1}^2(\|Du_1\|_0^2+\|Du_2\|_0^2) \tag{4.16}$$

A proof of inequalities (4.15)-(4.16) appears in Embid [7]. From (4.13), (4.15), it follows that

$$\begin{aligned}
\|f(u_1)-f(u_2)\|_0^2&=\|I_f(u_1,u_2)\,(u_1-u_2)\,\|_0^2\\
&\le|I_f(u_1,u_2)|_{L^\infty}^2\|u_1-u_2\|_0^2\le\big|\frac{df}{du}\big|_{0,\overline{G}_1}^2\|u_1-u_2\|_0^2
\end{aligned}$$

From (4.13), (4.15), (4.16) and the estimate for $\|f(u_1)-f(u_2)\|_0^2$, it follows that

$$\begin{aligned}
&\|f(u_1)-f(u_2)\|_1^2\\
&=\|f(u_1)-f(u_2)\|_0^2+\|D(f(u_1)-f(u_2))\|_0^2\\
&\le\|f(u_1)-f(u_2)\|_0^2+C\|D(I_f(u_1,u_2))\,(u_1-u_2)\,\|_0^2+C\|I_f(u_1,u_2)D(u_1-u_2)\|_0^2
\end{aligned}$$

$$
\begin{aligned}
&\le \|f(u_1)-f(u_2)\|_0^2 + C\|D(I_f(u_1,u_2))\|_0^2|u_1-u_2|_{L^\infty}^2\\
&\quad + C|I_f(u_1,u_2)|_{L^\infty}^2\|D(u_1-u_2)\|_0^2\\
&\le \Big|\frac{df}{du}\Big|_{0,\overline{G}_1}^2\|u_1-u_2\|_0^2 + C\Big|\frac{df}{du}\Big|_{1,\overline{G}_1}^2(\|Du_1\|_0^2+\|Du_2\|_0^2)\|u_1-u_2\|_2^2\\
&\quad + C\Big|\frac{df}{du}\Big|_{0,\overline{G}_1}^2\|u_1-u_2\|_1^2\\
&\le C\Big|\frac{df}{du}\Big|_{1,\overline{G}_1}^2(1+\|Du_1\|_0^2+\|Du_2\|_0^2)\|u_1-u_2\|_2^2
\end{aligned}
$$

where $|u_1-u_2|_{L^\infty}^2 \le C\|u_1-u_2\|_2^2$ by Lemma 4.1. This completes the proof of the lemma. □

Lemma 4.6. *Let $Z^k(t)$ be a positive, integrable function for $0 \le t \le T$ and for $k = 1,2,3,\dots$, and let $Z^{k+1}(t) \le L\big(\epsilon Z^k(t) + \int_0^t Z^k(\tau_1)d\tau_1\big)$, where ϵ and L are positive constants which do not depend on k. Then*

$$
Z^{k+1}(t) \le (2\epsilon L)^k e^{t/\epsilon}\sup_{0\le t\le T} Z^1(t)
$$

Proof. We repeatedly apply the inequality given for Z^{k+1} as follows:

$$
\begin{aligned}
&Z^{k+1}(t)\\
&\le \epsilon L Z^k(t) + \int_0^t LZ^k(\tau_1)d\tau_1\\
&\le \epsilon L\Big(\epsilon L Z^{k-1}(t) + \int_0^t LZ^{k-1}(\tau_1)d\tau_1\Big) + \int_0^t L\Big(\epsilon L Z^{k-1}(\tau_1)\\
&\quad + \int_0^{\tau_1} LZ^{k-1}(\tau_2)d\tau_2\Big)d\tau_1\\
&= \epsilon^2L^2Z^{k-1}(t) + 2\epsilon\int_0^t L^2Z^{k-1}(\tau_1)d\tau_1 + \int_0^t\int_0^{\tau_1} L^2Z^{k-1}(\tau_2)d\tau_2d\tau_1\\
&\le \epsilon^2L^2\Big(\epsilon LZ^{k-2}(t) + \int_0^t LZ^{k-2}(\tau_1)d\tau_1\Big)\\
&\quad + 2\epsilon\int_0^t L^2\Big(\epsilon LZ^{k-2}(\tau_1) + \int_0^{\tau_1} LZ^{k-2}(\tau_2)d\tau_2\Big)d\tau_1\\
&\quad + \int_0^t\int_0^{\tau_1} L^2\Big(\epsilon LZ^{k-2}(\tau_2) + \int_0^{\tau_2} LZ^{k-2}(\tau_3)d\tau_3\Big)d\tau_2d\tau_1\\
&= \epsilon^3L^3Z^{k-2}(t) + 3\epsilon^2\int_0^t L^3Z^{k-2}(\tau_1)d\tau_1\\
&\quad + 3\epsilon\int_0^t\int_0^{\tau_1} L^3Z^{k-2}(\tau_2)d\tau_2d\tau_1 + \int_0^t\int_0^{\tau_1}\int_0^{\tau_2} L^3Z^{k-2}(\tau_3)d\tau_3d\tau_2d\tau_1\\
&\le L^k\epsilon^kZ^1(t) + L^k\sum_{j=1}^k\binom{k}{j}\epsilon^{k-j}\int_0^t\int_0^{\tau_1}\int_0^{\tau_2}\dots\int_0^{\tau_{j-1}} Z^1(\tau_j)d\tau_j\dots d\tau_3d\tau_2d\tau_1
\end{aligned}
\tag{4.17}
$$

Then we have the inequality

$$\begin{aligned}
&\int_0^t \int_0^{\tau_1} \int_0^{\tau_2} \dots \int_0^{\tau_{j-1}} Z^1(\tau_j) d\tau_j \dots d\tau_3\, d\tau_2\, d\tau_1 \\
&\leq \sup_{0\leq t\leq T} Z^1(t) \int_0^t \int_0^{\tau_1} \int_0^{\tau_2} \dots \int_0^{\tau_{j-1}} d\tau_j \dots d\tau_3 d\tau_2 d\tau_1 \\
&= \sup_{0\leq t\leq T} Z^1(t) \big(\frac{t^j}{j!}\big)
\end{aligned} \tag{4.18}$$

Substituting estimate (4.18) into inequality (4.17), and using the inequality $\binom{k}{j} \leq \sum_{j=0}^{k} \binom{k}{j} = 2^k$ for each $0 \leq j \leq k$ (see Abramowitz and Stegun [1]), yields

$$\begin{aligned}
Z^{k+1} &\leq L^k \sum_{j=0}^{k} \binom{k}{j} \epsilon^{k-j} \sup_{0\leq t\leq T} Z^1(t) \big(\frac{t^j}{j!}\big) \\
&= (\epsilon L)^k \sup_{0\leq t\leq T} Z^1(t) \sum_{j=0}^{k} \binom{k}{j} \Big(\frac{(\frac{t}{\epsilon})^j}{j!}\Big) \\
&\leq (\epsilon L)^k \sup_{0\leq t\leq T} Z^1(t) \sum_{j=0}^{k} 2^k \Big(\frac{(\frac{t}{\epsilon})^j}{j!}\Big) \\
&\leq (2\epsilon L)^k \sup_{0\leq t\leq T} Z^1(t) \sum_{j=0}^{\infty} \Big(\frac{(\frac{t}{\epsilon})^j}{j!}\Big) \\
&= (2\epsilon L)^k e^{t/\epsilon} \sup_{0\leq t\leq T} Z^1(t)
\end{aligned}$$

This completes the proof. □

The next lemma on the existence of a solution to an elliptic equation is standard.

Lemma 4.7. *Let $a \in C^2(\Omega) \cap H^s(\Omega)$, $f \in C^1(\Omega) \cap H^{s-1}(\Omega)$, and $\mathbf{g} \in C^2(\Omega) \cap H^s(\Omega)$, where $s > \frac{N}{2} + 2$, $\Omega = \mathbb{T}^N$, $N = 2, 3$. Let $a(\mathbf{x}) \geq 1$ for $\mathbf{x} \in \Omega$. And let $\int_\Omega (f + \nabla \cdot \mathbf{g}) d\mathbf{x} = 0$. Then there exists a solution $h \in C^3(\Omega) \cap H^{s+1}(\Omega)$ of*

$$\nabla \cdot (a \nabla h) = f + \nabla \cdot \mathbf{g} \tag{4.19}$$

which is unique up to a constant. And the zero-mean function $\overline{h} = h - \frac{1}{|\Omega|} \int_\Omega h d\mathbf{x}$ is also a solution. And the following inequality holds for $r \geq 1$

$$\|\nabla h\|_r^2 \leq C \sum_{j=0}^{r} \|Da\|_{r_1}^{2j} (\|f\|_{r-1}^2 + \|\mathbf{g}\|_r^2) \tag{4.20}$$

where $r_1 = \max\{r - 1, s_0\}$ and $s_0 = \left[\frac{N}{2}\right] + 1 = 2$, for $N = 2, 3$, and where C depends on r.

A proof of the above lemma appears in Denny [5]. The next lemma on the existence of a solution to a linear system of equations is standard.

Lemma 4.8. *Given $\mathbf{w}_0 \in H^s(\Omega)$, $\mathbf{b}_1 \in C([0, T], H^0(\Omega)) \cap L^\infty([0, T], H^s(\Omega))$, $b_2 \in C([0, T], H^0(\Omega)) \cap L^\infty([0, T], H^s(\Omega))$, $\mathbf{g} \in C([0, T], H^0(\Omega)) \cap L^\infty([0, T], H^s(\Omega))$,* *where* $s > \frac{N}{2} + 2$, $\Omega = \mathbb{T}^N$, $N = 2, 3$, $0 \le t \le T$, *there exists a unique classical solution* $\mathbf{w} \in C([0,T], C^2(\Omega)) \cap L^\infty([0,T], H^s(\Omega))$ *of*

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{b}_1 \cdot \nabla \mathbf{w} + b_2 \mathbf{w} = \mathbf{g}$$
$$\mathbf{w}(\mathbf{x}, 0) = \mathbf{w}_0(\mathbf{x})$$

And for $r \ge 1$, $\mathbf{w}$ *satisfies*

$$\|\mathbf{w}\|_r^2 \le e^{\beta t}(\|\mathbf{w}_0\|_r^2 + C \int_0^t \|\mathbf{g}\|_r^2 d\tau) \tag{4.21}$$

where $\beta = C(1 + \|\mathbf{b}_1\|_{r_1+1,T} + \|b_2\|_{r_1+1,T})$ *and* C *depends on* r. *Here,* $r_1 = \max\{r - 1, s_0\}$, *where* $s_0 = [\frac{N}{2}] + 1 = 2$ *for* $N = 2, 3$.

The proof of the above lemma is standard; see, for example, Embid [6].

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