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RELATIONSHIP BETWEEN SOLUTIONS TO A QUASILINEAR ELLIPTIC EQUATION IN ORLICZ SPACES

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Abstract. In this article, we consider three types of solutions in Orlicz spaces for the quasilinear elliptic problem

$$-\operatorname{div}(a(|\nabla u|)\nabla u) = 0.$$

By applying a comparison principle, we establish the relationships between viscosity supersolutions, weak supersolutions, and superharmonic functions.

1. Introduction

In this article, we study three types of solutions, in Orlicz-Sobolev spaces, to the quasilinear elliptic problem

$$-\operatorname{div}(a(|\nabla u|)\nabla u) = 0\,. \tag{1.1}$$

The operator $-\operatorname{div}(a|\nabla u|\nabla u)$ will be denoted by $-\Delta_P$. As special case, when $a(t) = |t|^{p-2}$, the operator $-\Delta_P$ is the usual p-Laplacian.

In the past decades, there have been many publications about p-Laplacian equations; see [2, 3, 7, 8, 9, 15, 19]. With the background of Orlicz-Sobolev spaces, the operator $-\Delta_P$ has been studied widely; see for example [5, 12, 13, 14, 24]. The operator $-\Delta_P$ plays an important role in geometry and physics. The nonlinear Hodge Theorem is generated by the operator $-\Delta_P$, see [16] (each cohomology class of a Manifold has a unique P-harmonic representative).

Also the operator $-\Delta_P$ has important physical background, for example:

(1) nonlinear elasticity: $P(t) = (1 + t^2)^\gamma - 1$, $\gamma > \frac{1}{2}$;
(2) plasticity: $P(t) = t^\alpha(\log(1 + t))^\beta$, $\alpha \geq 1$, $\beta > 0$;
(3) generalized Newtonian fluids: $P(t) = \int_0^t s^{1-\alpha}(\sinh^{-1} s)^\beta ds$, $0 \leq \alpha \leq 1$, $\beta > 0$.

Obviously, the operator $-\operatorname{div}(a|\nabla u|\nabla u)$ is nonhomogeneous. To deal with this situation, as in [4, 5, 12, 13, 14], we introduce an Orlicz-Sobolev space setting for the operator. Set

$$p(t) := \begin{cases} a(|t|)t, & t \neq 0, \\ 0, & t = 0, \end{cases} \tag{1.2}$$

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where $p(t):\mathbb{R}\to\mathbb{R}$ is an increasing homeomorphism from $\mathbb{R}$ onto itself (such functions are called Young or N-functions). If we set

$$P(t) := \int_0^t p(s)ds, \quad \tilde{P}(t) := \int_0^t p^{-1}(s)ds,$$

then P and $\tilde{P}$ are complementary N-functions (see [1, 22, 23]).

To construct an Orlicz-Sobolev space setting for operator $-\Delta_P$, we Assume the following conditions on $p(t)$:

(P0) $a(t)\in C^1(0,+\infty)$, $a(t)>0$,

(P1) $1<p^- := \inf_{t>0}\frac{tp(t)}{P(t)} \le p^+ := \sup_{t>0}\frac{tp(t)}{P(t)} < +\infty$,

(P2) $0<a^- := \inf_{t>0}\frac{tp'(t)}{p(t)} \le a^+ := \sup_{t>0}\frac{tp'(t)}{p(t)} < +\infty$.

Under condition (P1), the function $P(t)$ satisfies Δ_2-condition; i.e.,

$$P(2t) \le kP(t), \quad t>0,$$

for some constant $k>0$. Under the conditions (P0) and (P1), the Orlicz space L^P coincides with the set (equivalence classes) of measurable functions $u:\Omega\to R$ such that

$$\int_\Omega P(|u|)\,dx < +\infty. \tag{1.3}$$

The space $L^P(\Omega)$ is a Banach space endowed with the Luxemburg norm

$$|u|_P := \inf\big\{k>0, \int_\Omega P\left(\frac{|u|}{k}\right)dx < 1\big\}.$$

We shall denote by $W^{1,P}(\Omega)$ the corresponding Orlicz-Sobolev space with the norm

$$\|u\|_{W^{1,P}(\Omega)} := |u|_P + ||\nabla u||_P\,.$$

We denote by $W_0^{1,P}(\Omega)$ the closure of C_0^∞ in $W^{1,P}(\Omega)$. The Orlicz-Sobolev conjugate function P_* of P is given by

$$P_*^{-1}(t) := \int_0^t \frac{P^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}d\tau. \tag{1.4}$$

Define a new norm

$$\|u\|_{1,P} = \|u\| := \inf\big\{k>0 : \int_\Omega P(\frac{|\nabla u|}{k})dx < 1\big\}.$$

One can prove the above two norms are equivalent. The reader is referred to [1, 22, 23] for more information on Orlicz-Sobolev spaces. In the proofs of our results we shall use the following results.

Lemma 1.1 ([1, 22, 23])**.** *Under assumptions* (P0) *and* (P1)*, the spaces* $L^P(\Omega)$*,* $W_0^{1,P}(\Omega)$ *and* $W^{1,P}(\Omega)$ *are separable and reflexive Banach spaces.*

Lemma 1.2 ([1, 22, 23])**.** *Let P and Q be N-functions.*

(1) If $\int_1^{+\infty}\frac{P^{-1}(t)}{t^{\frac{N+1}{N}}} = \infty$ *and Q grow essentially more slowly than P_*, then the embedding* $W^{1,P}(\Omega)\hookrightarrow L^Q(\Omega)$ *is compact and the embedding* $W^{1,P}(\Omega)\hookrightarrow L^{P_*}(\Omega)$ *is continuous.*

(2) If $\int_1^{+\infty}\frac{P^{-1}(t)}{t^{\frac{N+1}{N}}} < \infty$*, then the embedding* $W^{1,P}(\Omega)\hookrightarrow L^Q(\Omega)$ *is compact and the embedding* $W^{1,P}(\Omega)\hookrightarrow L^\infty(\Omega)$ *is continuous.*

Lemma 1.3. *Under conditions* (P0)–(P2)*, we have*

(1) *if* $0 < t < 1$*, then* $P(1)t^{p^+} \le P(t) \le P(1)t^{p^-}$*;*
(2) *if* $t > 1$*, then* $P(1)t^{p^-} \le P(t) \le P(1)t^{p^+}$*.*

Lemma 1.4 ([12])**.** *Let* $\rho(u) = \int_\Omega P(u)dx$*, we have*

(1) *if* $|u|_P < 1$*, then* $|u|_P^{p^+} \le \rho(u) \le |u|_P^{p^-}$*;*
(2) *if* $|u|_P > 1$*, then* $|u|_P^{p^-} \le \rho(u) \le |u|_P^{p^+}$*;*
(3) *if* $0 < t < 1$*, then* $t^{p^+}P(u) \le P(tu) \le t^{p^-}P(u)$*;*
(4) *if* $t > 1$*, then* $t^{p^-}P(u) \le P(tu) \le t^{p^+}P(u)$*.*

Lemma 1.5 ([12])**.** *Let* $\tilde\rho(u) = \int_\Omega \tilde P(u)dx$*, we have*

(1) *if* $|u|_{\tilde P} < 1$*, then* $|u|_{\tilde P}^{p^-/(p^--1)} \le \tilde\rho(u) \le |u|_{\tilde P}^{p^+/(p^+-1)}$*;*
(2) *if* $|u|_{\tilde P} > 1$*, then* $|u|_{\tilde P}^{p^+/(p^+-1)} \le \tilde\rho(u) \le |u|_{\tilde P}^{p^-/(p^--1)}$*;*
(3) *if* $0 < t < 1$*, then* $t^{p^-/(p^--1)}\tilde P(u) \le \tilde P(tu) \le t^{p^+/(p^+-1)}\tilde P(u)$*;*
(4) *if* $t > 1$*, then* $t^{p^+/(p^+-1)}\tilde P(u) \le \tilde P(tu) \le t^{p^-/(p^--1)}\tilde P(u)$*.*

Lemma 1.6 ([14, 22, 23])**.** *Assuming that* $A(t)$ *and* $\tilde A(t)$ *are complementary* N*-functions, we have:*

(1) *Young inequalities:* $uv \le A(u) + \tilde A(v)$*;*
(2) *Hölder inequalities:* $|\int_\Omega u(x)v(x)dx| \le 2|u|_A|v|_{\tilde A}$*;*
(3) $\tilde A(\frac{A(u)}{u}) \le A(u)$*;*
(4) $\tilde A_*(\frac{A_*(u)}{u}) \le A_*(u)$*.*

2. Main results and their proofs

As in [21], we define define: weak solutions, viscous solutions, and P-harmonic functions, for

$$-\Delta_P u := -\operatorname{div}(a(|\nabla u|)\nabla u) = 0. \tag{2.1}$$

Definition 2.1. A function $u \in W^{1,P}_{\rm loc}(\Omega)$ is a weak supersolution (subsolution) of (2.1) if

$$\int_\Omega a(|\nabla u|)\nabla u \nabla\phi dx \ge (\le) 0, \tag{2.2}$$

for any $\phi \in C_0^\infty(\Omega)$.

Definition 2.2. A function $u : \Omega \to (-\infty, \infty]$ is called a P-superharmonic function, if u satisfies

(1) u is lower semicontinuous,
(2) u is bounded almost everywhere,
(3) the comparison principle holds: if v is a weak solution of (2.1) in $D \subset \Omega$, u is continuous on $\bar D$, and $u \ge v$ on ∂D, then

$$u \ge v \quad \text{on } D.$$

A function $u : \Omega \to [-\infty, \infty)$ is P-subharmonic, if $-u$ is P-superharmonic.

Definition 2.3. A function $u : \Omega \to (-\infty, \infty]$ is a viscous supersolution of (2.1), if u satisfies

(1) u is lower semicontinuous,

(2) u is bounded almost everywhere,
(3) If there exist $\phi \in C^2(\Omega)$, such that $u(x_0) = \phi(x_0)$, $u(x) > \phi(x)$ and $D\phi(x_0) \neq 0$, when $x \neq x_0$, then
$$-\Delta_P \phi(x_0) \geq 0.$$

A function $u : \Omega \to [-\infty, +\infty)$ is a viscosity subsolution to (2.1) if it is upper semicontinuous, finite a.e., and (3) holds with the inequalities reversed. Hence, a function u is a viscosity solution if it is both a viscosity supersolution and subsolution. The viscosity solution concept was introduced in the early 1980s by Lions and Crandall as a generalization of the classical concept of what is meant by a "solution" to a PDE. It has been found that the viscosity solution is the natural solution concept to use in many applications of PDE's, including first-order equations arising in optimal control (the Hamilton-Jacobi equation), differential games (the Isaacs equation) or front evolution problems [10], as well as second-order equations such as the ones arising in stochastic optimal control or stochastic differential games.

Now, we give the following results on the relationships among the viscous solutions, weak solutions of (2.1), and P-harmonic functions.

Theorem 2.4. *If u is a local bounded P-superharmonic function, then u is a supersolution of* (2.1)*.*

Theorem 2.5. *If u is a supersolution of* (2.1)*, and u satisfies*
$$u(x) = \operatorname{ess\,lim\,inf}_{y\to x} u(y) \tag{2.3}$$
for all $x \in \Omega$, then u is a P-superharmonic function.

Theorem 2.6. *u is a viscous supersolution of* (2.1) *if and only if u is a P-superharmonic function.*

Proof of Theorem 2.4. By a similar discussion as in [17], we know that the following two facts hold.

Fact 1: Let u be a P-superharmonic function in Ω and $D \subset \Omega$ an open set. Then there exists an increasing sequence of continuous supersolutions $\{u_i\}$ in D such that $u = \lim_{i\to\infty} u_i$ everywhere in D.

Fact 2: Assume $\{u_i\}$ is an increasing sequence of supersolutions in and that $u = \lim_{i\to\infty} u_i$ is locally bounded. Then u is a supersolution in Ω.

So, by Fact 1, for a P-superharmonic function, we can find an an increasing sequence $\{u_i\}$ of continuous supersolutions such that $u = \lim_{i\to\infty} u_i$. Thus u is a supersolution by Fact 2. □

Next we prove Theorems 2.5 and 2.6. First have a lemma whose proof can be founded in [18, pp. 61-62].

Lemma 2.7. *Assume that u, v are supersolution and subsolution of* (2.1) *respectively, and $u \geq v$ on $\partial\Omega$ in Sobolev sense. Then $u \geq v$ a.e. in Ω.*

Proof of Theorem 2.5. Using a similar method as in [17, Theorem 6.1], we know that u, as a supersolution, is locally bounded below. By condition (2.3), u is lower semicontinuous. Since $u \in W_0^{1,P}(\Omega)$, u is bounded almost everywhere. Now we will prove (3), i.e., the comparison principle holds. Let $D \subset$ be an open subset of Ω. Assume that h is a solution of (2.1) in D, which is upper-continuous in $\bar{D}$, and $u \geq h$ on ∂D. For any open set $G \subset D$, taking $\varepsilon > 0$, we let $u + \varepsilon \geq h$ on $D \setminus G$. By lower semicontinuity of u, the set $\{u-h\le -\varepsilon\}$ is closed. Hence, $\min\{u+\varepsilon-h,0\}$ has compact support in G. From the Lemma 2.7, we know that $u+\varepsilon\ge h$ hold a.e. in G, then $u+\varepsilon\ge h$ in D. By condition (2.3), we get that $u+\varepsilon\ge h$ in D. Let $\varepsilon\to 0$, the conclusion follows. □

To prove Theorem 2.6, we give some useful lemmas.

Lemma 2.8 ([12])**.** *There exists $k_0>0$, such that*

$$(a(|\xi|)\xi-a(|\eta|)\eta)\cdot(\xi-\eta)\ge k_0\frac{P(|\xi-\eta|)^{\frac{p^-+1}{p^-}}}{(P(|\xi|)+P(|\eta|))^{\frac{1}{p^-}}}\ge 0. \tag{2.4}$$

for any $\xi,\eta\in\mathbb{R}^N,\xi\ne 0$

Lemma 2.9. *Let $u,v\in W^{1,P}(\Omega)$ and $(u-v)_+\in W_0^{1,P}(\Omega)$. If*

$$\int_\Omega a(|\nabla u|)\nabla u\cdot\nabla\phi dx\le\int_\Omega a(|\nabla v|)\nabla v\cdot\nabla\phi dx, \tag{2.5}$$

for any positive function $\phi\in W_0^{1,P}(\Omega)$, then $u\le v$ a.e. in Ω.

Proof. By the assumption and inequality (2.4), we have

$$0\le\int_\Omega(a(|\nabla u|)\nabla u-a(|\nabla u|)\nabla v)\cdot\nabla(u-v)_+dx\le 0. \tag{2.6}$$

Since $\nabla(u-v)_+$ has zero boundary value, we have $\nabla(u-v)_+=0$. □

Lemma 2.10. *Let $u\in W^{1,P}(\Omega),u_\varepsilon\in W_0^{1,P}(\Omega)$ are solutions of*

$$-\operatorname{div}(a(|\nabla u|)\nabla u)=0, \tag{2.7}$$

$$-\operatorname{div}(a(|\nabla u_\varepsilon|)\nabla u_\varepsilon)=\varepsilon,\varepsilon>0 \tag{2.8}$$

respectively, and $u-u_\varepsilon\in W_0^{1,P}(\Omega)$, then u_ε converges to u, locally uniformly in Ω as $\varepsilon\to 0$.

Proof. Similar to (2.6), we obtain

$$\int_\Omega(a(|\nabla u_\varepsilon|)\nabla u_\varepsilon-a(|\nabla u|)\nabla u)\cdot\nabla(u_\varepsilon-u)dx=\varepsilon\int_\Omega(u_\varepsilon-u)dx. \tag{2.9}$$

For the right-hand term in (2.9), by Lemma 1.2, we deduce

$$\varepsilon\int_\Omega(u_\varepsilon-u)dx\le C\varepsilon|\nabla u_\varepsilon-\nabla u|_P. \tag{2.10}$$

To estimate the left-hand term in (2.9), by inequalities (2.4) and (2.10), Lemma 1.4, and Hölder inequality, we have

$$\begin{aligned}
&\int_\Omega P(|\nabla u_\varepsilon-\nabla u|)dx\\
&\le\Big\{\int_\Omega\frac{P(|\nabla u_\varepsilon-\nabla u|)^{\frac{p^-+1}{p^-}}}{(P(|\nabla u_\varepsilon|)+P(|\nabla u|))^{\frac{1}{p^-}}}dx\Big\}^{\frac{p^-}{p^-+1}}\Big\{\int_\Omega(P(|\nabla u_\varepsilon|)+P(|\nabla u|))dx\Big\}^{\frac{1}{p^-+1}}\\
&\le M\Big\{\frac{1}{k_0}\int_\Omega(a(|\nabla u_\varepsilon|)\nabla u_\varepsilon-a(|\nabla u|)\nabla u)\cdot\nabla(u_\varepsilon-u)dx\Big\}^{\frac{p^-}{p^-+1}}\\
&\le CM\frac{1}{k_0}\varepsilon\{|\nabla u_\varepsilon-\nabla u|_P\}^{\frac{p^-}{p^-+1}}
\end{aligned}$$

$$\le \begin{cases} CM\frac{1}{k_0}\varepsilon\big\{\int_\Omega P(|\nabla u_\varepsilon - \nabla u|)dx\big\}^{\frac{1}{p^-+1}}, & \text{if } |\nabla u_\varepsilon - \nabla u|_P > 1,\\ CM\frac{1}{k_0}\varepsilon\big\{\int_\Omega P(|\nabla u_\varepsilon - \nabla u|)dx\big\}^{\frac{p^-}{p^+(p^-+1)}}, & \text{if } |\nabla u_\varepsilon - \nabla u|_P < 1. \end{cases}$$

for a constant $C > 0$.

Then, we obtain that u_ε converges to u in $W^{1,P}(\Omega)$ as $\varepsilon \to 0$. So $u_\varepsilon \to u$ a.e. in Ω. By the regularity argument [24], we easily know that u_ε converges to u, uniformly. □

Lemma 2.11. *If $v_\varepsilon \in W^{1,P}(\Omega)$ is a weak solution of*

$$-\Delta_P v = \varepsilon, \tag{2.11}$$

and $\phi \in C^2(\Omega)$ satisfies $v_\varepsilon(x_0) = \phi(x_0), v_\varepsilon > \phi(x), x \neq x_0$, where x_0 is isolated critical point of ϕ, or $\nabla\phi(x_0) \neq 0$, then

$$\limsup_{x\to x_0, x\neq x_0} (-\Delta_P\phi(x)) \ge \varepsilon.$$

Proof. Without loss of generality, assuming $x_0 = 0$. If the conclusion does not hold, then there exists $r > 0$ such that

$$\nabla\phi(x) \neq 0 \quad \text{and} \quad -\Delta_P\phi(x) < 0,$$

for any $0 < |x| < r$.

Next, we prove that ϕ is a weak subsolution of (2.11) in $B_r = B(0, r)$. Let $0 < \rho < r$, for any positive $\eta \in C_0^\infty(B_r)$, integrating over $B_r \setminus B_\rho$, we obtain

$$-\int_{|x|=\rho} \eta a(|\nabla\phi|)\nabla\phi\cdot\frac{x}{\rho}dS = \int_{\rho<|x|<r} a(|\nabla\phi|)\nabla\phi\cdot\nabla\eta dx + \int_{\rho<|x|<r}(\Delta_P\phi)\eta dx$$

It is easy to show that the left-hand term converges to 0 as $\rho \to 0$ by noticing that

$$\Big| -\int_{|x|=\rho} \eta a(|\nabla\phi|)\nabla\phi\cdot\frac{x}{\rho}dS\Big| \le \|\eta\|_\infty \max\{a(|\nabla\phi|), |\nabla\phi|\}\rho^{n-1}. \tag{2.12}$$

By the assumptions, we have

$$\int_{\rho<|x|<r}(\Delta_P\phi)\eta dx \ge -\varepsilon\int_{\rho<|x|<r}\eta dx \ge -\varepsilon\int_{B_r}\eta dx. \tag{2.13}$$

Let $\rho \to 0$, we obtain

$$\int_{B_r} a(|\nabla\phi|)\nabla\phi\cdot\nabla\eta dx \le \varepsilon\int_{B_r}\eta dx.$$

This means that ϕ is a weak subsolution.

Let $m = \inf_{\partial B_r}(v_\varepsilon - \phi) > 0$, then $\tilde\phi := \phi + m$ is a weak solution of (2.11), Moreover, $\tilde\phi \le v_\varepsilon$ in ∂B_r. Moreover, Lemma 2.9 implies that $\tilde\phi \le v_\varepsilon$ in B_r, which contradicts with $\tilde\phi(0) > v_\varepsilon(0)$. The lemma holds. □

To prove Lemma 2.12, we decompose the operator Δ_P into two terms, namely,

$$\begin{aligned} -\Delta_P u &= -a(|\nabla u|)\Delta u - a'(|\nabla u|)|\nabla u|\frac{\nabla^2 u\nabla u\cdot\nabla u}{|\nabla u|^2}\\ &= -a(|\nabla u|)\Delta u - a'(|\nabla u|)|\nabla u|\Delta_\infty u, \end{aligned} \tag{2.14}$$

where $\Delta_\infty u := \frac{\nabla^2 u\nabla u\cdot\nabla u}{|\nabla u|^2}$.

Let X is a n order symmetric matrix, and

$$A(\xi) := a(|\xi|)I + a'(|\xi|)|\xi|\frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|},$$
$$F(\xi, X) := \operatorname{trace}(A(\xi)X).$$

Then

$$\Delta_P \phi = F(\nabla\phi, D^2\phi) = \operatorname{trace}(A(\nabla\phi)D^2\phi), \tag{2.15}$$

where $\nabla\phi(x) \neq 0$, $D^2\phi = (\frac{\partial^2\phi}{\partial x_i \partial x_j})_{n\times n}$ is Hessian matrix for ϕ.

Lemma 2.12. *Assume that u is a viscous subsolution of* (2.1)*, v is a weak solution of $-\Delta_P v = \varepsilon$, and $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in Ω.*

Proof. Without loss of generality, we assume $\varepsilon = 1$. To prove this conclusion, we argue by contradiction and assume that $u - v$ has a inner maximum, i.e.,

$$\sup_{\Omega}(u - v) > \sup_{\partial\Omega}(u - v). \tag{2.16}$$

Consider

$$w_j(x, y) = u(x) - v(y) - \Psi_j(x, y), \quad j = 1, 2, \dots, \tag{2.17}$$

where

$$\Psi_j(x, y) = \frac{j}{q}|x - y|^q, \quad q > \max\{\frac{p^-}{p^- - 1}, 2\}.$$

If $(x_j, y_j) \in \bar{\Omega} \times \bar{\Omega}$ is a maximum point of w_j, then by (2.16) and [6, Proposition 3.7], we have (x_j, y_j) is a inner point for j large enough. Since

$$u(x) - v(y) - \Psi_j(x, y) \leq u(x_j) - v(y_j) - \Psi_j(x_j, y_j), x, y \in \Omega,$$

and let $x = x_j$, we have

$$v(y) \geq -\Psi_j(x_j, y) + v(y_j) + \Psi_j(x_j, y_j), y \in \Omega.$$

Set

$$\phi_j(y) = -\Psi_j(x_j, y) + v(y_j) + \Psi(x_j, y_j) - \frac{1}{q+1}|y - y_j|^{q+1},$$

Obviously, $v - \phi_j$ has a strict local minimum at y_j. By Lemma 2.11, we obtain

$$\limsup_{y\to y_j, y\neq y_j} (-\Delta_P\phi_j(y)) \geq 1,$$

which means $x_j \neq y_j$. In fact, if $x_j = y_j$, by simple calculation, we can get $-\Delta_P\phi_j(y) \to 0$ as $y \to y_j$, which is a contradiction.

Next we use a method similar to the proof [21, Proposition 3.3] to complete the rest of proof. Since (x_j, y_j) is a local maximum of $w_j(x, y)$, then there exist n order symmetric matrixes of X_j, Y_j such that

$$(D_x\Psi_j(x_j, y_j), X_j) \in \bar{J}^{2,+}u(x_j),$$
$$-(D_y\Psi_j(x_j, y_j), Y_j) \in \bar{J}^{2,-}u(y_j),$$

and

$$\begin{bmatrix} X_j & 0 \\ 0 & -Y_j \end{bmatrix} \leq D^2\Psi_j(x_j, y_j) + \frac{1}{j}[D^2\Psi_j(x_j, y_j)]^2, \tag{2.18}$$

where $\bar{J}^{2,+}u(x_j), \bar{J}^{2,-}u(y_j)$ are the closure of the second order superjet of u at x_j and the second order subjet of v at y_j, respectively. One can refer to [6] for the definition and properties of jet. By (2.18), one has

$$X_j \leq Y_j,$$

in matrix sense. i.e., $\langle (Y_j - X_j\xi, \xi)\rangle \geq 0$ for all $\xi \in R^N$. According to [6], viscosity solutions can be defined using jets instead of test-functions as in Definition 2.3. Since $x_j \neq y_j$, we obtain

$$\eta_j \equiv D_x\Psi_j(x_j, y_j) = -D_y\Psi_j(x_j, y_j) \neq 0.$$

Therefore, $(\eta, X) \to F(\eta, X)$ is continuous in the neighbors of (η_j, X_j) and (η_j, Y_j). Since u is a subsolution of (2.1), we infer

$$-a(|\eta_j|)\big[\operatorname{trace}(X_j) + \frac{a'(|\eta_j|)\eta_j}{a(|\eta_j|)}\langle X_j\frac{\eta_j}{|\eta_j|}, \frac{\eta_j}{|\eta_j|}\rangle\big] \leq 0.$$

On the other hand, since $\eta_j \neq 0$, by definition of $\bar{J}^{2,-}$ and Lemma 2.11, we obtain

$$-a(|\eta_j|)\big[\operatorname{trace}(Y_j) + \frac{a'(|\eta_j|)\eta_j}{a(|\eta_j|)}\langle Y_j\frac{\eta_j}{|\eta_j|}, \frac{\eta_j}{|\eta_j|}\rangle\big] \geq 1.$$

So,

$$\begin{aligned}
0 < 1 &\leq -a(|\eta_j|)\big[\operatorname{trace}(Y_j) + \frac{a'(|\eta_j|)\eta_j}{a(|\eta_j|)}\langle Y_j\frac{\eta_j}{|\eta_j|}, \frac{\eta_j}{|\eta_j|}\rangle\big] \\
&\quad + a(|\eta_j|)\big[\operatorname{trace}(X_j) + \frac{a'(|\eta_j|)\eta_j}{a(|\eta_j|)}\langle X_j\frac{\eta_j}{|\eta_j|}, \frac{\eta_j}{|\eta_j|}\rangle\big] \\
&\leq 0,
\end{aligned}$$

where the last inequality follows from the fact $X_j \leq Y_j$. It means that our initial assumption is false, so

$$\sup_{\Omega}(u - v) = \sup_{\partial\Omega}(u - v) \leq 0.$$

□

Proof of Theorem 2.6. Firstly, we prove that the P-superharmonic function is the viscous supersolution of (2.1). Assuming v is superharmonic, and assuming by contradiction that v is a not viscous supersolution of (2.1), then there is $\phi \in C^2(\Omega)$ such that $v(x_0) = \phi(x_0), v(x) > \phi(x)$ and

$$-\Delta_P\phi(x_0) < 0$$

for all $x \neq x_0, \nabla\phi(x_0) \neq 0$. By continuity, there is $r > 0$ $\nabla\phi(x) \neq 0$ and

$$-\Delta_P\phi(x) < 0$$

for all $x \in B(x_0, r)$. Let

$$m = \inf_{|x-x_0|=r}(v(x) - \phi(x)) > 0,$$

$$\tilde{\phi} = \phi + m,$$

then $\tilde{\phi}$ is a weak subsolution of (2.1) in $B(x_0, r)$, and $\tilde{\phi} \leq v$ on $\partial B(x_0, r)$. By Lemma 2.9, $\tilde{\phi} \leq v$ in $B(x_0, r)$, thus

$$\tilde{\phi}(x_0) = \phi(x_0) + m > v(x_0),$$

which is a contradiction.

On the other hand, we assume that v is a viscous supersolution of (2.1), and we will show that v is also a P-superharmonic function. Let $D \subset \Omega$ and let $h \in C(\bar{D})$ is a weak solution of (2.1) such that $v \geq h$ on ∂D. By the lower semicontinuity of v, for each $\delta > 0$, there exists a smooth domain $D' \subset D$ such that $h \leq v + \delta$ in $D \setminus D'$. Here the reason for taking D is that h can be considered as boundary value. Hence, h belongs to some Orlicz-Sobolev space instead of $W^{1,P}_{\rm loc}(D)$.

Given $\varepsilon > 0$, let h_ε be the unique weak solution of the equation

$$-\Delta_P h_\varepsilon = -\varepsilon, \varepsilon > 0,$$

such that $h_\varepsilon - h \in W_0^{1,P}(D')$. Then h_ε is local Lipschitz continuous in D' (see[24]). Owing to the smoothness of D', we have $v + \delta \geq h_\varepsilon$ on $\partial D'$. From Lemma 2.10, we easily know that h_ε converges uniformly to h locally in D' as $\varepsilon \to 0$. Finally, Lemma 2.12 implies that $v + \delta \geq h_\varepsilon$ in D', and so $v \geq h$ in D. This completes our proof. □

Remark 2.13. The equivalence of weak and viscosity solutions was firstly obtained by Juutinen, Lindqvist and Manfredi [21] for the p-Laplace equation. In [20], Julin and Juutinen gave a new proof for this result.

Remark 2.14. Obviously, our results are extension of [20, 21]. Moreover, in [20], Julin and Juutinen suggest to consider the more generalized equation

$$-\operatorname{div} A(x, u) = 0$$

and hope to obtain the similar results. Here we believe that if the operator $-\operatorname{div} A(x, \cdot)$ is equipped a Musielak-Sobolev space, then the similar results can be obtained. The reader is referred to [11] for more details on Musielak-Sobolev space theory.

Remark 2.15. Following the method in [20] or [21], we can also obtain the similar results for the following parabolic equation in Orlicz-Sobolev space

$$u_t - \operatorname{div}(a(|\nabla u|)\nabla u) = 0,$$

and we omit it.

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