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NAVIER-STOKES PROBLEM IN VELOCITY-PRESSURE FORMULATION: NEWTON LINEARIZATION AND CONVERGENCE

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Abstract. In this article we study the nonlinear Navier-Stokes problem in velocity-pressure formulation. We construct a sequence of a Newton-linearized problems and we show that the sequence of weak solutions converges towards the solution of the nonlinear one in a quadratic way.

1. Introduction

The stationary Navier-Stokes problem may be written in the form

$$\begin{gathered}
-\nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad \text{in } \Omega\\
\operatorname{div} u = 0 \quad \text{in } \Omega\\
u = 0 \quad \text{on } \Gamma = \partial\Omega
\end{gathered} \tag{1.1}$$

This equation describes the motion of an incompressible fluid contained in Ω and subjected to an outside forces f, u is the velocity of fluid flow, p is the pressure and ν its viscosity.

The variational formulation of the Navier Stokes equations in the classic form is well studied in [8, 9, 16]. In most publications they uses a trilinear form in the variational formulation for studying the nonlinear term presented in the equation of momentum.

This paper is devoted to give another idea: we construct a sequence of a Newton-linearized problems and we show, using Lax-Milgramm theorem, that the variational formulation of each one has an unique solution. We show then that the sequence of weak solutions converges towards the solution of the nonlinear one in a quadratic way.

The outline of the paper is as follows: In Section 2 we start by a Newton-linearisation of the Navier Stokes equations. We obtain a sequence of linear problems and we show the existence of a weak solution. In Section 3 we show the quadratic convergence of the sequence of the solutions in Theorem 3.3. In section 4 the nonhomogeneous problem is treated.

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2. Linearized problems

Linearization. Let Ω a bounded domain of $\mathbb{R}^2$ with Lipschitz-continuous boundary Γ, and let

$$V = \{v \in (H_0^1(\Omega))^2, \operatorname{div} v = 0\}$$

with norm $\|u\|_V = \max\{\|u_1\|_{H_0^1}, \|u_2\|_{H_0^1}\}$. We set $L_0^2 = (L_0^2(\Omega))^2$, and $H_0^1(\Omega) = (H_0^1(\Omega))^2$ with norm $\|u\|_{H_0^1} = \max\{\|u_1\|_{H_0^1}, \|u_2\|_{H_0^1}\}$ and $W = H_0^1(\Omega) \times L_0^2$.

The nonlinear term

$$(u \cdot \nabla)u = \begin{pmatrix} u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} \\ u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} \end{pmatrix}$$

can be written as

$$(u \cdot \nabla)u = \frac{1}{2}\nabla|u|^2 + \operatorname{rot} u \wedge u.$$

To solve (1.1) we construct a sequence of Newton-linearized problems. Starting from an arbitrary $u_0 \in H_0^1(\Omega)$ and $p_0 \in L_0^2$ we consider the iterative scheme:

$$\begin{gathered} -\nu\Delta u_{n+1} + (u_{n+1} \cdot \nabla)u_n + (u_n \cdot \nabla)u_{n+1} + \nabla p_{n+1} = f_n \quad \text{in } \Omega \\ \operatorname{div} u_{n+1} = 0 \quad \text{in } \Omega \\ u_{n+1} = 0 \quad \text{on } \Gamma = \partial\Omega \end{gathered} \tag{2.1}$$

where $f_n = f + (u_n \cdot \nabla)u_n$. Problem (2.1) is linear.

Variational formulation. The variational formulation of (2.1) is

$$\begin{gathered} \text{Find } (u_{n+1}, p_{n+1}) \in W \text{ such that} \\ a_0(u_{n+1}, v) + a_n(u_{n+1}, v) + a^n(u_{n+1}, v) + b(p_{n+1}, v) = L_n(v) \quad \forall v \in H_0^1(\Omega) \\ b(q, u_{n+1}) = 0 \quad \forall q \in L_0^2 \end{gathered} \tag{2.2}$$

where the bilinear forms a_0, a_n , a^n are given for $v, u \in H_0^1(\Omega)$ and $p \in L_0^2$ by

$$a_0(u,v) = \nu \int_\Omega \nabla u \nabla v \, dx, \quad a^n(u,v) = \int_\Omega (u \cdot \nabla u_n) v \, dx,$$

$$a_n(u,v) = \int_\Omega (u_n \cdot \nabla u) \, v \, dx, \quad b(p,v) = \int_\Omega \nabla p v \, dx = -\int_\Omega p \operatorname{div} v \, dx$$

and $L_n(v) = \langle f_n, v\rangle$ Using Green formula and $\operatorname{div} v = 0$ we have $b(p,v) = 0$.

Then we associate to (2.2) the problem

$$\begin{gathered} \text{Find } u_{n+1} \in V \text{ such that} \\ a_0(u_{n+1}, v) + a_n(u_{n+1}, v) + a^n(u_{n+1}, v) = L_n(v) \quad \forall v \in V \,. \end{gathered} \tag{2.3}$$

Lemma 2.1. *Problem* (2.2) *is equivalent to problem* (2.3).

Proof. Indeed, if (u_{n+1}, p_{n+1}) is a solution of problem (2.2) then u_{n+1} is a solution of (2.3). Reciprocally, if u_{n+1} is a solution of the problem (2.3) then we apply de Rham's theorem: Let Ω a bounded regular domain of $\mathbb{R}^2$ and $\mathcal{K}$ a continuous linear form on $(H_0^1(\Omega))^2$. Then the linear form $\mathcal{K}$ vanishes on V if and only if there exists a unique function $p_{n+1} \in L^2(\Omega)/\mathbb{R}$ such that for all $v \in H_0^1(\Omega)$,

$$\mathcal{K}(v) = \int_\Omega p_{n+1} \operatorname{div} v \, dx \,.$$

Let the linear form satisfies

$$\mathcal{K}(v) = a_0(u_{n+1}, v) + a_n(u_{n+1}, v) + a^n(u_{n+1}, v) - L_n(v).$$

Therefore we have $\mathcal{K}(v) = 0$ for all $v \in V$, then de Rham's theorem implies that there exists a unique function $p_{n+1} \in \mathrm{L}^2(\Omega)/\mathbb{R}$ such that

$$a_0(u_{n+1}, v) + a_n(u_{n+1}, v) + a^n(u_{n+1}, v) - L_n(v) = \int_\Omega p_{n+1} \operatorname{div} v \, dx \quad \forall v \in H_0^1(\Omega);$$

therefore,

$$a_0(u_{n+1}, v) + a_n(u_{n+1}, v) + a^n(u_{n+1}, v) - \int_\Omega p_{n+1} div v dx = L_n(v) \quad \forall v \in H_0^1(\Omega)\,.$$

Which gives the desired result. □

Let us now show that problem (2.3) has an unique solution for each n. For this, we need the following lemma.

Lemma 2.2. *For fixed $u_n \in V$ the form $(u, v) \to a_n(u, v)$ and $(u, v) \to a^n(u, v)$ are continuous on $H_0^1(\Omega)$.*

Proof. We have

$$a_n(u, v) = \sum_{i,j=1}^{2} \int_\Omega u_{n,j} \frac{\partial u_i}{\partial x_j} v_i \, dx,$$

$$a^n(u, v) = \sum_{i,j=1}^{2} \int_\Omega u_j \frac{\partial u_{n,i}}{\partial x_j} v_i \, dx$$

by Holder's inequality we have

$$\Big| \int_\Omega u_{n,j} \frac{\partial u_i}{\partial x_j} v_i \, dx \Big| \le \|u_{n,j}\|_{L^4} \|v_i\|_{L^4} \Big\|\frac{\partial u_i}{\partial x_j}\Big\|_{L^2} \tag{2.4}$$

According to the Sobolev Imbedding Theorem, the space $H^1(\Omega)$ is continuously embedded in $L^4(\Omega)$. Then there exists $C_1 > 0$ such that

$$|a_n(u, v)| \le C_1 \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \|u_n\|_{H_0^1(\Omega)}. \tag{2.5}$$

The same result holds with the term a^n,

$$|a^n(u, v)| \le C_2 \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \|u_n\|_{H_0^1(\Omega)}. \tag{2.6}$$

□

To show the coercivity of the form $a = a_0 + a_n + a^n$ we have the following lemma.

Lemma 2.3. *We have $a_n(u, u) = 0$ for all $u \in V$.*

Proof. Note that

$$a_n(u, u) = \int_\Omega (u_n.\nabla) u \; u \, dx = \frac{1}{2} \int_\Omega u_n \nabla(|u|^2) \, dx \tag{2.7}$$

where

$$\nabla(|u|^2) = \begin{pmatrix} \frac{\partial(u_1^2 + u_2^2)}{\partial x} \\ \frac{\partial(u_1^2 + u_2^2)}{\partial y} \end{pmatrix}$$

Using Green's formula and $\operatorname{div} u_n = 0$ and boundary conditions we have

$$2a_n(u, u) = \int_\Omega \nabla|u|^2 u_n \, dx = -\int_\Omega \operatorname{div} u_n |u|^2 \, dx = 0. \tag{2.8}$$

□

For $\alpha > 0$, let $B_\alpha = \{v \in V : \|v\|_{H_0^1(\Omega)} \le \alpha\}$.

Lemma 2.4. *We have*

$$a(u,u) \ge (\nu C_3 - \alpha C_2)\|u\|^2_{H_0^1(\Omega)} \quad \forall u \in V,\ \forall u_n \in B_\alpha \tag{2.9}$$

with $C_3 = \min[\frac{1}{2(C_p(\Omega))^2}, \frac{1}{2}]$.

Proof. Using (2.8) we obtain $a(u,u) = a_0(u,u) + a^n(u,u)$. By the Poincare inequality,

$$\|u\|_{L^2(\Omega)} \le C_p(\Omega)\|\nabla u\|_{L^2(\Omega)}, \tag{2.10}$$

we obtain

$$a_0(u,u) = \nu\|\nabla u\|^2_{L^2(\Omega)} \ge \nu \min[\frac{1}{2(C_p(\Omega))^2}, \frac{1}{2}]\|u\|^2_{H_0^1(\Omega)}\,.$$

Then

$$a_0(u,u) \ge \nu C_3\|u\|^2_{H_0^1(\Omega)}\,. \tag{2.11}$$

Using (2.6) we have

$$a^n(u,u) \le C_2\alpha\|u\|^2_{H_0^1(\Omega)} \quad \forall u \in V,\ u_n \in B_\alpha \tag{2.12}$$

which gives

$$a^n(u,u) \ge -C_2\alpha\|u\|^2_{H_0^1(\Omega)} \quad \forall u \in V,\ u_n \in B_\alpha \tag{2.13}$$

with (2.11) we have the result. □

Lemma 2.5. *For* $\|f\|_{(L^2(\Omega))^2}$ *small enough or* ν *large enough there is* $\alpha^* > 0$ *independent of* n *such that* $\|u_n\|_{H_0^1(\Omega)} \le \alpha^*$ *for all* $n \in \mathbb{N}$ *where* u_n *is solution of* (2.3) *with* n *instead of* $n+1$.

Proof. We must have $(\nu C_3 - C_2\alpha^*) > 0$. So we choose $\alpha^* < \frac{\nu C_3}{C_2}$.

Remains to show by induction that if u_n is solution of (2.3) with n instead of $n+1$, then $\|u_n\|_{H_0^1(\Omega)} \le \alpha^*$ for all $n \in \mathbf{N}$. Let $u_0 \in B_{\alpha^*}$ and assume that $u_n \in B_{\alpha^*}$. We note $u = u_{n+1}$ is a solution of (2.3) and $\|f\|_2 = \|f\|_{(L^2(\Omega))^2}$. We have

$$a(u,u) = L_n(u) = \int_\Omega (f + (u_n\nabla)u_n)u\,dx\,. \tag{2.14}$$

Then $a(u,u) \le (\|f\|_2 + C{\alpha^*}^2)\|u\|_{H_0^1(\Omega)}$.

From (2.9) we obtain $(\nu C_3 - C_2\alpha^*)\|u\|_{H_0^1}(\Omega) \le (\|f\|_2 + C{\alpha^*}^2)$ which gives

$$\|u\|_{H_0^1(\Omega)} \le \frac{\|f\|_2 + C{\alpha^*}^2}{(\nu C_3 - C_2\alpha^*)}\,.$$

So to deduce the result we must have

$$\frac{\|f\|_2 + C{\alpha^*}^2}{(\nu C_3 - C_2\alpha^*)} \le \alpha^*\,.$$

We put

$$P(\alpha^*) = (C + C_2){\alpha^*}^2 - \nu C_3\alpha^* + \|f\|_2 \le 0, \quad \alpha^* < \frac{\nu C_3}{C_2}$$

Therefore, the discriminant of the polynomial $P(\alpha^*)$ must verify

$$\Delta = \nu^2{C_3}^2 - 4(C + C_2)\|f\|_2 > 0\,. \tag{2.15}$$

Then

$$\|f\|_2 < \frac{\nu^2 {C_3}^2}{4(C + C_2)} \tag{2.16}$$

and hence $P(\alpha^*)$ has two roots

$$\alpha_1 = \frac{\nu C_3 - \sqrt{\Delta}}{2(C + C_2)}, \quad \alpha_2 = \frac{\nu C_3 + \sqrt{\Delta}}{2(C + C_2)}$$

Since $\alpha_2 > 0$ we can choose $0 < \alpha^* < \min(\frac{\nu C_3}{C_2}, \alpha_2)$. □

Theorem 2.6. *(1) For $f \in (L^2(\Omega))^2$ satisfying (2.14), problem* (2.3) *has a unique solution $u_{n+1} \in V \cap B_{\alpha^*}$.*

(2) If $u_0 \in B_{\alpha^*} \cap H^2(\Omega)$, then $u_{n+1} \in H^2(\Omega)$.

Proof. (1) Since $u_n \in B_{\alpha^*}$, we have

$$|L_n(v)| \le (\|f\|_2 + C{\alpha^*}^2)\|v\|_{H_0^1(\Omega)}$$

which gives the continuity of L_n and using Lemma 2.1, Lemma 2.3 and Lemma 2.4 with Lax-Milgram Theorem we obtain the result.

(2) We assume that $u_n \in H^2(\Omega)$ then $(u_n \nabla)u_n \in (L^2(\Omega)^2$, which implies that $f_n = f + (u_n \nabla)u_n \in (L^2(\Omega)^2$ for $f \in (L^2(\Omega)^2$, and by classical regularity Theorem we have $u_{n+1} \in H^2(\Omega)$. □

3. Convergence

The sequence $(u_n)_{n\in\mathbb{N}}$, solutions of (2.3) with n instead of $n + 1$, satisfy

$$\|u_n\|_{H_0^1(\Omega)} \le \alpha^* \quad \forall n \ge 0, \tag{3.1}$$

which implies that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Then there exist a subsequence that converges weakly to ϕ in $H_0^1(\Omega)$. Since the injection of $H_0^1(\Omega)$ in $(L^2(\Omega))^2$ is compact, there exists a subsequence still noted u_n which converges strongly to ϕ in $(L^2(\Omega))^2$.

We need the following result.

Lemma 3.1. *For $v \in V$, we have:*

1. $\lim_{n\to\infty} a_0(u_{n+1}, v) = a_0(\phi, v)$;
2. $\lim_{n\to\infty} a_n(u_{n+1}, v) = a_\infty(\phi, v) = \int_\Omega (\phi\dot{\nabla})\phi v dx$;
3. $\lim_{n\to\infty} a^n(u_{n+1}, v) = a^\infty(\phi, v) = \int_\Omega (\phi\dot{\nabla})\phi v dx$;
4. *We have* $\lim_{n\to\infty} L_n(v) = L_\infty(v) = \int_\Omega [f + (\phi\dot{\nabla})\phi] v dx$.

Proof. (1) Since $u_n \rightharpoonup \phi$, and by linearity of $u \to a_0(u, v)$ we have $a_0(u_{n+1}, v) \to a_0(\phi, v)$ for all $v \in V$.

(2) Let

$$E = |a^n(u_{n+1}, v) - a^\infty(\phi, v)| = \Big| \int_\Omega \{(u_{n+1}\dot{\nabla})u_n - (\phi\dot{\nabla})\phi\} v \, dx \Big| \tag{3.2}$$

We can write

$$(u_{n+1} \cdot \nabla)u_n - (\phi.\nabla)\phi = ((u_{n+1} - \phi) \cdot \nabla)u_n + (\phi \cdot \nabla)(u_n - \phi) \tag{3.3}$$

which gives with $u_n \in H^2(\Omega)$ and using Green's theorem,

$$E \le C[\|u_{n+1} - \phi\|_2 \|u_n\|_{H^1} \|v\|_{H^1} + \|u_n - \phi\|_2 (\|\nabla v\|_{H^1} \|\phi\|_{H^1} + \|\nabla\phi\|_{H^1} \|v\|_{H^1})]. \tag{3.4}$$

Since u_n converges strongly to ϕ in $(L^2(\Omega))^2$, it follows that $E \to 0$.

(3) Let

$$F = |a_n(u_{n+1}, v) - a_\infty(\phi, v)| = \int_\Omega \{(u_n \cdot \nabla)u_{n+1} - (\phi \cdot \nabla)\phi\} v \, dx \,.$$

Then

$$F \le C[\|u_n - \phi\|_2 \|u_{n+1}\|_{H^1} \|v\|_{H^1} + \|u_{n+1} - \phi\|_2 (\|v\|_{H^1} \|\nabla\phi\|_{H^1} + \|\phi\|_{H^1} \|\nabla v\|_{H^1})]; \tag{3.5}$$

thus $F \to 0$.

(4) Let

$$G = |L_n(v) - L_\infty(v)| \le \int_\Omega |(u_n \nabla u_n) - (\phi \nabla \phi)| |\nabla v| \, dx.$$

Then

$$G \le C[\|u_n - \phi\|_2 (\|u_n\|_{H^1} \|v\|_{H^1} + \|\phi\|_{H^1} \|\nabla v\|_{H^1} + \|\nabla\phi\|_{H^1} \|v\|_{H^1})]. \tag{3.6}$$

Then Lemma 3.1 gives the desired result. □

For using de Rham's Theorem, let $\mathcal{L}$ a continuous linear form on $(H_0^1(\Omega))^2$ which vanishes on V if and only if there exists a unique function $\varphi \in L^2(\Omega)/\mathbb{R}$ such that for all $v \in H_0^1(\Omega)$,

$$\mathcal{L}(v) = \int_\Omega \varphi \operatorname{div} v \, dx.$$

Theorem 3.2. *We have* $\lim_{n\to\infty} u_n = \phi$ *in* V *then* ϕ *is a solution of* (1.1).

Proof. It follows from Lemma 3.1 that

$$\lim_{n\to\infty} a_0(u_{n+1}, v) + a_n(u_{n+1}, v) + a^n(u_{n+1}, v) = a_0(\phi, v) + 2a_\infty(\phi, v) = L_\infty(v) \,.$$

Let the linear form $\mathcal{L}(v) = a_0(\phi, v) + a_\infty(\phi, v) + a^\infty(\phi, v) - L_\infty(v)$. Therefore $\mathcal{L}(v) = 0$ for all $v \in V$, then de Rham's theorem implies that there exists a unique function $p \in \mathrm{L}^2(\Omega)/\mathbb{R}$ such that

$$a_0(\phi, v) + 2a_\infty(\phi, v) - L_\infty(v) = \int_\Omega p \operatorname{div} v \, dx \quad \forall v \in H_0^1(\Omega) \tag{3.7}$$

which gives

$$\nu \int_\Omega \nabla\phi \ \nabla v \, dx + \int_\Omega (\phi \dot{\nabla})\phi v \, dx - \int_\Omega p \operatorname{div} v \, dx = \int_\Omega f v \, dx \quad \forall v \in H_0^1(\Omega), \tag{3.8}$$

$$\int_\Omega (-\nu\Delta\phi + (\phi\dot{\nabla})\phi + \nabla p - f) v \, dx = 0 \quad \forall v \in H_0^1(\Omega) \,. \tag{3.9}$$

Then in $\mathcal{D}'(\Omega)$,

$$-\nu\Delta\phi + (\phi\dot{\nabla})\phi + \nabla p - f = 0 \,. \tag{3.10}$$

Since $\phi \in V$ we conclude that ϕ is the solution of (1.1). □

Theorem 3.3. *Let* u_{n+1} *be the solution of* (2.2), *and* ϕ *be the solution of* (1.1). *Then convergence of the sequence* $(u_{n+1})_{n\in\mathbb{N}}$ *towards* ϕ *is quadratic; i.e.,*

$$\|u_{n+1} - \phi\|_{H_0^1(\Omega)} \le C_2 \|u_n - \phi\|^2_{H_0^1(\Omega)} \tag{3.11}$$

Proof. Let $\omega_n = u_n - \phi$ and $\chi_n = p_n - p$. Subtracting problem (2.1) from (1.1) we obtain

$$
\begin{gathered}
-\nu\Delta\omega_{n+1} + (\omega_{n+1}\nabla)u_n + (u_n\nabla)\omega_{n+1} + \nabla\chi_{n+1} = (\omega_n\nabla)\omega_n \quad \text{in } \Omega \\
\operatorname{div}\omega_{n+1} = 0 \quad \text{in } \Omega \\
\omega_{n+1} = 0 \quad \text{on } \Gamma
\end{gathered}
\tag{3.12}
$$

The variational formulation of (3.12) is

$$
\begin{gathered}
\text{Find } (\omega_{n+1}, \chi_{n+1}) \in W \text{ such that} \\
a(\omega_{n+1}, v) + b(\chi_{n+1}, v) = F_n(v) \quad \forall v \in H_0^1(\Omega) \\
b(q, \omega_{n+1}) = 0 \quad \forall q \in L_0^2,
\end{gathered}
\tag{3.13}
$$

where $a = a_0 + a^n + a_n$ and

$$
b(q, \omega_{n+1}) = -\int_\Omega q \operatorname{div}\omega_{n+1}\,dx, \quad F_n(v) = \int_\Omega (\omega_n\nabla)\omega_n v\,dx.
$$

Since $\operatorname{div}\omega_{n+1} = 0$, using Lemma2.1 and Lemma 2.5, for $u_n \in B_{\alpha^*}$ and $v = \omega_{n+1}$, we obtain

$$
\begin{aligned}
(\nu C_1 - C\alpha^*)\|\omega_{n+1}\|^2_{H_0^1(\Omega)} &\le a(\omega_{n+1}, \omega_{n+1}) = F_(\omega_{n+1}) \\
&\le C\|\omega_n\|^2_{H_0^1(\Omega)}\|\omega_{n+1}\|_{H_0^1(\Omega)}\,.
\end{aligned}
\tag{3.14}
$$

This gives 3.10, with $C_2 = \frac{C}{(\nu C_1 - C\alpha^*)}$ and the convergence is quadratic. □

4. Nonhomogeneous problem

We are concerned now with the nonhomogeneous problem

$$
\begin{gathered}
-\nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad \text{in } \Omega \\
\operatorname{div} u = 0 \quad \text{in } \Omega \\
u = g \quad \text{on } \Gamma
\end{gathered}
\tag{4.1}
$$

Where the state u is sought in the space $(H^1(\Omega))^2 \cap V$.

Throughout this section Ω denotes a bounded domain in $\mathbb{R}^2$, with Lipschitz-continuous boundary $\Gamma = \cap\Gamma_i$ $i = 1, \dots, 4$. We assume in this section that

$$
\int_{\Gamma_i} g.n_i\,d\sigma = 0 \quad \text{with } g \in H = (H^{1/2}(\Gamma))^2 \text{ and } f \in K = (H^{-1}(\Omega))^2. \tag{4.2}
$$

We assume also that for a given $g \in H$ satisfying 3.11, for any $c > 0$ there exists a function $w_0 \in (H^1(\Omega))^2$ such that

$$
\operatorname{div} w_0 = 0, \quad w_0|\Gamma = g, \tag{4.3}
$$

$$
|a_n(w_0, u_n)| \le c\|u_n\|^2_{H_0^1(\Omega)} \quad \forall u_n \in V. \tag{4.4}
$$

The existence of w_0 satisfying 3.11, 3.14 is a technical result due to Hopf [11].

Theorem 4.1. *Given $(g, f) \in K \times H$ satisfying 3.14, there exists a pair $(u, p) \in (H^1(\Omega))^2 \times L_0^2(\Omega)$ which is a solution of (4.1).*

Proof. Let $\xi_0 = u_0 - w_0$ where w_0 verify 3.11, 3.14 and an arbitrary $u_0 \in V$. We consider the sequence of linear problems

$$
\begin{gathered}
-\nu\Delta\xi_{n+1} + (\xi_{n+1}.\nabla)\xi_n + (\xi_n\cdot\nabla)\xi_{n+1} + \nabla p_{n+1} = \mathfrak{f}_\mathfrak{n} \quad \text{in } \Omega \\
\operatorname{div}\xi_{n+1} = 0 \quad \text{in } \Omega \\
\xi_{n+1} = 0 \quad \text{on } \Gamma
\end{gathered}
\tag{4.5}
$$

with $\xi_{n+1} = u_{n+1} - w_0$, $u_{n+1} \in H_0^1(\Omega)$ and $\mathfrak{f_n} = f + (\xi_n \nabla)\xi_n + \nu \Delta w_0 - (w_0 \cdot \nabla) w_0$.

Then ξ_{n+1} is a solution of the variational problem

$$\begin{gathered} \text{Find } \xi_{n+1} \in V \text{ such that} \\ a(\xi_{n+1}, v) = L_n(v) \quad \forall v \in V\,, \end{gathered} \tag{4.6}$$

where $a(\xi, v) = a_0(\xi, v) + a_n(\xi, v) + a^n(\xi, v) + a_\star(\xi, v)$ with

$$a_\star(\xi, v) = \int_\Omega (\xi \dot{\nabla}) w_0 v \, dx + \int_\Omega (w_0 \dot{\nabla}) \xi v \, dx$$

and $L_n(v) = \langle \mathfrak{f_n}, v \rangle$.

Taking $c > \nu$ and using 2.9 we obtain

$$|a(\xi_{n+1}, \xi_{n+1})| \geq (\nu - c) \|\xi_{n+1}\|^2_{H_0^1(\Omega)} \tag{4.7}$$

Thus we have the coercivity and L is obviously continuous on V. We observe that problem (4.6) fits into the framework of section 1 and therefore the sequence ξ_n converges towards a solution of (4.1). $\square$

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