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EXISTENCE AND EXPONENTIAL DECAY OF SOLUTIONS FOR TRANSMISSION PROBLEMS WITH DELAY

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Abstract. In this article we consider a transmission problem in a bounded domain with a delay term in the first equation. Under suitable assumptions on the weight of the damping and the weight of the delay, we prove the existence and the uniqueness of the solution using the semigroup theory. Also we show the exponential stability of the solution by introducing a suitable Lyaponov functional.

1. Introduction

In this article, we consider the transmission problem with a delay term,

$$\begin{gathered} u_{tt}(x,t) - au_{xx}(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \quad (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, \quad (x,t) \in (L_1,L_2) \times (0,+\infty), \end{gathered} \tag{1.1}$$

where $0 < L_1 < L_2 < L_3$, $\Omega =]0, L_1[\cup]L_2, L_3[$, a, b, μ_1, μ_2 are positive constants, and $\tau > 0$ is the delay.

System (1.1) is subjected to the following boundary and transmission conditions:

$$\begin{gathered} u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t), \quad i = 1,2 \\ au_x(L_i,t) = bv_x(L_i,t), \quad i = 1,2 \end{gathered} \tag{1.2}$$

and the initial conditions:

$$\begin{gathered} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t-\tau) = f_0(x,t-\tau), \quad x \in \Omega,\ t \in [0,\tau], \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in]L_1,L_2[. \end{gathered} \tag{1.3}$$

For $\mu_2 = 0$, system (1.1)-(1.3) has been investigated in [3]; for $\Omega = [0, L_1]$, the authors showed the well-posedness and exponential stability of the total energy. Muñoz Rivera and Oquendo [11] studied the wave propagations over materials

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consisting of elastic and viscoelastic components; that is,

$$\begin{gathered}\rho_1 u_{tt} - \alpha_1 u_{xx} = 0, \quad x \in]0, L_0[,\ t > 0, \\ \rho_2 v_{tt} - \alpha_2 v_{xx} + \int_0^t g(t-s) v_{xx}(s) ds = 0, \quad x \in]L_0, L[,\ t > 0,\end{gathered} \tag{1.4}$$

with the boundary and initial conditions:

$$\begin{gathered}u(0,t) = v(L,t), \quad u(L_0,t) = v(L_0,t), \quad t > 0, \\ \alpha_1 u_x(L_0,t) = \alpha_2 v_x(L_0,t) - \int_0^t g(t-s) v_x(s) ds, \quad t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in [0, L_0], \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in [L_0, L],\end{gathered} \tag{1.5}$$

where ρ_1 and ρ_2 are densities of the materials and α_1, α_2 are elastic coefficients and g is positive exponential decaying function. They showed that the dissipation produced by the viscoelastic part is strong enough to produce an exponential decay of the solution, no matter how small is its size. Ma and Oquendo [6] considered transmission problem involving two Euler-Bernoulli equations modeling the vibrations of a composite beam. By using just one boundary damping term in the boundary, they showed the global existence and decay property of the solution. Marzocchi et al [7] investigated a 1-D semi-linear transmission problem in classical thermoelasticity and showed that a combination of the first, second and third energies of the solution decays exponentially to zero, no matter how small the damping subdomain is. A similar result has sheen shown by Messaoudi and Said-Houari [9], where a transmission problem in thermoelasticity of type III has been investigated. See also Marzocchi et al [8] for a multidimensional linear thermoelastic transmission problem.

For $\mu_2 > 0$, problem (1.1) has a delay term in the internal feedback. This delay term may destabilize system (1.1)-(1.3) that is exponentially stable in the absence of delays [3]. The effect of the delay in the stability of hyperbolic systems has been investigated by many people. See for instance [4, 5].

In [10] the authors examined a system of wave equations with a linear boundary damping term with a delay:

$$\begin{gathered}u_{tt} - \Delta u = 0, \quad x \in \Omega,\ t > 0 \\ u(x,t) = 0, \quad x \in \Gamma_0,\ t > 0 \\ \frac{\partial u}{\partial \nu}(x,t) = \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) \quad x \in \Gamma_1,\ t > 0 \\ u(x,0) = u_0(x), \quad x \in \Omega, \\ u_t(x,0) = u_1(x) \quad x \in \Omega, \\ u_t(x, t-\tau) = g_0(x, t-\tau) \quad x \in \Omega,\ \tau \in (0,1)\end{gathered} \tag{1.6}$$

and under the assumption

$$\mu_2 < \mu_1 \tag{1.7}$$

they proved that the solution is exponentially stable. On the contrary, if (1.7) does not hold, they found a sequence of delays for which the corresponding solution of (1.6) will be unstable. We also recall the result by Xu et al [13], where the authors proved the same result as in [10] for the one space dimension by adopting the spectral analysis approach.

The aim of this article is to study the well-posedness and asymptotic stability of system (1.1)-(1.3) provided that (1.7) is satisfied. The paper is organized as follows. The well-posedness of the problem is analyzed in Section 2 using the semigroup theory. In Section 3, we prove the exponential decay of the energy when time goes to infinity.

2. Well-posedness of the problem

In this section, we prove the existence and the uniqueness of a local solution of system (1.1)-(1.3) by using the semi-group theory. So let us introduce the following new variable [10]

$$y(x,\rho,t)=u_t(x,t-\tau\rho). \tag{2.1}$$

Then, we obtain

$$\tau y_t(x,\rho,t)+y_\rho(x,\rho,t)=0,\quad \text{in } \Omega\times(0,1)\times(0,+\infty). \tag{2.2}$$

Therefore, problem (1.1) is equivalent to

$$\begin{gathered} u_{tt}(x,t)-au_{xx}(x,t)+\mu_1u_t(x,t)+\mu_2y(x,1,t)=0,\quad (x,t)\in\Omega\times]0,+\infty[\\ v_{tt}(x,t)-bv_{xx}(x,t)=0,\quad (x,t)\in]L_1,L_2[\times]0,+\infty[\\ \tau y_t(x,\rho,t)+y_\rho(x,\rho,t)=0,\quad \text{in } \Omega\times(0,1)\times(0,+\infty) \end{gathered} \tag{2.3}$$

which together with (1.3) can be rewritten as

$$\begin{gathered} U'=\mathscr{A}U,\\ U(0)=(u_0,v_0,u_1,v_1,f_0(.,-.\tau))^{\mathrm{T}}, \end{gathered} \tag{2.4}$$

where the operator $\mathscr{A}$ is defined by

$$\mathscr{A}\begin{pmatrix}u\\ v\\ \varphi\\ \psi\\ y\end{pmatrix}=\begin{pmatrix}\varphi\\ \psi\\ au_{xx}-\mu_1\varphi-\mu_2y(.,1)\\ bv_{xx}\\ -\frac{1}{\tau}y_\rho\end{pmatrix} \tag{2.5}$$

with the domain

$$D(\mathscr{A})=\big\{(u,v,\varphi,\psi,y)^{\mathrm{T}}\in\mathscr{H};y(.,0)=\varphi \text{ on } \Omega\big\},$$

where

$$\mathscr{H}=\big\{\big(H^2(\Omega)\times H^2(L_1,L_2)\big)\cap X_*\big\}\times H^1(\Omega)\times H^1(L_1,L_2)\times L^2(0,1,H^1(\Omega)).$$

Here the space X_* is defined by

$$\begin{aligned} X_*=\Big\{&(u,v)\in H^1(\Omega)\cap H^1(L_1,L_2):u(0,t)=u(L_3,t)=0,\\ &u(L_i,t)=v(L_i,t),\ au_x(L_i,t)=bv_x(L_i,t),\ i=1,2\Big\}. \end{aligned}$$

Now the energy space is defined by

$$\mathscr{K}=X_*\times L^2(\Omega)\times L^2(L_1,L_2)\times L^2((\Omega)\times(0,1)).$$

Let

$$U=(u,v,\varphi,\psi,y)^{\mathrm{T}},\quad \bar U=(\bar u,\bar v,\bar\varphi,\bar\psi,\bar y)^{\mathrm{T}}.$$

Then, for a positive constant ζ satisfying

$$\tau\mu_2\le\zeta\le\tau(2\mu_1-\mu_2), \tag{2.6}$$

we define the inner product in $\mathscr{K}$ as follows:

$$(U,\bar{U})_{\mathscr{K}} = \int_\Omega \{\varphi\bar{\varphi}+au_x\bar{u}_x\}\,dx+\int_{L_1}^{L_2}\{\psi\bar{\psi}+bv_x\bar{v_x}\}\,dx+\zeta\int_\Omega\int_0^1 y(x,\rho)\bar{y}(x,\rho)\,d\rho\,dx\,.$$

The existence and uniqueness result is stated as follows.

Theorem 2.1. *For any $U_0\in\mathscr{K}$ there exists a unique solution $U\in C([0,+\infty[,\mathscr{K})$ of problem* (2.4)*. Moreover, if $U_0\in D(\mathscr{A})$, then*

$$U\in C([0,+\infty[,D(\mathscr{A}))\cap C^1([0,+\infty[,\mathscr{K}).$$

Proof. To prove the result stated in Theorem 2.1, we use the semigroup theory, that is, we show that the operator $\mathscr{A}$ generates a C_0-semigroup in $\mathscr{K}$. In this step, we concern ourselves to prove that the operator $\mathscr{A}$ is dissipative. Indeed, for $U=(u,v,\varphi,\psi,y)^{\mathrm{T}}\in D(\mathscr{A})$, where $\varphi(L_2)=\psi(L_2)$ and ζ is a positive constant, we have

$$\begin{aligned}(\mathscr{A}U,U)_{\mathscr{K}} &= a\int_\Omega u_{xx}\varphi\,dx+b\int_{L_1}^{L_2}v_{xx}\psi\,dx-\mu_1\int_\Omega\varphi^2\,dx\\ &\quad-\mu_2\int_\Omega y(.,1)\varphi\,dx-\frac{\zeta}{\tau}\int_\Omega\int_0^1 y(x,\rho)y_\rho(x,\rho)\,d\rho\,dx\\ &\quad+a\int_\Omega u_x\varphi_x\,dx+b\int_{L_1}^{L_2}v_x\psi_x\,dx\,.\end{aligned}\tag{2.7}$$

Looking now at the last term of the right-hand side of (2.7), we have

$$\begin{aligned}\zeta\int_\Omega\int_0^1 y(x,\rho)y_\rho(x,\rho)\,d\rho\,dx &= \zeta\int_\Omega\frac{1}{2}\frac{\partial}{\partial\rho}y^2(x,\rho)\,d\rho\,dx\\ &= \frac{\zeta}{2}\int_\Omega(y^2(x,1)-y^2(x,0))\,dx.\end{aligned}\tag{2.8}$$

Integrating by parts in (2.7), keeping in mind the fact that $y(x,0,t)=\varphi(x,t)$ and using (2.8), we have from (2.7)

$$\begin{aligned}(\mathscr{A}U,U)_{\mathscr{K}} &= a[u_x\varphi]_{\partial\Omega}+b[v_x\psi]_{L_1}^{L_2}-\big(\mu_1-\frac{\zeta}{2\tau}\big)\int_\Omega\varphi^2\,dx\\ &\quad-\mu_2\int_\Omega y(.,1)\varphi\,dx-\frac{\zeta}{2\tau}\int_\Omega y^2(x,1)\,dx.\end{aligned}\tag{2.9}$$

Using Young's inequality, (1.2), and the equality $\varphi(L_2)=\psi(L_2)$, from (2.9), we obtain

$$\big(\mathscr{A}U,U\big)_{\mathscr{K}}\le-\big(\mu_1-\frac{\zeta}{2\tau}-\frac{\mu_2}{2}\big)\int_\Omega\varphi^2\,dx-\big(\frac{\zeta}{2\tau}-\frac{\mu_2}{2}\big)\int_\Omega y^2(x,1)\,dx.\tag{2.10}$$

Consequently, using (2.6), we deduce that $(\mathscr{A}U,U)_{\mathscr{K}}\le 0$. Thus, the operator $\mathscr{A}$ is dissipative.

Now to show that the operator $\mathscr{A}$ is maximal monotone, it is sufficient to show that the operator $\lambda I-\mathscr{A}$ is surjective for a fixed $\lambda>0$. Indeed, given

$(f_1, g_1, f_2, g_2, h)^{\mathrm{T}} \in \mathscr{K}$, we seek $U = (u, v, \varphi, \psi, y)^{\mathrm{T}} \in D(\mathscr{A})$ solution of

$$\begin{pmatrix} \lambda u - \varphi \\ \lambda v - \psi \\ \lambda\varphi - a u_{xx} + \mu_1 y(.,0) + \mu_2 y(.,1) \\ \lambda\psi - b v_{xx} \\ \lambda y + \frac{1}{\tau} y_\rho \end{pmatrix} = \begin{pmatrix} f_1 \\ g_1 \\ f_2 \\ g_2 \\ h \end{pmatrix} \tag{2.11}$$

suppose we have find (u, v) with the appropriate regularity, then

$$\begin{gathered} \varphi = \lambda u - f_1 \\ \psi = \lambda v - g_1. \end{gathered} \tag{2.12}$$

It is clear that $\varphi \in H^1(\Omega)$ and $\psi \in H^1(L_1, L_2)$, furthermore, by (2.11), we can find y as $y(x,0) = \varphi(x)$, $x \in \Omega$, using the approach as in Nicaise and Pignotti [10], we obtain, by using the equation in (2.11)

$$y(x,\rho) = \varphi(x)e^{-\lambda\rho\tau} + \tau e^{-\lambda\rho\tau} \int_0^\rho h(x,\sigma)e^{\lambda\sigma\tau} d\sigma\,.$$

From (2.12), we obtain

$$y(x,\rho) = \lambda u(x)e^{-\lambda\rho\tau} - f_1(x)e^{-\lambda\rho\tau} + \tau e^{-\lambda\rho\tau} \int_0^\rho h(x,\sigma)e^{\lambda\sigma\tau} d\sigma\,.$$

By using (2.11) and (2.12), the functions u, v satisfy the following equations:

$$\begin{gathered} \lambda^2 u - a u_{xx} + \mu_1 y(.,0) + \mu_2 y(.,1) = f_2 + \lambda f_1 \\ \lambda^2 v - b v_{xx} = g_2 + \lambda g_1\,. \end{gathered} \tag{2.13}$$

Since

$$\begin{aligned} y(x,1) &= \varphi(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau} \int_0^1 h(x,\sigma)e^{\lambda\tau} d\sigma \\ &= \lambda u e^{-\lambda\tau} + y_0(x), \end{aligned}$$

for $x \in \Omega$, we have

$$y_0(x) = -f_1(x) + \tau e^{-\lambda\tau} \int_0^1 h(x,\sigma)e^{\lambda\tau} d\sigma \tag{2.14}$$

The problem (2.13) can be reformulated as

$$\begin{gathered} \int_\Omega (\lambda^2 u - a u_{xx} + \mu_1 \lambda u + +\mu_2 \lambda u e^{-\lambda\tau})\omega_1 \, dx \\ = \int_\Omega (f_2 + \lambda f_1 - \mu_2 \lambda y_0(x))\omega_1 \, dx, \\ \int_{L_1}^{L_2} (\lambda^2 v - b v_{xx})\omega_2 \, dx = \int_{L_1}^{L_2} (g_2 + \lambda g_1)\omega_2 \, dx, \end{gathered} \tag{2.15}$$

for any $(\omega_1, \omega_2) \in X_*$.

Integrating the first equation in (2.15) by parts, we obtain

$$\begin{aligned}
&\int_\Omega (\lambda^2 u - a u_{xx} + \mu_1 u + \mu_2 \lambda u e^{-\lambda\tau})\omega_1\,dx \\
&= \int_\Omega \lambda^2 u\omega_1\,dx - a\int_\Omega u_{xx}\omega_1\,dx + \mu_1\int_\Omega \lambda u\,dx + \mu_2\int_\Omega \lambda u e^{-\lambda\tau}\omega_1\,dx \\
&= \int_\Omega \lambda^2 u\omega_1\,dx + a\int_\Omega u_x(\omega_1)_x\,dx - [a u_x\omega_1]_{\partial\Omega} \\
&\quad + \mu_1\int_\Omega \lambda u\,dx + \mu_2\int_\Omega \lambda u e^{-\lambda\tau}\omega_1\,dx \\
&= \int_\Omega (\lambda^2 + \mu_1\lambda + \mu_2\lambda e^{-\lambda\tau})u\omega_1\,dx + a\int_\Omega u_x(\omega_1)_x\,dx - [a u_x\omega_1]_{\partial\Omega}\,.
\end{aligned} \tag{2.16}$$

Integrating the second equation in (2.15) by parts, we obtain

$$\int_{L_1}^{L_2} (\lambda^2 v - b v_{xx})\omega_2\,dx = \int_{L_1}^{L_2} \lambda^2 v\omega_2\,dx + b\int_{L_1}^{L_2} v_x(\omega_2)_x\,dx - [b v_x\omega_2]_{L_1}^{L_2}\,. \tag{2.17}$$

Using (2.16) and (2.17), the problem (2.15) is equivalent to the problem

$$\Phi((u,v),(\omega_1,\omega_2)) = l(\omega_1,\omega_2) \tag{2.18}$$

where the bilinear form $\Phi : (X_* \times X_*) \to \mathbb{R}$ and the linear form $l : X_* \to \mathbb{R}$ are defined by

$$\begin{aligned}
\Phi((u,v),(\omega_1,\omega_2)) &= \int_\Omega (\lambda^2 + \mu_1\lambda + \mu_2\lambda e^{-\lambda\tau})u\omega_1\,dx + a\int_\Omega u_x(\omega_1)_x\,dx - [a u_x\omega_1]_{\partial\Omega} \\
&\quad + \int_{L_1}^{L_2} \lambda^2 v\ \omega_2\,dx + b\int_{L_1}^{L_2} v_x(\omega_2)_x\,dx - [b v_x\omega_2]_{L_1}^{L_2}
\end{aligned}$$

and

$$l(\omega_1,\omega_2) = \int_\Omega (f_2 + \lambda f_1 - \mu_2\lambda y_0(x))\omega_1\,dx + \int_{L_1}^{L_2} (g_2 + \lambda g_1)\omega_2\,dx\,.$$

Using the properties of the space X_*, it is clear that Φ is continuous and coercive, and l is continuous. So applying the Lax-Milgram theorem, we deduce that for all $(\omega_1,\omega_2) \in X_*$, problem (2.18) admits a unique solution $(u,v) \in X_*$. It follows from (2.16) and (2.17) that $(u,v) \in \{\big(H^2(\Omega) \times H^2(L_1,L_2)\big) \cap X_*\}$. Therefore, the operator $\lambda I - \mathscr{A}$ is dissipative for any $\lambda > 0$. Then the result in Theorem 2.1 follows from the Hille-Yoshida theorem. $\square$

3. Exponential decay of solutions

In this section we study the asymptotic behavior of the system (1.1)-(1.3). For any regular solution of (1.1)-(1.3), we define the energy as

$$E_1(t) = \frac{1}{2}\int_\Omega u_t^2(x,t)\,dx + \frac{a}{2}\int_\Omega u_x^2(x,t)\,dx, \tag{3.1}$$

$$E_2(t) = \frac{1}{2}\int_{L_1}^{L_2} v_t^2(x,t)\,dx + \frac{b}{2}\int_{L_1}^{L_2} v_x^2(x,t)\,dx\,. \tag{3.2}$$

The total energy is defined as

$$E(t) = E_1(t) + E_2(t) + \frac{\zeta}{2}\int_\Omega\int_0^1 y^2(x,\rho,t)\,d\rho\,dx \tag{3.3}$$

where ζ is the positive constant defined by (2.6). Our decay result reads as follows.

Theorem 3.1. *Let (u,v) be the solution of (1.1)-(1.3). Assume that $\mu_2 > \mu_1$ and*

$$\frac{a}{b} < \frac{L_3 + L_1 - L_2}{2(L_2 - L_1)}. \tag{3.4}$$

Then there exist two positive constants C and d, such that

$$E(t) \le Ce^{-dt}, \quad \forall t \ge 0. \tag{3.5}$$

Remark 3.2. Assumption (3.4) gives the relationship between the boundary regions and the transmission permitted. It can be also seen as a restriction on the wave speeds of the two equations and the damped part of the domain. It is known that for Timoshenko systems [12] and Bresse systems [1] that the wave speeds always control the decay rate of the solution. It is an interesting open question to show the behavior of the solution if (3.4) is not satisfied.

For the proof of Theorem 3.1 we use the following lemmas.

Lemma 3.3. *Let (u,v,y) be the solution of (2.3), (1.3). Assume that $\mu_1 \ge \mu_2$. Then we have the inequality*

$$\frac{dE(t)}{dt} \le \big(-\mu_1 + \frac{\mu_2}{2} + \frac{\zeta}{2\tau}\big) \int_\Omega y^2(x,0,t)\,dx + \big(\frac{\mu_2}{2} - \frac{\zeta}{2\tau}\big) \int_\Omega y^2(x,1,t)\,dx. \tag{3.6}$$

Proof. From (3.3) we have

$$\frac{dE_1(t)}{dt} = \int_\Omega u_{tt}(x,t)u_t(x,t)\,dx + a\int_\Omega u_{xt}(x,t)u_x(x,t)\,dx\,. \tag{3.7}$$

Using system (2.3), and integrating by parts, we obtain

$$\frac{dE_1(t)}{dt} = a[u_x u_t]_{\partial\Omega} - \mu_1 \int_\Omega u_t^2(x,t) - \mu_2 \int_\Omega u_t(x,t)y(x,1,t))\,dx\,. \tag{3.8}$$

On the other hand,

$$\frac{dE_2(t)}{dt} = b[v_x v_t]_{L_1}^{L_2}. \tag{3.9}$$

Using the fact that

$$\begin{aligned}
\frac{d}{dt}\frac{\zeta}{2}\int_\Omega\int_0^1 y^2(x,\rho,t)\,d\rho\,dx &= \zeta\int_\Omega\int_0^1 y(x,\rho,t)y_t(x,\rho,t)\,d\rho\,dx\\
&= -\frac{\zeta}{\tau}\int_\Omega\int_0^1 y_\rho(x,\rho,t)y(x,\rho,t)d\rho\,dx\\
&= -\frac{\zeta}{2\tau}\int_\Omega\int_0^1 \frac{d}{d\rho}y^2(x,\rho,t)\,d\rho\,dx\\
&= -\frac{\zeta}{2\tau}\int_\Omega (y^2(x,1,t) - y^2(x,0,t))\,dx\,,
\end{aligned} \tag{3.10}$$

collecting (3.8), (3.9), (3.10), using (1.2) and applying Young's inequality, we show that (3.6) holds. The proof is complete. □

Following [2], we define the functional

$$I(t) = \int_\Omega\int_{t-\tau}^t e^{s-t}u_t^2(x,s)\,ds\,dx\,,$$

and state the following lemma.

Lemma 3.4. *Let (u,v) be the solution of* (1.1)-(1.3). *Then*

$$\frac{dI(t)}{dt} \le \int_\Omega u_t^2(x,t)\,dx - e^{-\tau}\int_\Omega u_t^2(x,t-\tau)\,dx - e^{-\tau}\int_\Omega\int_{t-\tau}^t u_t^2(x,s)\,ds\,dx. \tag{3.11}$$

The proof of the above Lemma is straightforward, so we omit it. Now, we define the functional $\mathscr{D}(t)$ as follows

$$\mathscr{D}(t) = \int_\Omega uu_t\,dx + \frac{\mu_1}{2}\int_\Omega u^2\,dx + \int_{L_1}^{L_2} vv_t\,dx. \tag{3.12}$$

Then, we have the following estimate.

Lemma 3.5. *The functional $\mathscr{D}(t)$ satisfies*

$$\begin{aligned}\frac{d}{dt}\mathscr{D}(t) &\le -(a-\epsilon_0 c_0^2)\int_\Omega u_x^2\,dx - b\int_{L_1}^{L_2} v_x^2\,dx \\ &\quad + \int_\Omega u_t^2\,dx + \int_{L_1}^{L_2} v_t^2\,dx + C(\epsilon_0)\int_\Omega y^2(x,1,t)\,dx\end{aligned} \tag{3.13}$$

Proof. Taking the derivative of $\mathscr{D}(t)$ with respect to t and using (1.1), we find that

$$\begin{aligned}\frac{d}{dt}\mathscr{D}(t) &= \int_\Omega u_t^2\,dx + \int_{L_1}^{L_2} v_t^2\,dx - a\int_\Omega u_x^2\,dx - b\int_{L_1}^{L_2} v_x^2\,dx \\ &\quad - \mu_2\int_\Omega u(x,t)y(x,1,t)\,dx + [au_xu]_{\partial\Omega} + [bv_xv]_{L_1}^{L_2}.\end{aligned} \tag{3.14}$$

Applying young's inequality and using the boundary conditions (1.2), we have

$$\begin{aligned}[au_xu]_{\partial\Omega} + [bv_xv]_{L_1}^{L_2} &= au_x(L_1,t)u(L_1,t) - au_x(L_2,t)u(L_2,t) \\ &\quad + bv_x(L_2,t)v(L_2,t) - bv_x(L_1,t)v(L_1,t) = 0.\end{aligned} \tag{3.15}$$

On the other hand, we have by Poincaré's inequality and Young's inequality,

$$\mu_2\int_\Omega u(x,t)y(x,1,t)\,dx \le \epsilon_0 c_0^2\int_\Omega u_x^2\,dx + C(\epsilon_0)\int_\Omega y^2(x,1,t)\,dx \tag{3.16}$$

where ϵ_0 is a positive constants and c_0 is the Poincaé's constant. Consequently, plugging the above estimates into (3.14), we find (3.13). $\square$

Now, inspired by [7], we introduce the functional

$$q(x) = \begin{cases} x - \frac{L_1}{2}, & x\in[0,L_1], \\ x - \frac{L_2+L_3}{2}, & x\in[L_2,L_3], \\ \frac{L_2-L_3-L_1}{2(L_2-L_1)}(x-L_1) + \frac{L_1}{2}, & x\in[L_1,L_2] \end{cases} \tag{3.17}$$

Next, we define the functionals

$$\mathscr{F}_1(t) = -\int_\Omega q(x)u_xu_t\,dx, \quad \mathscr{F}_2(t) = -\int_{L_1}^{L_2} q(x)v_xv_t\,dx.$$

Then, we have the following estimates.

Lemma 3.6. *For any $\epsilon_2 > 0$, we have the estimates:*

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_1(t) &\le C(\epsilon_2)\int_\Omega u_t^2\,dx + \big(\frac{a}{2}+\epsilon_2\big)\int_\Omega u_x^2\,dx + C(\epsilon_2)\int_\Omega y^2(x,1,t)\,dx \\ &\quad - \frac{a}{4}[(L_3-L_2)u_x^2(L_2,t) + L_1u_x^2(L_1,t)]\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_2(t) \le \frac{L_2-L_3-L_1}{4(_2-L_1)}\Big(\int_{L_1}^{L_2} v_t^2\,dx + \int_{L_1}^{L_2} bv_x^2\,dx\Big) \\ + \frac{b}{4}\big((L_3-L_2)v_x^2(L_2,t)+L_1v_x^2(L_1,t)\big).\end{aligned} \tag{3.19}$$

Proof. Taking the derivative of $\mathscr{F}_1(t)$ with respect to t and using (1.1), we obtain

$$\begin{aligned}&\frac{d}{dt}\mathscr{F}_1(t)\\ &= -\int_\Omega q(x)u_{tx}u_t\,dx - \int_\Omega q(x)u_xu_{tt}\,dx \\ &= -\int_\Omega q(x)u_{tx}u_t\,dx - \int_\Omega q(x)u_x(au_{xx}(x,t)-\mu_1u_t(x,t)-\mu_2y(x,1,t))\,dx.\end{aligned} \tag{3.20}$$

Integrating by parts,

$$\int_\Omega q(x)u_{tx}u_t\,dx = -\frac{1}{2}\int_\Omega q'(x)u_t^2\,dx + \frac{1}{2}[q(x)u_t^2]_{\partial\Omega}. \tag{3.21}$$

On the other hand,

$$\int_\Omega aq(x)u_{xx}u_x\,dx = -\frac{1}{2}\int_\Omega aq'(x)u_x^2\,dx + \frac{1}{2}[aq(x)u_x^2]_{\partial\Omega}. \tag{3.22}$$

Substituting (3.21) and (3.22) in (3.20), we find that

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_1(t) = \frac{1}{2}\int_\Omega q'(x)u_t^2\,dx + \frac{1}{2}\int_\Omega aq'(x)u_x^2\,dx - \frac{1}{2}[q(x)u_t^2]_{\partial\Omega} \\ - \frac{1}{2}[aq(x)u_x^2]_{\partial\Omega} + \int_\Omega q(x)u_x\big(\mu_1u_t(x,t)+\mu_2y(x,1,t)\big)\,dx\,.\end{aligned} \tag{3.23}$$

Using Young's inequality and (3.17), equation (3.23) becomes

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_1(t) \le C(\epsilon_2)\int_\Omega u_t^2\,dx + \big(\frac{a}{2}+\epsilon_2\big)\int_\Omega u_x^2\,dx - \frac{1}{2}[q(x)u_t^2]_{\partial\Omega} \\ - \frac{a}{2}[q(x)u_x^2]_{\partial\Omega} + C(\epsilon_2)\int_\Omega y^2(x,1,t)\,dx.\end{aligned} \tag{3.24}$$

for any $\epsilon_2 > 0$. Since $q(L_1) > 0$ and $q(L_2) < 0$, by using the boundary conditions (1.2), we have

$$\frac{1}{2}[q(x)u_t^2]_{\partial\Omega} \ge 0. \tag{3.25}$$

Also, we have

$$\begin{aligned}-\frac{a}{2}[q(x)u_x^2]_{\partial\Omega} = -\frac{aL_1}{4}[u_x^2(L_1,t)+u_x^2(0,t)] \\ - \frac{a(L_3-L_2)}{4}[u_x^2(L_3,t)+u_x^2(L_2,t)].\end{aligned} \tag{3.26}$$

Taking into account (3.25) and (3.26), then (3.24) gives (3.18).

By the same method, taking the derivative of $\mathscr{F}_2(t)$ with respect to t, we obtain

$$\begin{aligned}
\frac{d}{dt}\mathscr{F}_2(t) &= -\int_{L_1}^{L_2} q(x)v_{tx}v_t\,dx - \int_{L_1}^{L_2} q(x)v_x v_{tt} \\
&= \frac{1}{2}\int_{L_1}^{L_2} q'(x)v_t^2\,dx - \frac{1}{2}[q(x)v_t^2]_{L_1}^{L_2} + \frac{1}{2}\int_{L_1}^{L_2} bq'(x)v_x^2\,dx - \frac{b}{2}[q(x)v_x^2]_{L_1}^{L_2} \\
&\le \frac{L_2-L_3-L_1}{4(L_2-L_1)}\Big(\int_{L_1}^{L_2} v_t^2\,dx + \int_{L_1}^{L_2} bv_x^2\,dx\Big) \\
&\quad + \frac{b}{4}\big((L_3-L_2)v_x^2(L_2,t) + L_1 v_x^2(L_1,t)\big).
\end{aligned} \tag{3.27}$$

which is exactly (3.19). □

Proof of Theorem 3.1. We define the Lyapunov functional

$$\mathscr{L}(t) = NE(t) + I(t) + \gamma_2\mathscr{D}(t) + \gamma_3\mathscr{F}_1(t) + \gamma_4\mathscr{F}_2(t), \tag{3.28}$$

where N, γ_2, γ_3 and γ_4 are positive constants that will be fixed later.

Now, it is clear from the boundary conditions (1.2), that

$$a^2u_x^2(L_i,t) = b^2v_x^2(L_i,t), \quad i=1,2. \tag{3.29}$$

Taking the derivative of (3.28) with respect to t and making use of (3.6), (3.11), (3.13), (3.18), (3.18) and taking into account (3.29), we obtain

$$\begin{aligned}
\frac{d}{dt}\mathscr{L}(t) &\le \big\{N\big(-\mu_1 + \frac{\mu_2}{2} + \frac{\zeta}{2\tau}\big) + 1 + \gamma_2 + \gamma_3C(\epsilon_2)\big\}\int_\Omega u_t^2\,dx \\
&\quad + \big\{N\big(\frac{\mu_2}{2} - \frac{\zeta}{2\tau}\big) - e^{-\tau} + \gamma_2C(\epsilon_0) + C(\epsilon_2)\gamma_3\big\}\int_\Omega y^2(x,1,t)\,dx \\
&\quad + \big\{\gamma_2(-a+\epsilon_0c_0^2) + \gamma_3\epsilon_2 + \frac{\gamma_3 a}{2}\big\}\int_\Omega u_x^2\,dx \\
&\quad + \big\{b\frac{L_2-L_3-L_1}{4(_2-L_1)}\gamma_4 - \gamma_2 b\big\}\int_{L_1}^{L_2} v_x^2\,dx \\
&\quad + \big\{\frac{L_2-L_3-L_1}{4(L_2-L_1)}\gamma_4 + \gamma_2\big\}\int_{L_1}^{L_2} v_t^2\,dx - e^{-\tau}\int_\Omega\int_{t-\tau}^{t} u_t^2(x,s)\,ds\,dx \\
&\quad - (\gamma_3 - \frac{a}{b}\gamma_4)\frac{a(L_3-L_2)}{4}u_x^2(L_2,t) - (\gamma_3 - \frac{a}{b}\gamma_4)\frac{aL_1}{4}u_x^2(L_1,t).
\end{aligned} \tag{3.30}$$

At this point, we choose our constants in (3.30), carefully, such that all the coefficients in (3.30) will be negative. Indeed, under the assumption (3.4), we can always find γ_2, γ_3 and γ_4 such that

$$\frac{L_2-L_3-L_1}{4(L_2-L_1)}\gamma_4 + \gamma_2 < 0, \quad \gamma_3 > \frac{a}{b}\gamma_4, \quad \gamma_2 > \frac{\gamma_3}{2}. \tag{3.31}$$

Once the above constants are fixed, we may choose ϵ_2 and ϵ_0 small enough such that

$$\epsilon_0c_0^2 + \gamma_3\epsilon_2 < a(\gamma_2 - \gamma_3/2).$$

Finally, keeping in mind (2.6) and choosing N large enough such that the first and the second coefficients in (3.30) are negatives.

Consequently, from the above, we deduce that there exist a positive constant η_1, such that (3.30) becomes

$$\begin{aligned} \frac{d\mathscr{L}(t)}{dt} &\le -\eta_1 \int_\Omega (u_t^2(x,t) + u_x^2(x,t) + u_t^2(x,t-\tau))\,dx \\ &\quad - \eta_1 \int_{L_1}^{L_2} (v_t^2(x,t) + v_x^2(x,t))\,dx - \eta_1 \int_\Omega \int_{t-\tau}^t u_t^2(x,s)\,ds\,dx\,. \end{aligned} \tag{3.32}$$

Consequently, recalling (3.3), we deduce that there exist also $\eta_2 > 0$, such that

$$\frac{d\mathscr{L}(t)}{dt} \le -\eta_2 E(t), \quad \forall t \ge 0. \tag{3.33}$$

On the other hand, it is not hard to see that from (3.28) and for N large enough, there exist two positive constants β_1 and β_2 such that

$$\beta_1\ E(t) \le \mathscr{L}(t) \le \beta_2 E(t), \quad \forall t \ge 0. \tag{3.34}$$

Combining (3.33) and (3.33), we deduce that there exists $\Lambda > 0$ for which the estimate

$$\frac{d\mathscr{L}(t)}{dt} \le -\Lambda \mathscr{L}(t), \quad \forall t \ge 0, \tag{3.35}$$

holds. Integrating (3.33) over $(0,t)$ and using ((3.33) once again, then (3.5) holds. Then, the proof is complete. □

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