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LIPSCHITZ STABILITY FOR LINEAR PARABOLIC SYSTEMS WITH INTERIOR DEGENERACY

IDRISS BOUTAAYAMOU, GENNI FRAGNELLI, LAHCEN MANIAR

Abstract. In this article, we study an inverse problem for linear degenerate parabolic systems with one force. We establish Lipschitz stability for the source term from measurements of one component of the solution at a positive time and on a subset of the space domain, which contains degeneracy points. The key ingredient is the derivation of a Carleman-type estimate.

1. Introduction

The null controllability and inverse problems of parabolic equations and parabolic coupled systems have attracted much interest in these last years, see [3, 4, 5, 6, 8, 15, 16, 17, 18, 23, 24, 25, 28, 29, 30]. The main result in these papers is the development of suitable Carleman estimates, which are crucial tools to obtain observability inequalities and Lipschitz stability for term sources, initial data, potentials and diffusion coefficients. The above systems are considered to be non degenerate. In other words, the diffusion coefficients are uniformly coercive. On the contrary, the case of degenerate coefficients at the boundary is also considered in several papers by developing adequate Carleman estimates. The null controllability and inverse problems of degenerate parabolic equations are studied in [10, 12, 13, 14, 27, 32], and for the coupled degenerate parabolic systems in [1, 2, 7, 11, 26]. In these papers, the degeneracy considered is at the boundary of the spatial domain.

After the pioneering works [19, 20], there has been substantial progress in understanding the null controllability of parabolic equations with interior degeneracy (see, e.g., [21]). In this scope, the goal of this paper is to study an inverse source problem of a 2×2 parabolic systems with interior degeneracy and different diffusion coefficients

$$\begin{gathered}
u_t-(a_1u_x)_x+b_{11}u+b_{12}v=f,\quad (t,x)\in Q,\\
v_t-(a_2v_x)_x+b_{22}v=0,\quad (t,x)\in Q,\\
u(t,0)=u(t,1)=v(t,0)=v(t,1)=0,\quad t\in(0,T),\\
u(0,x)=u_0(x),\quad v(0,x)=v_0(x),\quad x\in(0,1),
\end{gathered}\tag{1.1}$$

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where $u_0, v_0 \in L^2(0,1)$, $T > 0$ fixed, $Q := (0,T) \times (0,1)$, $b_{ij} \in L^\infty(0,1)$, $i,j = 1,2$, and every a_i, $i = 1,2$, degenerates at an interior point x_i of the spatial domain $(0,1)$ (for the precise assumptions we refer to Section 2). For $t_0 \in (0,T)$ given, let $Q_{t_0}^T = (t_0,T) \times (0,1)$ and $T' := \frac{T+t_0}{2}$. For a given $C_0 > 0$, we denote by $S(C_0)$ the space

$$S(C_0) := \{f \in H^1(0,T;L^2(0,1)) : |f_t(t,x)| \le C_0|f(T',x)|, \text{ a.e. } (t,x) \in Q\}.$$

More precisely, we want to establish Lipschitz stability for the source term f from measurements of the component u at time T' and on a subset $\omega \subset (0,1)$, which contains the degeneracy points.

The main ingredient to obtain Lipschitz stability is Carleman estimates for degenerate equations. For null controllability of a parabolic equation with interior degeneracy, Carleman estimates were obtained in [21] and in [20]. For inverse problems, these estimates are not sufficient, and one needs also some additional estimates on the term u with a special weight and the derivative term u_t. We prove first these for parabolic equations with interior degeneracy similar to the ones obtained in [10, 32] in the case of a boundary degeneracy. This will lead to obtain our Carleman estimates for system (1.1). At the end having these Carleman estimates in hand, we follow the method developed in [5, 8, 24] to obtain the Lipschitz stability for the source term f. The main task here is to estimate the source f by the measurements, on the domain ω, of the first component u of the solutions of system (1.1).

To prove our Carleman estimates, we use the following Hardy-Poincaré inequality proved in [20, Proposition 2.1]

$$\int_0^1 \frac{p(x)}{(x-x_0)^2} w^2(x)\,dx \le C_{HP} \int_0^1 p(x)|w_x(x)|^2\,dx \tag{1.2}$$

for all functions w such that

$$w(0) = w(1) = 0 \quad \text{and} \quad \int_0^1 p(x)|w_x(x)|^2\,dx < \infty.$$

Here p is any continuous function in $[0,1]$, with $p > 0$ on $[0,1] \setminus \{x_0\}$, $p(x_0) = 0$, for some x_0 in $(0,1)$, and such that there exists $\vartheta \in (1,2)$ so that the function $x \mapsto p(x)/|x-x_0|^\vartheta$ is non-increasing on the left of x_0 and nondecreasing on the right of x_0.

This article is organized as follows: in Section 2, we discuss the well-posedness of the system (1.1). Then, in Section 3, we establish different Carleman estimates for parabolic equations and parabolic systems (1.1). Finally, in Section 4, we apply the Carleman estimates to prove the Lipschitz stability result.

2. Assumptions and well-posedness

To study the well-posedness of system (1.1), we consider two situations, namely the weakly degenerate (WD) and the strongly degenerate (SD) cases. The associated weighted spaces and assumptions on diffusion coefficients are the following:
Case (WD): for $i = 1,2$, let

$$H_{a_i}^1(0,1) := \big\{u \text{ abs. cont. in } [0,1],\ \sqrt{a_i}u_x \in L^2(0,1),\ u(0) = u(1) = 0\big\},$$

where the functions a_i satisfy

$$\begin{aligned}&\text{there exists } x_i \in (0,1),\ i=1,2 \text{ such that } a_i(x_i)=0,\ a_i>0 \text{ in } [0,1]\setminus\{x_i\},\ a_i \in C^1([0,1]\setminus\{x_i\});\\ &\text{and there exists } K_i \in (0,1) \text{ such that } (x-x_i)a_i' \le K_i a_i, \text{ a.e. in } [0,1].\end{aligned} \tag{2.1}$$

Case (SD): for $i=1,2$, let

$$\begin{aligned}H^1_{a_i}(0,1) := \Big\{&u \in L^2(0,1) : u \text{ is locally abs. cont. in } [0,1]\setminus\{x_i\},\\ &\sqrt{a_i}u_x \in L^2(0,1),\ u(0)=u(1)=0\Big\},\end{aligned}$$

where the functions a_i satisfy

$$\begin{aligned}&\text{there exists } x_i \in (0,1),\ i=1,2 \text{ such that } a_i(x_i)=0,\ a_i>0 \text{ in } [0,1]\setminus\{x_i\},\ a_i \in C^1([0,1]\setminus\{x_i\})\cap W^{1,\infty}(0,1); \text{ there exists}\\ &K_i \in [1,2) \text{ such that } (x-x_i)a_i' \le K_i a_i \text{ a.e. in } [0,1], \text{ and if } K_i > 4/3, \text{ there exists } \gamma \in (0,K_i] \text{ such that } a_i/|x-x_i|^\gamma \text{ is non-}\\ &\text{increasing on the left of } x_i \text{ and nondecreasing on the right of } x_i.\end{aligned} \tag{2.2}$$

In both cases, for $i=1,2$, we consider the space

$$H^2_{a_i}(0,1) := \big\{u \in H^1_{a_i}(0,1) : a_i u_x \in H^1(0,1)\big\}$$

with the norms

$$\|u\|^2_{H^1_{a_i}} := \|u\|^2_{L^2(0,1)} + \|\sqrt{a_i}u_x\|^2_{L^2(0,1)}, \quad \|u\|^2_{H^2_{a_i}} := \|u\|^2_{H^1_{a_i}} + \|(a_i u_x)_x\|^2_{L^2(0,1)}.$$

We recall from [21] that, for $i=1,2$, the operator $(A_i, D(A_i))$ defined by $A_i u := (a_i u_x)_x$, $u \in D(A_i) = H^2_{a_i}(0,1)$ is closed negative self-adjoint with dense domain in $L^2(0,1)$. In the Hilbert space $\mathbb{H} := L^2(0,1)\times L^2(0,1)$, the system (1.1) can be transformed into the Cauchy problem

$$\begin{gathered}X'(t) = \mathcal{A}X(t) - BX(t) + F(t), \quad t \in (0,T),\\ X(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix},\end{gathered}$$

where $X(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$, $\mathcal{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, $D(\mathcal{A}) = D(A_1)\times D(A_2)$, $F(t) = \begin{pmatrix} f(t) \\ 0 \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}$.

Since the operator $\mathcal{A}$ is diagonal and B is a bounded perturbation, the following well-posedness and regularity results hold.

Proposition 2.1. *(i) The operator $\mathcal{A}$ generates a contraction strongly continuous semigroup.*

(ii) For all $(u_0,v_0) \in D(\mathcal{A})$ and $f \in H^1(0,T;L^2(0,1))$, the problem (1.1) *has a unique solution $(u,v) \in C\big([0,T], D(\mathcal{A})\big) \cap C^1(0,T;\mathbb{H})$.*

(iii) For all $f \in L^2(Q)$, $u_0, v_0 \in L^2(0,1)$, and $\varepsilon \in (0,T)$, there exists a unique mild solution $(u,v) \in X_T := H^1\big([\varepsilon,T],\mathbb{H}\big) \cap L^2\big(\varepsilon,T;D(\mathcal{A})\big)$ of (1.1) *satisfying*

$$\|(u,v)\|_{X_T} \le C_T\Big(\|(u_0,v_0)\|^2_{\mathbb{H}} + \|(F,G)\|^2_{\mathbb{H}}\Big).$$

Moreover, for $f \in H^1(0,T;L^2(0,1))$ and $\varepsilon \in (0,T)$, we have $(u,v) \in Y_T := C\big([\varepsilon,T], D(\mathcal{A})\big) \cap C^1(\varepsilon,T;\mathbb{H})$.

3. Carleman estimate

The main topic of this section is to establish a Carleman estimate for a degenerate parabolic single equation with a boundary observation on the right hand side. Then, we will deduce the one for the degenerate system (1.1) with distributed observation of u on the subdomain ω.

Part of these estimates were obtained in [20, 21] under two different assumptions on the degenerate diffusion coefficient for a null controllability purpose. In the forthcoming theorems we will prove additional estimates on u and u_t, that are crucial to prove Lipschitz stability results.

Throughout this section, we set $\omega = (\lambda, \beta)$ and assume, without loss of generality, $x_1 < x_2$. Set also $\omega' := \tilde{\omega} \cup \overline{\omega} := (\lambda, \beta_1) \cup (\lambda_2, \beta)$ and $\omega'' := \tilde{\tilde{\omega}} \cup \overline{\overline{\omega}} := (\lambda, \frac{\lambda_1+2\beta_1}{3}) \cup (\frac{\lambda_2+2\beta_2}{3}, \beta)$, where λ_i and β_i, for $i = 1, 2$, satisfy $0 < \lambda < \lambda_1 < \beta_1 < x_1 < x_2 < \lambda_2 < \beta_2 < \beta < 1$.

3.1. Carleman estimate for an equation.

In this subsection we shall derive the Carleman estimate for the solution of the problem

$$\begin{gathered} u_t - (a(x)u_x)_x + cu = h, \quad (t, x) \in Q, \\ u(t, 0) = u(t, 1) = 0, \quad t \in (0, T), \\ u(0, x) = u_0(x), \quad x \in (0, 1), \end{gathered} \tag{3.1}$$

where the diffusion coefficient a satisfies (2.1) or (2.2) in $x_0 \in (0, 1)$ with K in place of K_i, $i = 1, 2$, $h \in L^2(Q)$ and $c \in L^\infty(Q)$. As usual this aim relies on the introduction of some suitable weight functions. Towards this end, as in [20, 21], we define the following time and space weight functions

$$\begin{gathered} \theta(t) := \frac{1}{[(t - t_0)(T - t)]^4}, \quad \eta(t) := T + t_0 - 2t, \\ \psi(x) = c_1\Big(\int_{x_0}^x \frac{y - x_0}{a(y)}\, dy - c_2\Big), \quad \varphi(t, x) := \theta(t)\psi(x), \end{gathered}$$

where $t_0 \in (0, T)$ is given, $c_1 > 0$ and $c_2 > \max\big\{\frac{(1-x_0)^2}{a(1)(2-K)}, \frac{x_0^2}{a(0)(2-K)}\big\}$. For this choice it is easy to prove that $-c_1c_2 \le \psi(x) < 0$ for every $x \in [0, 1]$, and that η is positive if $0 < t < \frac{T+t_0}{2}$ and negative if $\frac{T+t_0}{2} < t < T$.

Now we are ready to state the Carleman estimate related to (3.1).

Theorem 3.1. *Assume* (2.1) *or* (2.2) *and let $T > 0$. Then there exist two positive constants C and s_0 such that the solution u of* (3.1) *in $H^1\big([\varepsilon, T], L^2(0, 1)\big) \cap L^2\big(\varepsilon, T; H^2_a(0, 1)\big)$ satisfies, for all $s \ge s_0$,*

$$\begin{aligned} &\int_{Q_{t_0}^T} \Big(s\theta a(x)u_x^2 + s^3\theta^3 \frac{(x - x_0)^2}{a(x)} u^2 + s\theta^{3/2}|\eta\psi|u^2 + \frac{1}{s\theta} u_t^2\Big) e^{2s\varphi}\, dx\, dt \\ &\le C\Big(\int_{Q_{t_0}^T} h^2 e^{2s\varphi}\, dx\, dt + sc_1 \int_{t_0}^T \Big[\theta a(x - x_0)u_x^2 e^{2s\varphi}\Big]_{x=0}^{x=1} dt\Big). \end{aligned} \tag{3.2}$$

Proof. Let u be the solution of (3.1). For $s > 0$, the function $w = e^{s\varphi}u$ satisfies

$$\underbrace{-(aw_x)_x - s\varphi_t w - s^2 a\varphi_x^2 w}_{L_s^+ w} + \underbrace{w_t + 2sa\varphi_x w_x + s(a\varphi_x)_x w}_{L_s^- w} = \underbrace{he^{s\varphi} - cw}_{h_s}.$$

Moreover, $w(t_0,x) = w(T,x) = 0$. This property allows us to apply the Carleman estimates established in [21] to w with $Q_{t_0}^T$ in place of $(0,T)\times(0,1)$

$$\begin{aligned} &\|L_s^+ w\|^2 + \|L_s^- w\|^2 + \int_{Q_{t_0}^T} \Big(s^3\theta^3 \frac{(x-x_0)^2}{a(x)} w^2 + s\theta a(x) w_x^2\Big)\,dx\,dt \\ &\le C\Big(\int_{Q_{t_0}^T} h^2 e^{2s\varphi}\,dx\,dt + sc_1\int_{t_0}^T \Big[\theta a(x-x_0)w_x^2\Big]_{x=0}^{x=1} dt\Big). \end{aligned} \tag{3.3}$$

The operators L_s^+ and L_s^- are not exactly the ones of [20, 21]. However, one can prove that the Carleman estimates do not change. Using the previous estimate we will bound the integral $\int_{Q_{t_0}^T}\big(\frac{1}{s\theta}u_t^2 + s\theta^{3/2}|\eta\psi|u^2\big)e^{2s\varphi}\,dx\,dt$. In fact, we have

$$\begin{aligned} &\int_{Q_{t_0}^T} s\theta^{3/2}|\eta\psi|w^2\,dx\,dt \\ &\le C\int_{Q_{t_0}^T} s\theta^{3/2}w^2\,dx\,dt \\ &= sC\Big|\int_{Q_{t_0}^T}\Big(\theta\frac{a^{1/3}}{|x-x_0|^{2/3}}w^2\Big)^{3/4}\Big(\theta^3\frac{|x-x_0|^2}{a}w^2\Big)^{1/4}\,dx\,dt\Big| \\ &\le sC\frac{3}{2}\int_{Q_{t_0}^T}\theta\frac{a^{1/3}}{|x-x_0|^{2/3}}w^2\,dx\,dt + s^3C\frac{1}{2}\int_{Q_{t_0}^T}\theta^3\frac{|x-x_0|^2}{a}w^2\,dx\,dt, \end{aligned}$$

since $|\eta| \le T + t_0$ and $|\psi| \le c_1c_2$. Now, if $K \le 4/3$, we consider the function $p(x) = |x-x_0|^{4/3}$. Obviously, there exists $q \in (1,4/3)$ such that the function $\frac{p(x)}{|x-x_0|^q}$ is non-increasing on the left of x_0 and nondecreasing on the right of x_0. Then, we can apply the Hardy-Poincaré inequality (1.2), obtaining

$$\begin{aligned} \int_0^1 \frac{a^{1/3}}{|x-x_0|^{2/3}}w^2\,dx &\le \max_{x\in[0,1]} a^{1/3}(x)\int_0^1\frac{1}{|x-x_0|^{2/3}}w^2\,dx \\ &= \max_{x\in[0,1]} a^{1/3}(x)\int_0^1\frac{p(x)}{|x-x_0|^2}w^2\,dx \\ &\le \max_{x\in[0,1]} a^{1/3}(x)C_{HP}\int_0^1 p(x)w_x^2\,dx \\ &= \max_{x\in[0,1]} a^{1/3}(x)C_{HP}\int_0^1 a\frac{|x-x_0|^{4/3}}{a}w_x^2\,dx \\ &= \max_{x\in[0,1]} a^{1/3}(x)C_{HP}C_1\int_0^1 aw_x^2\,dx, \end{aligned}$$

where

$$C_1 = \max\Big(\frac{x_0^{4/3}}{a(0)}, \frac{(1-x_0)^{4/3}}{a(1)}\Big).$$

In the previous inequality, we have used the property that the map $x \mapsto |x - x_0|^\gamma/a(x)$ is non-increasing on the left of x_0 and nondecreasing on the right of x_0 for all $\gamma > K$, see [20, Lemma 2.1]. If $K > 4/3$, we can consider the function

$p(x) = (a(x)|x-x_0|^4)^{1/3}$. Then $p(x) = a(x)\Big(\frac{(x-x_0)^2}{a(x)}\Big)^{2/3} \le C_1 a(x)$, where

$$C_1 := \max\Big\{\Big(\frac{x_0^2}{a(0)}\Big)^{2/3}, \Big(\frac{(1-x_0)^2}{a(1)}\Big)^{2/3}\Big\}, \quad \frac{a^{1/3}}{|x-x_0|^{2/3}} = \frac{p(x)}{(x-x_0)^2}.$$

Moreover, using hypothesis (2.2), one has that the function $\frac{p(x)}{|x-x_0|^q}$, with $q := \frac{4+\gamma}{3}$ in $(1,2)$, is non-increasing on the left of x_0 and nondecreasing on the right of x_0. The Hardy-Poincaré inequality implies

$$\begin{aligned}\int_0^1 \frac{a^{1/3}}{|x-x_0|^{2/3}} w^2 dx &= \int_0^1 \frac{p}{(x-x_0)^2} w^2 dx \le C_{HP} \int_0^1 p(w_x)^2 dx \\ &\le C_{HP} C_1 \int_0^1 a(w_x)^2 dx.\end{aligned}$$

Thus, in every case,

$$\int_{Q_{t_0}^T} \theta \frac{a^{1/3}}{|x-x_0|^{2/3}} w^2 \, dx\, dt \le C \int_{Q_{t_0}^T} \theta a w_x^2 \, dx\, dt \tag{3.4}$$

for a positive constant C. Then, for s large enough, we have

$$\int_{Q_{t_0}^T} s\theta^{3/2} |\eta\psi| w^2 \, dx\, dt \le C \int_{Q_{t_0}^T} \big(s\theta a w_x^2 + s^3\theta^3 \frac{|x-x_0|^2}{a} w^2\big) \, dx\, dt, \tag{3.5}$$

$$\int_{Q_{t_0}^T} s\theta^{3/2} |\eta\psi| w^2 \, dx\, dt \le C\Big(\int_{Q_{t_0}^T} h^2 e^{2s\varphi} \, dx\, dt + sc_1 \int_{t_0}^T \Big[\theta a(x-x_0) w_x^2\Big]_{x=0}^{x=1} dt\Big). \tag{3.6}$$

On the other hand, we have

$$\frac{1}{\sqrt{s\theta}} L_s^- w = \frac{1}{\sqrt{s\theta}} w_t + 2c_1\sqrt{s\theta}(x-x_0) w_x + c_1\sqrt{s\theta} w.$$

Therefore,

$$\begin{aligned}\int_{Q_{t_0}^T} \frac{1}{s\theta} w_t^2 \, dx\, dt &\le C\Big(\|L_s^- w\|^2 + \int_{Q_{t_0}^T} s\theta \frac{|x-x_0|^2}{a} a w_x^2 \, dx\, dt + \int_{Q_{t_0}^T} s\theta w^2 \, dx\, dt\Big) \\ &\le C\Big(\|L_s^- w\|^2 + \int_{Q_{t_0}^T} s\theta a w_x^2 \, dx\, dt + \int_{Q_{t_0}^T} s\theta w^2 \, dx\, dt\Big),\end{aligned} \tag{3.7}$$

since $1/\sqrt{\theta}$ is bounded and

$$\frac{|x-x_0|^2}{a(x)} \le \max\Big\{\frac{x_0^2}{a(0)}, \frac{(1-x_0)^2}{a(1)}\Big\}$$

(see [20, Lemma 2.1]). Proceeding as in the proof of (3.4), we can estimate $\int_{Q_{t_0}^T} s\theta w^2 \, dx\, dt$ thanks to the Hardy-Poincaré inequality (1.2),

$$\begin{aligned}\int_{Q_{t_0}^T} s\theta w^2 \, dx\, dt &= s\Big|\int_{Q_{t_0}^T} \Big(\theta \frac{a^{1/3}}{|x-x_0|^{2/3}} w^2\Big)^{3/4} \Big(\theta \frac{|x-x_0|^2}{a} w^2\Big)^{1/4} \, dx\, dt\Big| \\ &\le s\frac{3}{2} \int_{Q_{t_0}^T} \theta \frac{a^{1/3}}{|x-x_0|^{2/3}} w^2 \, dx\, dt + \frac{s}{2} \int_{Q_{t_0}^T} \theta \frac{|x-x_0|^2}{a} w^2 \, dx\, dt\end{aligned}$$

$$\le C \int_{Q_{t_0}^T} \Big(s\theta a w_x^2 + s^3\theta^3 \frac{(x-x_0)^2}{a} w^2 \Big) dx\, dt.$$

Hence, taking s large enough, one has

$$\int_{Q_{t_0}^T} \frac{1}{s\theta} w_t^2 \, dx\, dt \le C\Big(\int_{Q_{t_0}^T} h^2 e^{2s\varphi}\, dx\, dt + sc_1 \int_{t_0}^T \Big[\theta a(x-x_0) w_x^2\Big]_{x=0}^{x=1} dt\Big). \tag{3.8}$$

Now from (3.6) and (3.8) we can get the estimate of u_t as follows: from the definition of w, we have $w_t = u_t e^{s\varphi} + s\varphi_t w$. Hence

$$\int_{Q_{t_0}^T} \frac{1}{s\theta} u_t^2 e^{2s\varphi}\, dx\, dt \le 2\Big(\int_{Q_{t_0}^T} \frac{1}{s\theta} w_t^2\, dx\, dt + \int_{Q_{t_0}^T} \frac{s^2\varphi_t^2}{s\theta} w^2\, dx\, dt\Big).$$

The second term in the above right-hand side is estimated as follows:

$$\begin{aligned}\int_{Q_{t_0}^T} \frac{s^2\varphi_t^2}{s\theta} w^2\, dx\, dt &= 16 \int_{Q_{t_0}^T} s\theta^{3/2}\eta^2\psi^2 w^2\, dx\, dt \\ &\le 16(T+t_0)c_1c_2 \int_{Q_{t_0}^T} s\theta^{3/2} |\eta\psi| w^2\, dx\, dt.\end{aligned}$$

Hence using (3.6) and (3.8) we conclude that

$$\int_{Q_{t_0}^T} \frac{1}{s\theta} u_t^2 e^{2s\varphi}\, dx\, dt \le C\Big(\int_{Q_{t_0}^T} h^2 e^{2s\varphi}\, dx\, dt + sc_1 \int_{t_0}^T \Big[\theta a(x-x_0) w_x^2\Big]_{x=0}^{x=1} dt\Big). \tag{3.9}$$

Thus (3.2) follows by (3.3), (3.4) and (3.9). □

3.2. **Carleman estimate for systems.** By the above Carleman estimate (3.2), we are able to show the main result of this section, which is the ω-Carleman estimate for the system (1.1). For $x \in [0,1]$, let us define

$$\varphi_i(t,x) := \theta(t)\psi_i(x), \quad \theta(t) := \frac{1}{[(t-t_0)(T-t)]^4}, \quad \psi_i(x) = c_i\Big[\int_{x_i}^x \frac{y-x_i}{a_i(y)}\, dy - d_i\Big],$$

and, for $x \in [-1,1]$,

$$\Phi_i(t,x) := \theta(t)\Psi_i(x), \quad \Psi_i(x) = e^{2\rho_i} - e^{r_i\zeta_i(x)},$$

where

$$\zeta_i(x) = \int_x^1 \frac{dy}{\sqrt{\tilde a_i(y)}}, \quad \rho_i = r_i\zeta_i(-1), \quad \tilde a_i(x) := \begin{cases} a_i(x), & x \in [0,1], \\ a_i(-x), & x \in [-1,0]. \end{cases}$$

Here the functions a_i, $i = 1,2$, satisfy hypothesis (2.1) or (2.2) and the positive constants c_i, d_i, and r_i are chosen such that

$$d_2 > \frac{16A}{16A-15} \max\Big\{ \frac{x_2^2}{(2-K_2)a_2(0)}, \frac{(1-x_2)^2}{(2-K_2)a_2(1)}, d_2^\star \Big\}, \quad \frac{15}{16} < A < 1 \tag{3.10}$$

$$\begin{gathered} d_1 > \max\Big\{ \frac{x_1^2}{(2-K_1)a_1(0)}, \frac{(1-x_1)^2}{(2-K_1)a_1(1)} \Big\}, \\ \rho_2 > 2\ln(2), \quad e^{2\rho_1} - e^{r_1\zeta_1(0)} \ge e^{2\rho_2} - 1, \end{gathered} \tag{3.11}$$

$$\begin{aligned} &\max\Big\{ \frac{e^{2\rho_2}-1}{d_2 - d_2^\star}, \frac{(2-K_2)a_2(1)(e^{2\rho_2}-1)}{(2-K_2)a_2(1)d_2 - (1-x_2)^2}, \frac{(2-K_2)a_2(0)(e^{2\rho_2}-1)}{(2-K_2)a_2(0)d_2 - x_2^2} \Big\} \\ &\le c_2 < \frac{4A}{3d_2}(e^{2\rho_2} - e^{r_2\zeta_2(0)}) \end{aligned} \tag{3.12}$$

$$\begin{aligned} c_1 \geq \max\Big\{ &\frac{e^{2\rho_1}-1}{d_1-d_1^\star}, \frac{(2-K_1)a_1(1)(e^{2\rho_1}-1)}{(2-K_1)a_1(1)d_1-(1-x_1)^2}, \\ &\frac{(2-K_1)a_1(0)(e^{2\rho_1}-1)}{(2-K_1)a_1(0)d_1-x_1^2}, \frac{c_2d_2}{d_1-d_1^\star}\Big\}, \end{aligned} \tag{3.13}$$

where

$$A = \frac{\min \Psi_2(-x)}{\max \Psi_2(x)}, \quad d_i^\star := \max\Big\{ \int_{x_i}^1 \frac{y-x_i}{a_i(y)}\,dy, \int_{x_i}^0 \frac{y-x_i}{a_i(y)}\,dy \Big\}.$$

Remark 3.2. The interval

$$\begin{aligned} \Big[&\max\Big\{\frac{e^{2\rho_2}-1}{d_2-d_2^\star}, \frac{(2-K_2)a_2(1)(e^{2\rho_2}-1)}{(2-K_2)a_2(1)d_2-(1-x_2)^2}, \frac{(2-K_2)a_2(0)(e^{2\rho_2}-1)}{(2-K_2)a_2(0)d_2-x_2^2}\Big\}, \\ &\frac{4A(e^{2\rho_2}-e^{r_2\zeta(0)})}{3d_2}\Big] \end{aligned}$$

is not empty. In fact, from $\rho_2 > 2\ln 2$, $A > 15/16$ and $d_2 > 16Ad_2^\star/(16A-15)$, we have

$$\begin{aligned} \frac{d_2^\star}{d_2} < 1 - \frac{15}{16A} &\Leftrightarrow \frac{5}{4} \leq \frac{4A}{3}(1-\frac{d_2^\star}{d_2}) \\ &\Leftrightarrow 1+e^{-\rho_2} < \frac{4A}{3}(1-\frac{d_2^\star}{d_2}) \\ &\Leftrightarrow \frac{e^{2\rho_2}-1}{d_2-d_2^\star} < \frac{4A}{3d_2}(e^{2\rho_2}-e^{\rho_2}) < \frac{4A}{3d_2}(e^{2\rho_2}-e^{r_2\zeta_2(0)}). \end{aligned}$$

Similarly for

$$d_2 > \frac{16A}{16A-15}\max\Big\{\frac{x_2^2}{(2-K_2)a_2(0)}, \frac{(1-x_2)^2}{(2-K_2)a_2(1)}\Big\}$$

one has

$$\begin{aligned} &\max\Big\{\frac{(2-K_2)a_2(1)(e^{2\rho_1}-1)}{(2-K_2)a_2(1)d_2-(1-x_2)^2}, \frac{(2-K_2)a_2(0)(e^{2\rho_1}-1)}{(2-K_2)a_2(0)d_2-x_2^2}\Big\} \\ &< \frac{4A}{3d_2}(e^{2\rho_2}-e^{r_2\zeta_2(0)}). \end{aligned}$$

From (3.10)-(3.13), we have the following results.

Lemma 3.3. *(i) For* $(t,x) \in [0,T]\times[0,1]$,

$$\varphi_1 \leq \varphi_2, \quad -\Phi_1 \leq -\Phi_2, \quad \varphi_i \leq -\Phi_i. \tag{3.14}$$

(ii) For $(t,x) \in [0,T]\times[0,1]$,

$$-\Phi_2(t,x) \leq -\Phi_2(t,-x), \quad 4\Phi_2(t,-x)+3\varphi_2(t,x) > 0. \tag{3.15}$$

Proof. (i)

(1) $\varphi_1 \leq \varphi_2$: since $\theta \geq 0$ it is sufficient to prove $\psi_1 \leq \psi_2$. By the choice of c_1, we have $c_1 \geq \frac{c_2d_2}{d_1-d_1^\star}$. Then $\max\{\psi_1(0), \psi_1(1)\} \leq -c_2d_2$. Hence, $\psi_1(x) \leq \psi_2(x)$.

(2) $-\Phi_1 \leq -\Phi_2$:
since Ψ_i is increasing, it is sufficient to prove that $\min\Psi_1(x) \geq \max\Psi_2(x)$. Indeed $\Psi_1(0) = e^{2\rho_1} - e^{r_1\zeta_1(0)} \geq e^{2\rho_2} - 1 = \Psi_2(1)$.

(3) $\varphi_i \le -\Phi_i$: since $c_i \ge \frac{e^{2\rho_i}-1}{d_i-d_i^\star}$, then $\max\{\psi_i(0),\psi_i(1)\} \le -\Psi_i(1)$ and the thesis follows immediately.

(ii)

(1) The first inequality follows from $-\Psi_2(x) \le -\Psi_2(-x)$ for all $x \in [0,1]$.

(2) $4\Psi_2(-x)+3\psi_2(x) > 0$: by definition of A, we have $A\Psi_2(x) \le \Psi_2(-x)$ and, obviously, $4\Psi_2(-x) + 3\psi_2(x) \ge 4A\Psi_2(x) + 3\psi_2(x)$. Thus, to obtain the thesis, it is sufficient to prove that $4A\Psi_2(x)+3\psi_2(x) > 0$. This follows easily observing that, by the assumption $3c_2d_2 < 4A\Psi_2(0)$, $-3\psi_2(x_0) < 4A\Psi_2(0)$. Hence $-3\psi_2(x) \le -3\psi_2(x_0) < 4A\Psi_2(0) \le 4A\Psi_2(x)$ for all $x \in [0,1]$.

□

We show first an intermediate Carleman estimate with distributed observation of u and v.

Theorem 3.4. *Let $T > 0$. There exist two positive constants C and s_0 such that, for every $(u_0, v_0) \in \mathbb{H}$ and all $s \ge s_0$, the solution of (1.1) satisfies*

$$
\begin{aligned}
&\int_{Q_{t_0}^T} \Big(s\theta a_1 u_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{a_1} u^2 + s\theta^{3/2}|\eta\psi_1|u^2 + \frac{1}{s\theta}u_t^2\Big) e^{2s\varphi_1}\,dx\,dt \\
&+ \int_{Q_{t_0}^T} \Big(s\theta a_2 v_x^2 + s^3\theta^3 \frac{(x-x_2)^2}{a_2} v^2 + s\theta^{3/2}|\eta\psi_2|v^2 + \frac{1}{s\theta}v_t^2\Big) e^{2s\varphi_2}\,dx\,dt \\
&\le C\Big(\int_{Q_{t_0}^T} f^2 e^{-2s\Phi_2(t,-x)} dx\,dt + \int_{t_0}^T \int_{\omega'} s^2\theta^2(u^2+v^2) e^{-2s\Phi_2(t,-x)} dx\,dt\Big).
\end{aligned}
$$

For the proof we shall use the following classical Carleman estimate (see [20]).

Proposition 3.5. *Let z be the solution of*

$$
\begin{gathered}
z_t - (az_x)_x = h, \quad x \in (A,B),\ t \in (0,T), \\
z(t,A) = z(t,B) = 0, \quad t \in (0,T),
\end{gathered}
$$

where $a \in C^1([A,B])$ is a strictly positive function. Then there exist two positive constants r and s_0 such that for any $s \ge s_0$,

$$
\int_{t_0}^T \int_A^B s\theta e^{r\zeta} z_x^2 e^{-2s\Phi}\,dx\,dt + \int_{t_0}^T \int_A^B s^3\theta^3 e^{3r\zeta} z^2 e^{-2s\Phi}\,dx\,dt \tag{3.16}
$$

$$
\le c\Big(\int_{t_0}^T \int_A^B h^2 e^{-2s\Phi}\,dx\,dt - c\int_{t_0}^T \big[\sigma(t,\cdot) z_x^2(t,\cdot) e^{-2s\Phi(t,\cdot)}\big]_{x=0}^{x=1} dt\Big) \tag{3.17}
$$

for some positive constant c. Here the functions Φ, σ and ζ are defined, for $r, s > 0$ and $(t,x) \in [0,T] \times [A,B]$, by

$$
\begin{gathered}
\phi(t,x) := \theta(t)\Psi(x), \quad \Psi(x) := e^{2r\zeta(A)} - e^{r\zeta(x)} > 0, \\
\zeta(x) := \int_x^B \frac{1}{\sqrt{a(y)}}\,dy, \quad \sigma(t,x) := rs\theta(t)e^{r\zeta(x)}.
\end{gathered}
$$

Proof of Theorem 3.4. Consider a cut-off function $\xi : [0,1] \to R$ such that

$$
\begin{gathered}
0 \le \xi(x) \le 1, \quad \text{for all } x \in [0,1], \\
\xi(x) = 1, \quad x \in [\lambda_1, \beta_2], \\
\xi(x) = 0, \quad x \in [0,1] \setminus \omega.
\end{gathered}
$$

Define $w := \xi u$ and $z := \xi v$, where (u, v) is the solution of (1.1). Hence, w and z satisfy the system

$$
\begin{gathered}
w_t - (a_1 w_x)_x + b_{11} w = \xi f - b_{12} z - (a_1 \xi_x u)_x - \xi_x a_1 u_x =: g, \quad (t,x) \in Q, \\
z_t - (a_2 z_x)_x + b_{22} z = -(a_2 \xi_x v)_x - \xi_x a_2 v_x, \quad (t,x) \in Q, \\
w(t,0) = w(t,1) = z(t,0) = z(t,1) = 0, \quad t \in (0,T).
\end{gathered}
$$

Applying the estimate (3.2) and using $w = w_x = 0$ in a neighborhood of $x = 0$ and $x = 1$, from the definition of ξ, we have

$$
\begin{aligned}
&\int_{Q_{t_0}^T} \Big(s\theta a_1 w_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{a_1} w^2 + s\theta^{3/2} |\eta \psi_1| w^2 + \frac{1}{s\theta} w_t^2 \Big) e^{2s\varphi_1} \, dx \, dt \\
&\le C \int_{Q_{t_0}^T} g^2 e^{2s\varphi_1} \, dx \, dt
\end{aligned}
$$

for all $s \ge s_0$. Then using the fact that ξ_x and ξ_{xx} are supported in ω'', we can write

$$
g^2 \le C(\xi^2 f^2 + b_{12}^2 z^2 + (u^2 + u_x^2)\chi_{\omega''}).
$$

Hence, applying Cacciopoli inequality (5.1) and the previous estimates, we obtain

$$
\begin{aligned}
&\int_{Q_{t_0}^T} \Big(s\theta a_1 w_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{a_1} w^2 + s\theta^{3/2} |\eta \psi_1| w^2 + \frac{1}{s\theta} w_t^2 \Big) e^{2s\varphi_1} \, dx \, dt \\
&\le C \Big(\int_{Q_{t_0}^T} \xi^2 f^2 e^{2s\varphi_1} \, dx \, dt + \int_{Q_{t_0}^T} b_{12}^2 z^2 e^{2s\varphi_1} \, dx \, dt \\
&\quad + \int_{t_0}^T \int_{\omega'} ((1 + s^2\theta^2) u^2 + f^2) e^{2s\varphi_1} \, dx \, dt \Big).
\end{aligned} \tag{3.18}
$$

Arguing as before and applying the estimate (3.2) to the second component z of the system, we obtain

$$
\begin{aligned}
&\int_{Q_{t_0}^T} \Big(s\theta a_2 z_x^2 + s^3\theta^3 \frac{(x-x_2)^2}{a_2} z^2 + s\theta^{3/2} |\eta \psi_2| z^2 + \frac{1}{s\theta} z_t^2 \Big) e^{2s\varphi_2} \, dx \, dt \\
&\le C \int_{t_0}^T \int_{\omega''} (v^2 + v_x^2) e^{2s\varphi_2} \, dx \, dt.
\end{aligned}
$$

Hence, Cacciopoli inequality (5.1) yields

$$
\begin{aligned}
&\int_{Q_{t_0}^T} \Big(s\theta a_2 z_x^2 + s^3\theta^3 \frac{(x-x_2)^2}{a_2} z^2 + s\theta^{3/2} |\eta \psi_2| z^2 + \frac{1}{s\theta} z_t^2 \Big) e^{2s\varphi_2} \, dx \, dt \\
&\le C \int_{t_0}^T \int_{\omega'} (1 + s^2\theta^2) v^2 e^{2s\varphi_2} \, dx \, dt.
\end{aligned} \tag{3.19}
$$

On the other hand, using the Poincaré inequality and the fact that $\varphi_1 < \varphi_2$, we have

$$
\begin{aligned}
\int_0^1 b_{12}^2 z^2 e^{2s\varphi_1} dx &\le \|b_{12}\|_\infty^2 \int_0^1 (z e^{s\varphi_2})^2 dx \\
&= \|b_{12}\|_\infty^2 \int_0^1 \Big(\frac{|x-x_2|^2}{a_2(x)} z^2 e^{2s\varphi_2} \Big)^{1/4} \Big(\frac{a_2^{1/3}(x)}{|x-x_2|^{2/3}} z^2 e^{2s\varphi_2} \Big)^{3/4} dx
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{\|b_{12}\|_\infty^2}{4}\int_0^1 \frac{|x-x_2|^2}{a_2(x)} z^2 e^{2s\varphi_2}dx\\
&\quad + \frac{3\|b_{12}\|_\infty^2}{4}\int_0^1 \frac{a_2^{1/3}(x)}{|x-x_2|^{2/3}} z^2 e^{2s\varphi_2}dx.
\end{aligned}
$$

Applying the Hardy-Poincaré inequality to $w(t,x) := e^{s\varphi(t,x)}z(t,x)$ and proceeding as in the proof of (3.4), one has

$$
\begin{aligned}
\int_0^1 \frac{a_2^{1/3}(x)}{|x-x_2|^{2/3}} z^2 e^{2s\varphi_2}dx &= \int_0^1 \frac{a_2^{1/3}(x)}{|x-x_2|^{2/3}} w^2 dx \le C\int_0^1 a(w_x)^2 dx\\
&\le C\int_0^1 s^2\theta^2 \frac{|x-x_2|^2}{a_2(x)} z^2 e^{2s\varphi}dx + C\int_0^1 a_2 z_x^2 e^{2s\varphi_2}dx,
\end{aligned}
$$

since $\psi_{2,x}(x) = c_2\frac{x-x_2}{a_2(x)}$. Hence, for a universal positive constant C, it results

$$
\int_{Q_{t_0}^T} b_{12}^2 z^2 e^{2s\varphi_1}dx\,dt \le C\int_{Q_{t_0}^T} (a_2 z_x^2 + s^2\theta^2\frac{(x-x_2)^2}{a_2}z^2)e^{2s\varphi_2}dx\,dt.
$$

Taking s such that $C \le \frac{s\theta}{2}$, we have

$$
\int_{Q_{t_0}^T} b_{12}^2 z^2 e^{2s\varphi_1}dx\,dt \le \frac{1}{2}\int_{Q_{t_0}^T} \Big(s\theta a_2 z_x^2 + s^3\theta^3\frac{(x-x_2)^2}{a_2}z^2\Big)e^{2s\varphi_2}\,dx\,dt. \tag{3.20}
$$

Combining (3.18), (3.19) and (3.20) we obtain for s large enough

$$
\begin{aligned}
&\int_{Q_{t_0}^T} \Big(s\theta a_1 w_x^2 + s^3\theta^3\frac{(x-x_1)^2}{a_1}w^2 + s\theta^{3/2}|\eta\psi_1|w^2 + \frac{1}{s\theta}w_t^2\Big)e^{2s\varphi_1}\,dx\,dt\\
&+ \int_{Q_{t_0}^T} \Big(s\theta a_2 z_x^2 + s^3\theta^3\frac{(x-x_2)^2}{a_2}z^2 + s\theta^{3/2}|\eta\psi_2|z^2 + \frac{1}{s\theta}z_t^2\Big)e^{2s\varphi_2}\,dx\,dt\\
&\le C\Big(\int_{Q_{t_0}^T} \xi^2 f^2 e^{2s\varphi_1}\,dx\,dt + \int_{t_0}^T\int_{\omega'} s^2\theta^2 v^2 e^{2s\varphi_2}\,dx\,dt\\
&\quad + \int_{t_0}^T\int_{\omega'} (s^2\theta^2 u^2 + f^2)e^{2s\varphi_1}\,dx\,dt\Big).
\end{aligned}
$$

By the previous inequality, the definition of w and z, it follows that

$$
\begin{aligned}
&\int_{t_0}^T\int_{\lambda_1}^{\beta_1} \Big(s\theta a_1 u_x^2 + s^3\theta^3\frac{(x-x_1)^2}{a_1}u^2 + s\theta^{3/2}|\eta\psi_1|u^2 + \frac{1}{s\theta}u_t^2\Big)e^{2s\varphi_1}\,dx\,dt\\
&+ \int_{t_0}^T\int_{\lambda_1}^{\beta_1} \Big(s\theta a_2 v_x^2 + s^3\theta^3\frac{(x-x_2)^2}{a_2}v^2 + s\theta^{3/2}|\eta\psi_2|v^2 + \frac{1}{s\theta}v_t^2\Big)e^{2s\varphi_2}\,dx\,dt\\
&= \int_{t_0}^T\int_{\lambda_1}^{\beta_1} \Big(s\theta a_1 w_x^2 + s^3\theta^3\frac{(x-x_1)^2}{a_1}w^2 + s\theta^{3/2}|\eta\psi_1|w^2 + \frac{1}{s\theta}w_t^2\Big)e^{2s\varphi_1}\,dx\,dt\\
&\quad + \int_{t_0}^T\int_{\lambda_1}^{\beta_1} \Big(s\theta a_2 z_x^2 + s^3\theta^3\frac{(x-x_2)^2}{a_2}z^2 + s\theta^{3/2}|\eta\psi_2|z^2 + \frac{1}{s\theta}z_t^2\Big)e^{2s\varphi_2}\,dx\,dt
\end{aligned} \tag{3.21}
$$

$$
\le \int_{Q_{t_0}^T} \Big(s\theta a_1 w_x^2 + s^3\theta^3\frac{(x-x_1)^2}{a_1}w^2 + s\theta^{3/2}|\eta\psi_1|w^2 + \frac{1}{s\theta}w_t^2\Big)e^{2s\varphi_1}\,dx\,dt
$$

$$
\begin{aligned}
&+\int_{Q_{t_0}^T}\Big(s\theta a_2 z_x^2+s^3\theta^3\frac{(x-x_2)^2}{a_2}z^2+s\theta^{3/2}|\eta\psi_2|z^2+\frac{1}{s\theta}z_t^2\Big)e^{2s\varphi_2}\,dx\,dt\\
&\le C\Big(\int_{Q_{t_0}^T}\xi^2 f^2 e^{2s\varphi_1}\,dx\,dt+\int_{t_0}^T\int_{\omega'}s^2\theta^2v^2e^{2s\varphi_2}\,dx\,dt\\
&+\int_{t_0}^T\int_{\omega'}(s^2\theta^2u^2+f^2)e^{2s\varphi_1}\,dx\,dt\Big).
\end{aligned}\tag{3.22}
$$

Now define $U=\chi u$ and $V=\chi v$, where (u,v) is the solution of (1.1) and $\chi:[0,1]\to\mathbb{R}$ is a cut-off function defined as

$$
\begin{gathered}
0\le\chi(x)\le 1,\quad x\in[0,1],\\
\chi(x)=1,\quad x\in[\beta_2,1],\\
\chi(x)=0,\quad x\in[0,\frac{\lambda_2+2\beta_2}{3}].
\end{gathered}
$$

Then U and V satisfy

$$
\begin{gathered}
U_t-(a_1U_x)_x+b_{11}U=\chi f-b_{12}V-(a_1\chi_x u)_x-\chi_x a_1 u_x,\quad (t,x)\in Q\\
V_t-(a_2V_x)_x+b_{22}V=-(a_2\chi_x v)_x-\chi_x a_2 v_x,\quad (t,x)\in Q\\
U(t,0)=U(t,1)=V(t,0)=V(t,1)=0,\quad t\in(0,T).
\end{gathered}
$$

Using (3.16) and a technique similar to the one used in (3.7)-(3.8), one has

$$
\begin{aligned}
&\int_{Q_{t_0}^T}(s\theta e^{r_1\zeta_1}U_x^2+s^3\theta^3e^{3r_1\zeta_1}U^2+\frac{1}{s\theta}U_t^2)e^{-2s\Phi_1}\,dx\,dt\\
&\le C\int_{Q_{t_0}^T}\chi^2f^2e^{-2s\Phi_1}dx\,dt+\tilde C\int_{Q_{t_0}^T}V^2e^{-2s\Phi_1}dx\,dt\\
&\quad+C\int_{t_0}^T\int_{\overline{\overline{\omega}}}(u^2+u_x^2)e^{-2s\Phi_1}\,dx\,dt.
\end{aligned}
$$

Analogously, one can prove that V satisfies

$$
\begin{aligned}
&\int_{Q_{t_0}^T}(s\theta e^{r_2\zeta_2}V_x^2+s^3\theta^3e^{3r_2\zeta_2}V^2+\frac{1}{s\theta}V_t^2)e^{-2s\Phi_2}\,dx\,dt\\
&\le C\int_{t_0}^T\int_{\overline{\overline{\omega}}}(v^2+v_x^2)e^{-2s\Phi_2}dx\,dt.
\end{aligned}
$$

Thus combining the last two inequalities,

$$
\begin{aligned}
&\int_{Q_{t_0}^T}(s\theta e^{r_1\zeta_1}U_x^2+s^3\theta^3e^{3r_1\zeta_1}U^2+\frac{1}{s\theta}U_t^2)e^{-2s\Phi_1}\,dx\,dt\\
&+\int_{Q_{t_0}^T}(s\theta e^{r_2\zeta_2}V_x^2+s^3\theta^3e^{3r_2\zeta_2}V^2+\frac{1}{s\theta}V_t^2)e^{-2s\Phi_2}\,dx\,dt\\
&\le C\int_{Q_{t_0}^T}\chi^2f^2e^{-2s\Phi_1}dx\,dt+\tilde C\int_{Q_{t_0}^T}V^2e^{-2s\Phi_1}dx\,dt\\
&\quad+C\int_{t_0}^T\int_{\overline{\overline{\omega}}}(u^2+u_x^2)e^{-2s\Phi_1}dx\,dt+C\int_{t_0}^T\int_{\overline{\overline{\omega}}}(v^2+v_x^2)e^{-2s\Phi_2}dx\,dt.
\end{aligned}
$$

Taking s such that $\tilde{C} \le \frac{1}{2}s^3\theta^3 e^{3r_2\zeta_2}$, using $-\Phi_1 \le -\Phi_2$ and Cacciopoli inequality (5.1), we obtain

$$\begin{aligned}
&\int_{Q_{t_0}^T} (s\theta e^{r_1\zeta_1} U_x^2 + s^3\theta^3 e^{3r_1\zeta_1} U^2 + \frac{1}{s\theta} U_t^2) e^{-2s\Phi_1}\,dx\,dt \\
&+ \int_{Q_{t_0}^T} (s\theta e^{r_2\zeta_2} V_x^2 + \frac{1}{2} s^3\theta^3 e^{3r_2\zeta_2} V^2 + \frac{1}{s\theta} V_t^2) e^{-2s\Phi_2}\,dx\,dt \\
&\le C \int_{Q_{t_0}^T} \chi^2 f^2 e^{-2s\Phi_1} dx\,dt + C\int_{t_0}^T \int_{\overline{\omega}} (s^2\theta^2 u^2 + f) e^{-2s\Phi_1} dx\,dt \\
&\quad + C\int_{t_0}^T \int_{\overline{\omega}} s^2\theta^2 v^2 e^{-2s\Phi_2} dx\,dt.
\end{aligned}$$

Then, by Lemma 3.3, one can prove that there exists a positive constant C such that for every $(t,x) \in [0,T] \times \operatorname{supp}(\chi)$,

$$a_i(x) e^{2s\varphi_i(t,x)} \le C e^{r_i\zeta_i} e^{-2s\Phi_i}, \quad \frac{(x-x_i)^2}{a_i(x)} e^{2s\varphi_i(t,x)} \le \frac{C}{2} e^{3r_i\zeta_i} e^{-2s\Phi_i}. \tag{3.23}$$

Consequently,

$$\begin{aligned}
&\int_{Q_{t_0}^T} \Big(s\theta a_1 U_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{a_1} U^2 + s\theta^{3/2}|\eta\psi_1| U^2 + \frac{1}{s\theta} U_t^2\Big) e^{2s\varphi_1}\,dx\,dt \\
&+ \int_{Q_{t_0}^T} \Big(s\theta a_2 V_x^2 + s^3\theta^3 \frac{(x-x_2)^2}{a_2} V^2 + s\theta^{3/2}|\eta\psi_2| V^2 + \frac{1}{s\theta} V_t^2\Big) e^{2s\varphi_2}\,dx\,dt \\
&\le C\Big(\int_{Q_{t_0}^T} \chi^2 f^2 e^{-2s\Phi_1} dx\,dt + \int_{t_0}^T \int_{\overline{\omega}} (s^2\theta^2 u^2 + f) e^{-2s\Phi_1} dx\,dt \\
&\quad + \int_{t_0}^T \int_{\overline{\omega}} s^2\theta^2 v^2 e^{-2s\Phi_2} dx\,dt\Big).
\end{aligned}$$

By the definitions of U and V, we obtain

$$\begin{aligned}
&\int_{t_0}^T \int_{\beta_2}^1 \Big(s\theta a_1 u_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{a_1} u^2 + s\theta^{3/2}|\eta\psi_1| u^2 + \frac{1}{s\theta} u_t^2\Big) e^{2s\varphi_1}\,dx\,dt \\
&+ \int_{t_0}^T \int_{\beta_2}^1 \Big(s\theta a_2 v_x^2 + s^3\theta^3 \frac{(x-x_2)^2}{a_2} v^2 + s\theta^{3/2}|\eta\psi_2| v^2 + \frac{1}{s\theta} v_t^2\Big) e^{2s\varphi_2}\,dx\,dt \\
&= \int_{t_0}^T \int_{\beta_2}^1 \Big(s\theta a_1 U_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{a_1} U^2 + s\theta^{3/2}|\eta\psi_1| U^2 + \frac{1}{s\theta} U_t^2\Big) e^{2s\varphi_1}\,dx\,dt \\
&\quad + \int_{t_0}^T \int_{\beta_2}^1 \Big(s\theta a_2 V_x^2 + s^3\theta^3 \frac{(x-x_2)^2}{a_2} V^2 + s\theta^{3/2}|\eta\psi_2| V^2 + \frac{1}{s\theta} V_t^2\Big) e^{2s\varphi_2}\,dx\,dt \\
&\le \int_{Q_{t_0}^T} \Big(s\theta a_1 U_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{a_1} U^2 + s\theta^{3/2}|\eta\psi_1| U^2 + \frac{1}{s\theta} U_t^2\Big) e^{2s\varphi_1}\,dx\,dt \\
&\quad + \int_{Q_{t_0}^T} \Big(s\theta a_2 V_x^2 + s^3\theta^3 \frac{(x-x_2)^2}{a_2} V^2 + s\theta^{3/2}|\eta\psi_2| V^2 + \frac{1}{s\theta} V_t^2\Big) e^{2s\varphi_2}\,dx\,dt \\
&\le C\Big(\int_{Q_{t_0}^T} \chi^2 f^2 e^{-2s\Phi_1} dx\,dt + \int_{t_0}^T \int_{\overline{\omega}} (s^2\theta^2 u^2 + f) e^{-2s\Phi_1} dx\,dt
\end{aligned}$$

$$+\int_{t_0}^T \int_{\overline{\omega}} s^2\theta^2 v^2 e^{-2s\Phi_2} dx\,dt\Big). \tag{3.24}$$

To complete the proof it is sufficient to prove a similar inequality on the interval $[0,\lambda_1]$. To this aim define the functions

$$W(t,x) := \begin{cases} u(t,x), & x\in[0,1],\\ -u(t,-x), & x\in[-1,0], \end{cases} \quad Z(t,x) := \begin{cases} v(t,x), & x\in[0,1],\\ -v(t,-x), & x\in[-1,0], \end{cases}$$

where (u,v) solves (1.1), and

$$\tilde f(t,x) := \begin{cases} f(t,x), & x\in[0,1],\\ -f(t,-x), & x\in[-1,0], \end{cases} \quad \tilde b_{ij}(x) := \begin{cases} b_{ij}(x), & x\in[0,1],\\ b_{ij}(-x), & x\in[-1,0]. \end{cases}$$

Therefore, (W,Z) solves

$$\begin{gathered} W_t - (\tilde a_1 W_x)_x + \tilde b_{11} W = \tilde f - \tilde b_{12} Z, \quad x\in(-1,1),\ t\in(0,T),\\ Z_t - (\tilde a_2 Z_x)_x + \tilde b_{22} Z = 0, \quad x\in(-1,1),\ t\in(0,T),\\ W(t,-1) = W(t,1) = Z(t,-1) = Z(t,1) = 0, \quad t\in(0,T). \end{gathered} \tag{3.25}$$

Consider a cut-off function $\rho:[-1,1]\to\mathbb{R}$ such that

$$\begin{gathered} 0\le \rho(x)\le 1, \quad x\in[-1,1],\\ \rho(x) = 1, \quad x\in[-\lambda_1,\lambda_1],\\ \rho(x) = 0, \quad x\in[-1,-\frac{\lambda_1+2\beta_1}{3}]\cup[\frac{\lambda_1+2\beta_1}{3},1]. \end{gathered}$$

The functions $p=\rho W$ and $q=\rho Z$, where (W,Z) is the solution of (3.25), satisfy

$$\begin{gathered} p_t - (\tilde a_1 p_x)_x + \tilde b_{11} p = \rho\tilde f - \tilde b_{12} q - (\tilde a_1\rho_x W)_x - \rho_x \tilde a_1 W_x, \quad x\in(-1,1),\ t\in(0,T),\\ q_t - (\tilde a_2 q_x)_x + \tilde b_{22} q = -(\tilde a_2\rho_x Z)_x - \rho_x\tilde a_2 Z_x, \quad x\in(-1,1),\ t\in(0,T),\\ p(t,-1) = p(t,1) = q(t,-1) = q(t,1) = 0, \quad t\in(0,T). \end{gathered}$$

Applying (3.16) for the first component p with $A=-\beta_1$, $B=\beta_1$ and proceeding as in (3.7)-(3.8), we obtain

$$\begin{aligned} &\int_{t_0}^T\int_{-\beta_1}^{\beta_1} (s\theta e^{r_1\zeta_1} p_x^2 + s^3\theta^3 e^{3r_1\zeta_1} p^2 + \frac{1}{s\theta} p_t^2) e^{-2s\Phi_1}\,dx\,dt\\ &\le C\int_{t_0}^T\int_{-\beta_1}^{\beta_1} \rho^2 \tilde f^2 e^{-2s\Phi_1} dx\,dt + \tilde C\int_{t_0}^T\int_{-\beta_1}^{\beta_1} q^2 e^{-2s\Phi_1} dx\,dt\\ &\quad + C\int_{t_0}^T\int_{\lambda_1}^{\frac{\lambda_1+2\beta_1}{3}} (W^2+W_x^2) e^{-2s\Phi_1} dx\,dt\\ &\quad + C\int_{t_0}^T\int_{-\frac{\lambda_1+2\beta_1}{3}}^{-\lambda_1} (W^2+W_x^2) e^{-2s\Phi_1} dx\,dt. \end{aligned}$$

Using the definition of W, $-\Phi_1\le-\Phi_2$ and $e^{-2s\Phi_2(t,x)}\le e^{-2s\Phi_2(t,-x)}$, a.e. $x\in[0,\beta_1]$, it follows that

$$\int_{t_0}^T\int_{-\beta_1}^{\beta_1} (s\theta e^{r_1\zeta_1} p_x^2 + s^3\theta^3 e^{3r_1\zeta_1} p^2 + \frac{1}{s\theta} p_t^2) e^{-2s\Phi_1}\,dx\,dt$$

$$\begin{aligned}
&\le C\int_{t_0}^{T}\int_{0}^{\beta_1}\rho^2 f^2 e^{-2s\Phi_2(t,-x)}dx\,dt+\tilde{C}\int_{t_0}^{T}\int_{-\beta_1}^{\beta_1}q^2e^{-2s\Phi_1}dx\,dt\\
&\quad+C\int_{t_0}^{T}\int_{\lambda_1}^{\frac{\lambda_1+2\beta_1}{3}}(u^2+u_x^2)e^{-2s\Phi_1(t,-x)}dx\,dt.
\end{aligned}$$

Similarly, for the second component q, we obtain

$$\begin{aligned}
&\int_{t_0}^{T}\int_{-\beta_1}^{\beta_1}(s\theta e^{r_2\zeta_2}q_x^2+s^3\theta^3e^{3r_2\zeta_2}q^2+\frac{1}{s\theta}q_t^2)e^{-2s\Phi_2}\,dx\,dt\\
&\le C\int_{t_0}^{T}\int_{\lambda_1}^{\frac{\lambda_1+2\beta_1}{3}}(v^2+v_x^2)e^{-2s\Phi_2(t,-x)}dx\,dt.
\end{aligned}$$

Thus it follows from the last two inequalities that

$$\begin{aligned}
&\int_{t_0}^{T}\int_{-\beta_1}^{\beta_1}(s\theta e^{r_1\zeta_1}p_x^2+s^3\theta^3e^{3r_1\zeta_1}p^2+\frac{1}{s\theta}p_t^2)e^{-2s\Phi_1}\,dx\,dt\\
&+\int_{t_0}^{T}\int_{-\beta_1}^{\beta_1}(s\theta e^{r_2\zeta_2}q_x^2+s^3\theta^3e^{3r_2\zeta_2}q^2+\frac{1}{s\theta}q_t^2)e^{-2s\Phi_2}\,dx\,dt\\
&\le C\int_{t_0}^{T}\int_{0}^{\beta_1}\rho^2 f^2 e^{-2s\Phi_2(t,-x)}dx\,dt+\tilde{C}\int_{t_0}^{T}\int_{-\beta_1}^{\beta_1}q^2e^{-2s\Phi_1}dx\,dt\\
&\quad+C\int_{t_0}^{T}\int_{\lambda_1}^{\frac{\lambda_1+2\beta_1}{3}}(v^2+v_x^2)e^{-2s\Phi_2(t,-x)}dx\,dt\\
&\quad+C\int_{t_0}^{T}\int_{\lambda_1}^{\frac{\lambda_1+2\beta_1}{3}}(u^2+u_x^2)e^{-2s\Phi_1(t,-x)}dx\,dt.
\end{aligned}$$

Taking s such that $\tilde{C}\le\frac{1}{2}s^3\theta^3e^{3r_2\zeta_2}$, using $-\Phi_1\le-\Phi_2$ and Cacciopoli inequality (5.1), we obtain

$$\begin{aligned}
&\int_{t_0}^{T}\int_{-\beta_1}^{\beta_1}(s\theta e^{r_1\zeta_1}p_x^2+s^3\theta^3e^{3r_1\zeta_1}p^2+\frac{1}{s\theta}p_t^2)e^{-2s\Phi_1}\,dx\,dt\\
&+\int_{t_0}^{T}\int_{-\beta_1}^{\beta_1}(s\theta e^{r_2\zeta_2}q_x^2+\frac{1}{2}s^3\theta^3e^{3r_2\zeta_2}q^2+\frac{1}{s\theta}q_t^2)e^{-2s\Phi_2}\,dx\,dt\\
&\le C\int_{t_0}^{T}\int_{0}^{\beta_1}\rho^2 f^2 e^{-2s\Phi_2(t,-x)}dx\,dt+C\int_{t_0}^{T}\int_{\tilde{\omega}}(s^2\theta^2u^2+f)e^{-2s\Phi_1(t,-x)}dx\,dt\\
&\quad+C\int_{t_0}^{T}\int_{\tilde{\omega}}s^2\theta^2v^2e^{-2s\Phi_2(t,-x)}dx\,dt.
\end{aligned}\tag{3.26}$$

Clearly, one can obtain the same estimates (3.23) in $[0,T]\times[0,\beta_1]$. Hence, by (3.14), (3.26), by the definitions of W, Z, p and q, and proceeding as in (3.5), we obtain

$$\begin{aligned}
&\int_{t_0}^{T}\int_{0}^{\lambda_1}\Big(s\theta a_1u_x^2+s^3\theta^3\frac{(x-x_1)^2}{a_1}u^2+s\theta^{3/2}|\eta\psi_1|u^2+\frac{1}{s\theta}u_t^2\Big)e^{2s\varphi_1}\,dx\,dt\\
&+\int_{t_0}^{T}\int_{0}^{\lambda_1}\Big(s\theta a_2v_x^2+s^3\theta^3\frac{(x-x_2)^2}{a_2}v^2+s\theta^{3/2}|\eta\psi_2|v^2+\frac{1}{s\theta}v_t^2\Big)e^{2s\varphi_2}\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_{t_0}^T \int_0^{\lambda_1} \Big(s\theta a_1 W_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{a_1} W^2 + s\theta^{3/2}|\eta\psi_1| W^2 + \frac{1}{s\theta} W_t^2 \Big) e^{2s\varphi_1}\, dx\, dt \\
&\quad + \int_{t_0}^T \int_0^{\lambda_1} \Big(s\theta a_2 Z_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{a_2} Z^2 + s\theta^{3/2}|\eta\psi_2| Z^2 + \frac{1}{s\theta} Z_t^2 \Big) e^{2s\varphi_2}\, dx\, dt \\
&= \int_{t_0}^T \int_0^{\lambda_1} \Big(s\theta a_1 p_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{a_1} p^2 + s\theta^{3/2}|\eta\psi_1| p^2 + \frac{1}{s\theta} p_t^2 \Big) e^{2s\varphi_1}\, dx\, dt \\
&\quad + \int_{t_0}^T \int_0^{\lambda_1} \Big(s\theta a_2 q_x^2 + s^3\theta^3 \frac{(x-x_2)^2}{a_2} q^2 + s\theta^{3/2}|\eta\psi_2| q^2 + \frac{1}{s\theta} q_t^2 \Big) e^{2s\varphi_2}\, dx\, dt \\
&\le C \int_{t_0}^T \int_{-\beta_1}^{\beta_1} (s\theta e^{r_1\zeta_1} p_x^2 + s^3\theta^3 e^{3r_1\zeta_1} p^2 + \frac{1}{s\theta} p_t^2) e^{-2s\Phi_1}\, dx\, dt \\
&\quad + C \int_{t_0}^T \int_{-\beta_1}^{\beta_1} (s\theta e^{r_2\zeta_2} q_x^2 + s^3\theta^3 e^{3r_2\zeta_2} q^2 + \frac{1}{s\theta} q_t^2) e^{-2s\Phi_2}\, dx\, dt \\
&\le C \int_{t_0}^T \int_0^{\beta_1} \rho^2 f^2 e^{-2s\Phi_2(t,-x)} dx\, dt + C \int_{t_0}^T \int_{\tilde\omega} (s^2\theta^2 u^2 + f) e^{-2s\Phi_1(t,-x)} dx\, dt \\
&\quad + C \int_{t_0}^T \int_{\tilde\omega} s^2\theta^2 v^2 e^{-2s\Phi_2(t,-x)} dx\, dt.
\end{aligned} \tag{3.27}
$$

Finally adding up (3.22), (3.24) and (3.27), the proof is complete. □

To estimate the term source with only the first component of solutions of (1.1), we need to show a Carleman estimate with one force.

Theorem 3.6. *Let $T > 0$. Moreover, assume that*

$$b_{12} \geqslant \mu > 0 \quad \text{on } \omega'.$$

There exist two positive constants C and s_0 such that, for every $(u_0, v_0) \in \mathbb{H}$ and for all $s \ge s_0$, the solution (u, v) of (1.1) satisfies

$$
\begin{aligned}
I(u,v) &:= \int_{Q_{t_0}^T} \Big(s\theta a_1 u_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{a_1} u^2 + s\theta^{3/2}|\eta\psi_1| u^2 + \frac{1}{s\theta} u_t^2 \Big) e^{2s\varphi_1}\, dx\, dt \\
&\quad + \int_{Q_{t_0}^T} \Big(s\theta a_2 v_x^2 + s^3\theta^3 \frac{(x-x_2)^2}{a_2} v^2 + s\theta^{3/2}|\eta\psi_2| v^2 + \frac{1}{s\theta} v_t^2 \Big) e^{2s\varphi_2}\, dx\, dt \\
&\le C \int_{Q_{t_0}^T} f^2 e^{-2s\Phi_2(t,-x)} dx\, dt + C \int_{t_0}^T \int_\omega u^2 dx\, dt := I(f,u).
\end{aligned} \tag{3.28}
$$

The above theorem is a consequence of Theorem 3.4 and of the following lemma.

Lemma 3.7. *Let $\omega_2 \Subset \omega_1$ such that $x_1,\, x_2 \notin \omega_1$. Moreover, assume that*

$$b_{12} \geqslant \mu > 0 \quad \text{on } \omega_1.$$

There exists $C > 0$ such that

$$\int_{t_0}^T \int_{\omega_2} s^2\theta^2 v^2 e^{-2s\Phi_2(t,-x)} dx\, dt \le \varepsilon J(v) + C \int_{Q_{t_0}^T} f^2 e^{2s\varphi_2}\, dx\, dt + C \int_{t_0}^T \int_\omega u^2\, dx\, dt,$$

where $\varepsilon > 0$ is small enough, s is large enough and

$$J(v) = \int_{Q_{t_0}^T} \Big(s\theta a_2 v_x^2 + s^3\theta^3 \frac{(x-x_2)^2}{a_2} v^2 \Big) e^{2s\varphi_2}\, dx\, dt.$$

Proof. The choice of the weight functions given in Lemma 3.3 will play a crucial role. We will adapt the technique used in [2]. Let $\chi \in C^\infty(0,1)$, such that $\mathrm{supp}(\chi) \subset \omega_1$ and $\chi \equiv 1$ on ω_2. Multiplying the first equation of (1.1) by $s^2\theta^2\chi e^{-2s\theta(t)\Psi_2(-x)} v$ and integrating over $Q_{t_0}^T$, we obtain

$$\begin{aligned}
&\int_{Q_{t_0}^T} s^2\theta^2 b_{12}\chi e^{-2s\theta(t)\Psi_2(-x)} v^2\, dx\, dt \\
&= \int_{Q_{t_0}^T} s^2\theta^2 \chi e^{-2s\theta(t)\Psi_2(-x)} f v\, dx\, dt - \int_{Q_{t_0}^T} s^2\theta^2 \chi e^{-2s\theta(t)\Psi_2(-x)} u_t v\, dx\, dt \\
&\quad + \int_{Q_{t_0}^T} s^2\theta^2 \chi e^{-2s\theta(t)\Psi_2(-x)} (a_1 u_x)_x v\, dx\, dt \\
&\quad - \int_{Q_{t_0}^T} s^2\theta^2 b_{11}\chi e^{-2s\theta(t)\Psi_2(-x)} u v\, dx\, dt.
\end{aligned} \tag{3.29}$$

Integrating by parts and using the second equation in (1.1), we obtain

$$\begin{aligned}
&\int_{Q_{t_0}^T} s^2\theta^2 \chi e^{-2s\theta(t)\Psi_2(-x)} v u_t\, dx\, dt \\
&= \int_{Q_{t_0}^T} s^2\theta^2 a_2 \chi e^{-2s\theta(t)\Psi_2(-x)} u_x v_x\, dx\, dt + \int_{Q_{t_0}^T} s^2\theta^2 a_2 (\chi e^{-2s\theta(t)\Psi_2(-x)})_x u v_x\, dx\, dt \\
&\quad + \int_{Q_{t_0}^T} (s^2\theta^2 b_{22} + 2s^3\theta^2\dot\theta\Psi_2(-x) - 2s^2\theta\dot\theta)\chi e^{-2s\theta(t)\Psi_2(-x)} v u\, dx\, dt,
\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}
&\int_{Q_{t_0}^T} s^2\theta^2 \chi e^{-2s\theta(t)\Psi_2(-x)} (a_1 u_x)_x v\, dx\, dt \\
&= -\int_{Q_{t_0}^T} s^2\theta^2 a_1 \chi e^{-2s\theta(t)\Psi_2(-x)} v_x u_x\, dx\, dt \\
&\quad + \int_{Q_{t_0}^T} s^2\theta^2 a_1 (\chi e^{-2s\theta(t)\Psi_2(-x)})_x v_x u\, dx\, dt \\
&\quad + \int_{Q_{t_0}^T} s^2\theta^2 (a_1 (\chi e^{-2s\theta(t)\Psi_2(-x)})_x)_x v u\, dx\, dt.
\end{aligned} \tag{3.31}$$

Combining (3.29)-(3.31), we obtain

$$\int_{Q_{t_0}^T} s^2\theta^2 b_{12}\chi e^{-2s\theta(t)\Psi_2(-x)} v^2\, dx\, dt = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \int_{Q_{t_0}^T} s^2\theta^2 \chi e^{-2s\theta(t)\Psi_2(-x)} v f\, dx\, dt,$$

$$I_2 = -\int_{Q_{t_0}^T} s^2\theta^2 (a_1 + a_2)\chi e^{-2s\theta(t)\Psi_2(-x)} u_x v_x\, dx\, dt,$$

$$\begin{aligned} I_3 &= \int_{Q_{t_0}^T} s^2\theta^2(a_1-a_2)(\chi e^{-2s\theta(t)\Psi_2(-x)})_x uv_x\,dx\,dt \\ &= \int_{Q_{t_0}^T} (s^2\theta^2\chi' + 2s^3\theta^3\Psi_{2,x}(-x)\chi)(a_1-a_2)e^{-2s\theta(t)\Psi_2(-x)}uv_x\,dx\,dt, \end{aligned}$$

$$\begin{aligned} I_4 &= \int_{Q_{t_0}^T} (2s^2\theta\dot\theta - s^2\theta^2(b_{11}+b_{22}) - 2s^3\theta^2\dot\theta\Psi_2(-x))\chi e^{-2s\theta(t)\Psi_2(-x)}uv\,dx\,dt \\ &\quad + \int_{Q_{t_0}^T} s^2\theta^2(a_1(\chi e^{-2s\theta(t)\Psi_2(-x)})_x)_x uv\,dx\,dt. \end{aligned}$$

For $\varepsilon > 0$, we have

$$\begin{aligned} |I_1| &= \int_{Q_{t_0}^T} (fe^{s\varphi_2})(s^2\theta^2\chi e^{-s(2\theta(t)\Psi_2(-x)+\varphi_2)}v)\,dx\,dt \\ &\le \frac12\int_{Q_{t_0}^T} f^2e^{2s\varphi_2}\,dx\,dt + C\int_{t_0}^T\int_{\omega_1} s^4\theta^4e^{-2s(2\theta(t)\Psi_2(-x)+\varphi_2)}v^2\,dx\,dt, \end{aligned}$$

$$\begin{aligned} |I_2| &= \int_{Q_{t_0}^T} (\sqrt{s\theta a_2}e^{s\varphi_2}v_x)((s\theta)^{3/2}(a_2)^{-\frac12}(a_1+a_2)\chi e^{-s(2\theta(t)\Psi_2(-x)+\varphi_2)}u_x)\,dx\,dt \\ &\le \varepsilon\int_{Q_{t_0}^T} s\theta a_2e^{2s\varphi_2}v_x^2\,dx\,dt \\ &\quad + \frac1\varepsilon\underbrace{\int_{Q_{t_0}^T} s^3\theta^3\frac{(a_1^2+a_2^2)}{a_2}\chi^2e^{-2s(2\theta(t)\Psi_2(-x)+\varphi_2)}u_x^2\,dx\,dt}_{L}. \end{aligned}$$

The integral L should be estimated by an integral in u^2. For this, we multiply the first equation in (1.1) by $s^3\theta^3\frac{(a_1^2+a_2^2)}{a_1a_2}\chi^2e^{-2s(2\theta(t)\Psi_2(-x)+\varphi_2)}u$ and we integrate by parts, obtaining

$$L = L_1 + L_2 + L_3 + L_4 + L_5,$$

where

$$\begin{aligned} L_1 &= \int_{Q_{t_0}^T} s^3\theta^3\frac{(a_1^2+a_2^2)}{a_1a_2}\chi^2e^{-2s(2\theta(t)\Psi_2(-x)+\varphi_2)}uf\,dx\,dt, \\ L_2 &= \frac12\int_{Q_{t_0}^T} s^3(3\theta^2 - 2s\theta^3(2\Psi_2(-x)+\psi_2))\dot\theta\frac{(a_1^2+a_2^2)}{a_1a_2}\chi^2 \\ &\quad\times e^{-2s(2\theta(t)\Psi_2(-x)+\varphi_2)}u^2\,dx\,dt, \\ L_3 &= \frac12\int_{Q_{t_0}^T} s^3\theta^3(a_1(\frac{(a_1^2+a_2^2)}{a_1a_2}\chi^2e^{-2s(2\theta(t)\Psi_2(-x)+\varphi_2)})_x)_x u^2\,dx\,dt, \\ L_4 &= -\int_{Q_{t_0}^T} s^3\theta^3\frac{(a_1^2+a_2^2)}{a_1a_2}\chi^2b_{11}e^{-2s(2\theta(t)\Psi_2(-x)+\varphi_2)}u^2\,dx\,dt, \\ L_5 &= -\int_{Q_{t_0}^T} s^3\theta^3\frac{(a_1^2+a_2^2)}{a_1a_2}\chi^2b_{12}e^{-2s(2\theta(t)\Psi_2(-x)+\varphi_2)}uv\,dx\,dt. \end{aligned}$$

Since $|\dot\theta| \le C\theta^2$ and $\operatorname{supp}(\chi) \subset \omega_1$, we obtain

$$
\begin{gathered}
|L_1| \le \frac{1}{2}\int_{Q_{t_0}^T} f^2 e^{2s\varphi_2}\,dx\,dt + C\int_{t_0}^T\int_{\omega_1} s^6\theta^6 e^{-2s(4\theta(t)\Psi_2(-x)+3\varphi_2)}u^2\,dx\,dt,\\
|L_2| \le C\int_{t_0}^T\int_{\omega_1} s^4\theta^5 e^{-2s(2\theta(t)\Psi_2(-x)+\varphi_2)}u^2\,dx\,dt,\\
|L_3| \le C\int_{t_0}^T\int_{\omega_1} s^5\theta^5 e^{-2s(2\theta(t)\Psi_2(-x)+\varphi_2)}u^2\,dx\,dt,\\
|L_4| \le C\int_{t_0}^T\int_{\omega_1} s^3\theta^3 e^{-2s(2\theta(t)\Psi_2(-x)+\varphi_2)}u^2\,dx\,dt,\\
\begin{aligned}
|L_5| &\le \varepsilon\int_{Q_{t_0}^T} s^3\theta^3\frac{(x-x_2)^2}{a_2}e^{2s\varphi_2}v^2\,dx\,dt\\
&\quad + \frac{C}{\varepsilon}\int_{t_0}^T\int_{\omega_1} s^3\theta^3 e^{-2s(4\theta(t)\Psi_2(-x)+3\varphi_2)}u^2\,dx\,dt.
\end{aligned}
\end{gathered}
$$

Hence,

$$
\begin{aligned}
|L| &\le C\int_{Q_{t_0}^T} f^2 e^{2s\varphi_2}\,dx\,dt + C_\varepsilon\int_{t_0}^T\int_{\omega_1} s^6\theta^3 e^{-2s(4\theta(t)\Psi_2(-x)+3\varphi_2)}u^2\,dx\,dt\\
&\quad + \varepsilon\int_{Q_{t_0}^T} s^3\theta^3\frac{(x-x_2)^2}{a_2}e^{2s\varphi_2}v^2\,dx\,dt.
\end{aligned}
$$

Furthermore,

$$
|I_2| \le C\int_{Q_{t_0}^T} f^2 e^{2s\varphi_2}\,dx\,dt + C_\varepsilon\int_{t_0}^T\int_{\omega_1} s^6\theta^6 e^{-2s(4\theta(t)\Psi_2(-x)+3\varphi_2)}u^2\,dx\,dt + \varepsilon J(v).
$$

Using the fact that χ' and χ are supported in ω' and $x_2 \notin \omega'$, proceeding as before, one has

$$
\begin{aligned}
|I_3| &\le C\int_{Q_{t_0}^T} s^3\theta^3(\chi'+\chi)e^{-2s\theta(t)\Psi_2(-x)}uv_x\,dx\,dt\\
&= C\int_{Q_{t_0}^T} (\sqrt{s\theta a_2}e^{s\varphi_2}v_x)((s\theta)^{5/2}(a_2)^{-\frac12}(\chi'+\chi)e^{-s(2\theta(t)\Psi_2(-x)+\varphi_2)}u)\,dx\,dt\\
&\le \varepsilon\int_{Q_{t_0}^T} s\theta a_2 v_x^2 e^{2s\varphi_2}\,dx\,dt + \frac{C}{\varepsilon}\int_{t_0}^T\int_{\omega_1} (s^5\theta^5 e^{-2s(2\theta(t)\Psi_2(-x)+\varphi_2)}u^2\,dx\,dt,
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &\le C\int_{Q_{t_0}^T} s^4\theta^4(\chi''+\chi'+\chi)e^{-2s\theta(t)\Psi_2(-x)}uv\,dx\,dt\\
&= C\int_{Q_{t_0}^T} (s^{3/2}\theta^{3/2}\frac{x-x_2}{\sqrt{a_2}}e^{s\varphi_2}v)((s\theta)^{5/2}\frac{\sqrt{a_2}}{x-x_2}(\chi''+\chi'+\chi)\\
&\quad \times e^{-s(2\theta(t)\Psi_2(-x)+\varphi_2)}u)\,dx\,dt\\
&\le \varepsilon\int_{Q_{t_0}^T} s^3\theta^3\frac{(x-x_2)^2}{a_2}v^2 e^{2s\varphi_2}\,dx\,dt
\end{aligned}
$$

$$+\frac{C}{\varepsilon}\int_{t_0}^{T}\int_{\omega_1}(s^5\theta^5 e^{-2s(2\theta(t)\Psi_2(-x)+\varphi_2)}u^2\,dx\,dt,$$

So, thanks to Lemma 3.3, we have

$$e^{-2s(2\theta(t)\Psi_2(-x)+\varphi_2)} \le e^{-2s(4\theta(t)\Psi_2(-x)+3\varphi_2)},$$
$$\sup_{(t,x)\in Q} s^r\theta^r(t)e^{-2s(4\theta(t)\Psi_2(-x)+3\varphi_2)} < \infty, \quad r\in\mathbb{R}.$$

Then, for ε small enough and s large enough, we have

$$\begin{aligned}&|\int_{Q_{t_0}^T} s^2\theta^2 b_{12}\chi e^{-2s\theta(t)\Psi_2(-x)}v^2\,dx\,dt|\\ &\le C\int_{Q_{t_0}^T} f^2 e^{2s\varphi_2}\,dx\,dt + C\int_{t_0}^T\int_\omega u^2\,dx\,dt + \varepsilon J(v).\end{aligned}$$

Finally, the definition of χ, (2.1) and the previous inequality give

$$\begin{aligned}&\mu\int_{t_0}^T\int_{\omega_2} s^2\theta^2 e^{-2s\theta(t)\Psi_2(-x)}v^2\,dx\,dt\\ &\le |\int_{t_0}^T\int_{\omega_2} s^2\theta^2 b_{12} e^{-2s\theta(t)\Psi_2(-x)}v^2\,dx\,dt|\\ &\le \int_{Q_{t_0}^T} |s^2\theta^2 b_{12}\chi e^{-2s\theta(t)\Psi_2(-x)}v^2|\,dx\,dt\\ &\le C\int_{Q_{t_0}^T} f^2 e^{2s\varphi_2}\,dx\,dt + C\int_{t_0}^T\int_\omega u^2\,dx\,dt + \varepsilon J(v).\end{aligned}$$

This completes the proof. □

4. The Lipschitz stability result

The object of this section is to recover a source term f from the knowledge of $(a_1u_x)_x(T',\cdot)$ and some additional observations of the component u. The main result of this paper reads as follows.

Theorem 4.1. *Let $C_0>0$. Then, there exists $C=C(T,t_0,x_1,x_2,C_0)>0$ such that, for all $f\in S(C_0)$ and $u_0\in L^2(0,1)$,*

$$\begin{aligned}\|f\|^2_{L^2(Q_{t_0}^T)} \le C\Big(&\|u\|^2_{L^2(\omega_{t_0}^T)} + \|u_t\|^2_{L^2(\omega_{t_0}^T)} + \|u(T',\cdot)\|^2_{L^2(0,1)}\\ &+\|(a_1u_x)_x(T',\cdot)\|^2_{L^2(0,1)}\Big),\end{aligned} \tag{4.1}$$

where $\omega_{t_0}^T := (t_0,T)\times\omega$.

Proof. The functions $y=u_t$ and $z=v_t$, where (u,v) is the solution of (1.1), are solutions of the system

$$\begin{gathered}y_t-(a_1y_x)_x+b_{11}y+b_{12}z=f_t,\quad (t,x)\in Q,\\ z_t-(a_2z_x)_x+b_{22}z=0,\quad (t,x)\in Q,\\ y(t,0)=y(t,1)=z(t,0)=z(t,1)=0,\quad t\in(0,T).\end{gathered}$$

When we apply Carleman estimate (3.28) to (y,z), we obtain

$$\begin{aligned} I(y,z) &:= \int_{Q_{t_0}^T} \Big(s\theta a_1 y_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{a_1} y^2 + s\theta^{3/2}|\eta\psi_1|y^2 + \frac{1}{s\theta} y_t^2 \Big) e^{2s\varphi_1}\,dx\,dt \\ &\quad + \int_{Q_{t_0}^T} \Big(s\theta a_2 z_x^2 + s^3\theta^3 \frac{(x-x_2)^2}{a_2} z^2 + s\theta^{3/2}|\eta\psi_2|z^2 + \frac{1}{s\theta} z_t^2 \Big) e^{2s\varphi_2}\,dx\,dt \\ &\le C \int_{Q_{t_0}^T} f_t^2 e^{-2s\Phi_2(t,-x)} dx\,dt + C \int_{t_0}^T \int_\omega u_t^2 dx\,dt \\ &:= I(f_t, y). \end{aligned} \tag{4.2}$$

The terms appearing in (4.1) are well defined, indeed, by Proposition 2.1, we have $y \in L^2(t_0,T; D(A_1)) \cap H^1(t_0,T; L^2(0,1))$.

As in [8], we divide the proof into three steps.

Step 1. We show first that there exists a constant $C>0$ such that

$$\begin{aligned} &I(f,u) + I(f_t,y) \\ &\le C\left(\frac{1}{\sqrt{s}} \int_0^1 f^2(T',x) e^{-2s\Phi_2(T',-x)} dx + \|u\|^2_{L^2(\omega_{t_0}^T)} + \|u_t\|^2_{L^2(\omega_{t_0}^T)} \right), \end{aligned} \tag{4.3}$$

where, we recall, $T' = (T+t_0)/2$. To obtain (4.3), it remains to prove that

$$\int_{Q_{t_0}^T} (f^2 + f_t^2) e^{-2s\Phi_2(t,-x)}\,dx\,dt \le \frac{C}{\sqrt{s}} \int_0^1 f^2(T',x) e^{-2s\Phi_2(T',-x)} dx.$$

Since $\Phi_{1,t}(T') = 0$ and $\Phi_{1,tt}(t) \ge \mu_0 > 0$, then Taylor's formula provides

$$-\Phi_2(t,-x) \le -\Phi_2(T',-x) - \frac{\mu_0}{2}(t-T')^2,$$

and then

$$\int_{t_0}^T e^{-2s\Phi_2(t,-x)} dt \le \frac{1}{\sqrt{\mu_0 s}} e^{-2s\Phi_2(T',-x)} \int_{-\infty}^\infty e^{-l^2} dl \le \frac{C}{\sqrt{s}} e^{-2s\Phi_2(T',-x)}.$$

So,

$$\int_{Q_{t_0}^T} f^2(T',x) e^{-2s\Phi_2(t,-x)} dx\,dt \le \frac{C}{\sqrt{s}} \int_0^1 f^2(T',x) e^{-2s\Phi_2(T',-x)} dx.$$

For $f \in S(C_0)$, one has

$$|f(t,x)| \le |f(T',x)| + \int_{T'}^t |f_t(s,x)| ds \le C|f(T',x)|. \tag{4.4}$$

Thus

$$\int_{Q_{t_0}^T} (f^2 + f_t^2)(t,x) e^{-2s\Phi_2(t,-x)} dx\,dt \le \frac{C}{\sqrt{s}} \int_0^1 f^2(T',x) e^{-2s\Phi_2(T',-x)} dx.$$

The purpose of the first step is acomplished.

Step 2. Now, let us show that there exists a constant $C>0$ such that

$$\int_0^1 (y(T',x) + b_{12} v(T',x))^2 e^{2s\varphi_1(T',x)} dx \le C(I(y,z) + I(u,v)). \tag{4.5}$$

Since, for a.e. $x \in (0,1)$,

$$\lim_{t\to t_0}(y(t,x)+b_{12}v(t,x))^2e^{2s\varphi_1(t,x)}=0.$$

Hence

$$\begin{aligned}&\int_0^1(y(T',x)+b_{12}v(T',x))^2e^{2s\varphi_1(T',x)}dx\\&=\int_0^1\int_{t_0}^{T'}\frac{\partial}{\partial t}\big((y+b_{12}v)^2e^{2s\varphi_1(t,x)}\big)\,dt\,dx\\&=\int_{t_0}^{T'}\int_0^1(2(y+b_{12}v)(y_t+b_{12}z)+2s\varphi_{1,t}(y+b_{12}v)^2)e^{2s\varphi_1(t,x)}dx\,dt.\end{aligned}\tag{4.6}$$

Using Young inequality, for s large enough, we obtain

$$\begin{aligned}&|\int_{t_0}^{T'}\int_0^1 2(y+b_{12}v)(y_t+b_{12}z)e^{2s\varphi_1}dx\,dt|\\&\le C\int_{Q_{t_0}^T}\big(s\theta y^2+s\theta z^2+s\theta v^2+\frac{1}{s\theta}y_t^2\big)e^{2s\varphi_1}dx\,dt\end{aligned}\tag{4.7}$$

and, by the Hardy inequality,

$$\begin{aligned}&\int_{Q_{t_0}^T}s\theta y^2e^{2s\varphi_1}dx\,dt\\&=s\Big|\int_{Q_{t_0}^T}\Big(\theta\frac{a_1^{1/3}}{|x-x_1|^{2/3}}y^2e^{2s\varphi_1}\Big)^{3/4}\Big(\theta\frac{|x-x_1|^2}{a_1}y^2e^{2s\varphi_1}\Big)^{1/4}dx\,dt\Big|\\&\le s\frac{3}{2}\int_{Q_{t_0}^T}\theta\frac{a_1^{1/3}}{|x-x_1|^{2/3}}y^2e^{2s\varphi_1}\,dx\,dt+\frac{s}{2}\int_{Q_{t_0}^T}\theta\frac{|x-x_1|^2}{a_1}y^2e^{2s\varphi_1}\,dx\,dt.\end{aligned}$$

Moreover, by the Hardy-Poincaré inequality applied to $ye^{s\varphi_1}$, one has

$$\begin{aligned}\int_{Q_{t_0}^T}s\theta y^2e^{2s\varphi_1}dx\,dt&\le C\int_{Q_{t_0}^T}\Big(s\theta a_1[y_x+2s\varphi_{1,x}y]^2+s^3\theta^3\frac{(x-x_1)^2}{a_1}y^2\Big)e^{2s\varphi_1}dx\,dt\\&\le C\int_{Q_{t_0}^T}\Big(s\theta a_1y_x^2+s^3\theta^3\frac{(x-x_1)^2}{a_1}y^2\Big)e^{2s\varphi_1}dx\,dt.\end{aligned}$$

Similarly, by $\varphi_1<\varphi_2$, we have

$$\begin{aligned}&\int_{Q_{t_0}^T}s\theta(z^2+v^2)e^{2s\varphi_1}dx\,dt\\&\le C\int_{Q_{t_0}^T}\Big(s\theta a_2(z_x^2+v_x^2)+s^3\theta^3\frac{(x-x_2)^2}{a_2}(z^2+v^2)\Big)e^{2s\varphi_2}dx\,dt.\end{aligned}\tag{4.8}$$

On the other hand, since $|\varphi_{1,t}|\le C|\eta\psi_1|\theta^{3/2}$, we have

$$\begin{aligned}&\int_{t_0}^{T'}\int_0^1 s\varphi_{1,t}(y+b_{12}v)^2e^{2s\varphi}\\&\le C\int_{Q_{t_0}^T}s\theta^{3/2}|\eta\psi_1|y^2e^{2s\varphi_1}+s\theta^{3/2}|\eta\psi_2|v^2e^{2s\varphi_2}dx\,dt.\end{aligned}\tag{4.9}$$

Thus, (4.6)-(4.9) yield the estimate (4.5).

Step 3. Combining (3.28), (4.2), (4.3) and (4.5), we deduce

$$\begin{aligned}&\int_0^1 (y(T',x)+b_{12}v(T',x))^2 e^{2s\varphi_1(T',x)}dx\\ &\le C\Big(\frac{1}{\sqrt{s}}\int_0^1 f^2(T',x)e^{-2s\Phi_2(T',-x)}dx + \|u\|^2_{L^2(\omega^T_{t_0})} + \|u_t\|^2_{L^2(\omega^T_{t_0})}\Big).\end{aligned} \tag{4.10}$$

Since $y + b_{12}v$ satisfies

$$y(T',x) + b_{12}v(T',x) = (a_1u_x)_x(T',x) - b_{11}u(T',x) + f(T',x),$$

it follows that

$$\begin{aligned}\int_0^1 f^2(T',x)e^{2s\varphi_1(T',x)}dx &\le C\Big(\int_0^1 (y(T',x)+b_{12}v(T',x))^2 e^{2s\varphi_1(T',x)}dx\\ &\quad + \|(a_1u_x)_x(T')\|^2_{L^2(0,1)} + \|u(T')\|^2_{L^2(0,1)}\Big).\end{aligned} \tag{4.11}$$

Hence, by (4.10) and (4.11) we obtain, for s large enough,

$$\begin{aligned}\int_0^1 f^2(T',x)dx &\le C\Big(\|u\|^2_{L^2(\omega^T_{t_0})} + \|u_t\|^2_{L^2(\omega^T_{t_0})} + \|u(T')\|^2_{L^2(0,1)}\\ &\quad + \|(au_x)_x(T')\|^2_{L^2(0,1)}\Big),\end{aligned}$$

which, together with (4.4), give the thesis. □

5. Appendix

In this section, we show a Cacciopoli's inequality for inhomogenous degenerate parabolic equations. This inequality is different from the one shown in [20] for homogenous case.

Proposition 5.1. *Let $\omega'' \subset \omega' \Subset \omega \subset (0,1)$ and $x_0 \notin \overline{\omega'}$. Let $s \ge s_0 > 0$, then there exists a positive constant $C = C(s_0, T, \inf_{\omega''} a(x), \|c\|_{L^\infty(Q)})$ such that every solution u of (3.1) satisfies*

$$\int_{t_0}^T \int_{\omega''} u_x^2 e^{2s\varphi}\,dx\,dt \le C\int_{t_0}^T\int_{\omega'} (s^2\theta^2u^2 + h^2)e^{2s\varphi}\,dx\,dt. \tag{5.1}$$

Proof. Define a smooth cut-off function $\xi \in C^\infty([0,1])$ such that $\xi \equiv 1$ in ω'' and $\operatorname{supp}(\xi) \subset \omega'$. Since u solves (3.1), we have

$$\begin{aligned}0 &= \int_{t_0}^T \frac{d}{dt}\Big(\int_0^1 \xi^2 e^{2s\varphi}u^2\,dx\Big)\,dt = \int_{t_0}^T\int_0^1 \Big(2s\xi^2\varphi_t e^{2s\varphi}u^2 + 2\xi^2 e^{2s\varphi}uu_t\Big)\,dx\,dt\\ &= \int_{t_0}^T\int_0^1 [2s\xi^2\varphi_t e^{2s\varphi}u^2 + 2\xi^2e^{2s\varphi}u((au_x)_x + h - cu)]\,dx\,dt\\ &= \int_{t_0}^T\int_0^1 [2\xi^2(s\varphi_t - c)e^{2s\varphi}u^2 + 2\xi^2e^{2s\varphi}uh - 2(\xi^2e^{2s\varphi})_x auu_x - 2\xi^2e^{2s\varphi}au_x^2]\,dx\,dt\\ &= -2\int_{t_0}^T\int_{\omega'} \xi^2e^{2s\varphi}au_x^2\,dx\,dt\\ &\quad + 2\int_{t_0}^T\int_{\omega'} [\xi^2(s\varphi_t - c)e^{2s\varphi}u^2 + \xi^2e^{2s\varphi}uh - (\xi^2e^{2s\varphi})_x auu_x]\,dx\,dt.\end{aligned}$$

Then, integrating by parts and using the Young inequality, we obtain

$$
\begin{aligned}
&2\int_{t_0}^T\int_{\omega'}\xi^2e^{2s\varphi}au_x^2\,dx\,dt\\
&=2\int_{t_0}^T\int_{\omega'}[\xi^2(s\varphi_t-c)e^{2s\varphi}u^2+\xi^2e^{2s\varphi}uh-(\xi^2e^{2s\varphi})_xauu_x]\,dx\,dt\\
&\le\int_{t_0}^T\int_{\omega'}\Big[\Big(2\xi^2(s\varphi_t-c)e^{2s\varphi}+\xi^2e^{2s\varphi}+\Big(\sqrt{a}\frac{(\xi^2e^{2s\varphi})_x}{\xi e^{s\varphi}}\Big)^2\Big)u^2+\xi^2e^{2s\varphi}h^2\Big]\,dx\,dt\\
&\quad+\int_{t_0}^T\int_{\omega'}\Big(\sqrt{a}\xi e^{s\varphi}\Big)^2u_x^2\,dx\,dt.
\end{aligned}
$$

Hence, for s large enough,

$$
\begin{aligned}
&\int_{t_0}^T\int_{\omega'}\xi^2e^{2s\varphi}au_x^2\,dx\,dt\\
&\le\int_{t_0}^T\int_{\omega'}\Big[\Big(2\xi^2(s\varphi_t-c)e^{2s\varphi}+\xi^2e^{2s\varphi}+\Big(\sqrt{a}\frac{(\xi^2e^{2s\varphi})_x}{\xi e^{s\varphi}}\Big)^2\Big)u^2+\xi^2e^{2s\varphi}h^2\Big]\,dx\,dt\\
&\le C\int_{t_0}^T\int_{\omega'}(s^2\theta^2u^2+h^2)e^{2s\varphi}\,dx\,dt.
\end{aligned}
$$

Since $x_0\notin\overline{\omega'}$,

$$
\begin{aligned}
\inf_{\omega''}a(x)\int_{t_0}^T\int_{\omega''}e^{2s\varphi}u_x^2\,dx\,dt&\le\int_{t_0}^T\int_{\omega'}\xi^2e^{2s\varphi}au_x^2\,dx\,dt\\
&\le C\int_{t_0}^T\int_{\omega'}(s^2\theta^2u^2+h^2)e^{2s\varphi}\,dx\,dt,
\end{aligned}
$$

and the proof is complete. □

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Idriss Boutaayamou
Département de Mathématiques, Faculté des Sciences Semlalia, LMDP, UMMISCO (IRD-UPMC), Université Cadi Ayyad, Marrakech 40000, B.P. 2390, Morocco
E-mail address: dsboutaayamou@gmail.com

Genni Fragnelli
Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro, Via E. Orabona 4, 70125 Bari - Italy
E-mail address: genni.fragnelli@uniba.it

Lahcen Maniar
Département de Mathématiques, Faculté des Sciences Semlalia, LMDP, UMMISCO (IRD-UPMC), Université Cadi Ayyad, Marrakech 40000, B.P. 2390, Morocco
E-mail address: maniar@uca.ma