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STEKLOV PROBLEMS INVOLVING THE $p(x)$-LAPLACIAN

GHASEM A. AFROUZI, ARMIN HADJIAN, SHAPOUR HEIDARKHANI

Abstract. Under suitable assumptions on the potential of the nonlinearity, we study the existence and multiplicity of solutions for a Steklov problem involving the $p(x)$-Laplacian. Our approach is based on variational methods.

1. Introduction

The aim of this article is to study the following Steklov problem involving the $p(x)$-Laplacian,

$$\begin{gathered} \Delta_{p(x)}u = a(x)|u|^{p(x)-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \lambda f(x,u) \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, λ is a positive parameter, $p \in C(\bar{\Omega})$, $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ denotes the $p(x)$-Laplace operator, $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $a \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega a > 0$ and ν is the outer unit normal to $\partial\Omega$.

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. It varies from nonlinear elasticity theory, electro-rheological fluids, and so on (see [24, 25]). Many results have been obtained on this kind of problems, for instance we here cite [1, 5, 6, 7, 9, 10, 11, 13, 14, 16, 18, 19].

The inhomogeneous Steklov problems involving the p-Laplacian has been the object of study in, for example, [22], in which the authors have studied this class of inhomogeneous Steklov problems in the cases of $p(x) \equiv p = 2$ and of $p(x) \equiv p > 1$, respectively.

In this paper, motivated by [1], at first, we prove the existence of a non-zero solution of the problem (1.1), without assuming any asymptotic condition neither at zero nor at infinity (see Theorem 3.1). Next, we obtain the existence of two solutions, possibly both non-zero, assuming only the classical Ambrosetti-Rabinowitz condition; that is, without requiring that the potential F satisfies the usual condition at zero (see Theorems 3.2 and 3.3). Finally, we present a three solutions existence result under appropriate condition on the potential F (see Theorem 3.4).

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Our approach is fully variational method and the main tools are critical point theorems contained in [3] and [8] (see Theorems 2.1 and 2.2 in the next section).

A special case of Theorem 3.4 is the following theorem.

Theorem 1.1. *Let $p(x) = p > N$ for every $x \in \Omega$ and let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function. Put $F(t) := \int_0^t f(\xi)d\xi$ for each $t \in \mathbb{R}$. Assume that $F(d) > 0$ for some $d \geq 1$ and, moreover,*

$$\liminf_{\xi \to 0} \frac{F(\xi)}{\xi^p} = \limsup_{|\xi| \to +\infty} \frac{F(\xi)}{\xi^p} = 0.$$

Then, there is $\lambda^\star > 0$ such that for each $\lambda > \lambda^\star$ the problem

$$\Delta_p u = a(x)|u|^{p-2}u \quad \text{in } \Omega,$$
$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda f(u) \quad \text{on } \partial\Omega,$$

admits at least three non-negative weak solutions.

2. Preliminaries

In this section, we recall definitions and theorems to be used in this paper. Let $(X, \|\cdot\|)$ be a real Banach space and Φ, $\Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals; put

$$I := \Phi - \Psi$$

and fix $r_1, r_2 \in [-\infty, +\infty]$, with $r_1 < r_2$. We say that the functional I satisfies the *Palais-Smale condition cut off lower at r_1 and upper at r_2* (${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition) if any sequence $\{u_n\} \in X$ such that

- $\{I(u_n)\}$ is bounded,
- $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,
- $r_1 < \Phi(u_n) < r_2 \quad \forall n \in \mathbb{N}$,

has a convergent subsequence.

If $r_1 = -\infty$ and $r_2 = +\infty$, it coincides with the classical (PS)-condition, while if $r_1 = -\infty$ and $r_2 \in \mathbb{R}$ it is denoted by $(\mathrm{PS})^{[r_2]}$-condition.

First we recall a result of local minimum obtained in [3], which is based on [2, Theorem 5.1].

Theorem 2.1 ([3, Theorem 2.3])**.** *Let X be a real Banach space and let Φ, Ψ : $X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r \in \mathbb{R}$ and $\bar{u} \in X$, with $0 < \Phi(\bar{u}) < r$, such that*

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})} \tag{2.1}$$

and, for each $\lambda \in \Lambda := \Big] \frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)} \Big[$ the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the $(\mathrm{PS})^{[r]}$-condition. Then, for each $\lambda \in \Lambda$, there is $u_\lambda \in \Phi^{-1}(]0, r[)$ (hence, $u_\lambda \neq 0$) such that $I_\lambda(u_\lambda) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(]0, r[)$ and $I'_\lambda(u_\lambda) = 0$.

Now we point out an other result, which insures the existence of at least three critical points, that has been obtained in [8] and it is a more precise version of [4, Theorem 3.2].

Theorem 2.2 ([8, Theorem 3.6]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, moreover*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r \in \mathbb{R}$ and $\bar{u} \in X$, with $0 < r < \Phi(\bar{u})$, such that

(i) $\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}$

(ii) *for each* $\lambda \in \Lambda := \Big]\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\Big[$ *the functional $I_\lambda = \Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda$, the functional I_λ has at least three distinct critical points in X.

Here and in the sequel, we suppose that $p \in C(\bar{\Omega})$ satisfies the following condition:

$$N < p^- := \inf_{x\in\Omega} p(x) \le p(x) \le p^+ := \sup_{x\in\Omega} p(x) < +\infty. \tag{2.2}$$

Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) := \Big\{u : \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}dx < +\infty\Big\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$\|u\|_{L^{p(x)}(\Omega)} = |u|_{p(x)} := \inf\big\{\lambda > 0 : \int_\Omega \big|\frac{u(x)}{\lambda}\big|^{p(x)}dx \le 1\big\}.$$

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) := \big\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\big\}$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} := |u|_{p(x)} + |\nabla u|_{p(x)}.$$

It is well known [17] that, in view of (2.2), both $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, with the respective norms, are separable, reflexive and uniformly convex Banach spaces.

When $a \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega a > 0$, for any $u \in W^{1,p(x)}(\Omega)$, define

$$\|u\|_a := \inf\Big\{\lambda > 0 : \int_\Omega \big(\big|\frac{\nabla u(x)}{\lambda}\big|^{p(x)} + a(x)\big|\frac{u(x)}{\lambda}\big|^{p(x)}\big)dx \le 1\Big\}.$$

Then, it is easy to see that $\|u\|_a$ is a norm on $W^{1,p(x)}(\Omega)$ equivalent to $\|u\|_{W^{1,p(x)}(\Omega)}$. In the following, we will use $\|u\|_a$ instead of $\|u\|_{W^{1,p(x)}(\Omega)}$ on $X = W^{1,p(x)}(\Omega)$.

As pointed out in [20] and [17], X is continuously embedded in $W^{1,p^-}(\Omega)$ and, since $p^- > N$, $W^{1,p^-}(\Omega)$ is compactly embedded in $C^0(\bar{\Omega})$. Thus, X is compactly embedded in $C^0(\bar{\Omega})$. So, in particular, there exists a positive constant $m > 0$ such that

$$\|u\|_{C^0(\bar{\Omega})} \le m\|u\|_a \tag{2.3}$$

for each $u \in X$. When Ω is convex, an explicit upper bound for the constant m is

$$m \le 2^{\frac{p^--1}{p^-}} \max\Big\{\big(\frac{1}{\|a\|_1}\big)^{\frac{1}{p^-}}, \frac{d}{N^{\frac{1}{p^-}}}\big(\frac{p^--1}{p^--N}|\Omega|\big)^{\frac{p^--1}{p^-}}\frac{\|a\|_\infty}{\|a\|_1}\Big\}\big(1 + |\Omega|\big)$$

where $d := \operatorname{diam}(\Omega)$ and $|\Omega|$ is the Lebesgue measure of Ω (for details, see [10]), $\|a\|_1 := \int_\Omega a(x)dx$ and $\|a\|_\infty := \sup_{x\in\Omega} a(x)$.

Lemma 2.3 ([17]). *Let $I(u) = \int_\Omega (|\nabla u|^{p(x)} + a(x)|u|^{p(x)})dx$. For $u \in X$ we have*

(i) $\|u\|_a < 1(= 1; > 1) \Leftrightarrow I(u) < 1(= 1; > 1)$;
(ii) *If* $\|u\|_a < 1 \Rightarrow \|u\|_a^{p^+} \le I(u) \le \|u\|_a^{p^-}$;
(iii) *If* $\|u\|_a > 1 \Rightarrow \|u\|_a^{p^-} \le I(u) \le \|u\|_a^{p^+}$.

We refer the reader to [15, 17] for the basic properties of the variable exponent Lebesgue and Sobolev spaces.

Throughout this article, we assume the following condition on the Carathéodory function $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$:

(F0) $|f(x,s)| \le \alpha(x) + b|s|^{\beta(x)-1}$ for all $(x,s) \in \partial\Omega \times \mathbb{R}$, where $\alpha \in L^{\frac{\beta(x)}{\beta(x)-1}}(\partial\Omega)$, $b \ge 0$ is a constant and $\beta \in C(\partial\Omega)$ such that
$$1 < \beta^- := \inf_{x\in\bar\Omega} \beta(x) \le \beta(x) \le \beta^+ := \sup_{x\in\bar\Omega} \beta(x) < p^-. \tag{2.4}$$

We recall that $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function if $x \mapsto f(x,\xi)$ is measurable for all $\xi \in \mathbb{R}$ and $\xi \mapsto f(x,\xi)$ is continuous for a.e. $x \in \partial\Omega$. Put
$$F(x,t) := \int_0^t f(x,\xi)d\xi,$$
for all $(x,t) \in \partial\Omega \times \mathbb{R}$.

Theorem 2.4 ([1, Theorem 2.9]). *Let $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying* (F0). *For each $u \in X$ set $\Psi(u) = \int_{\partial\Omega} F(x,u(x))d\sigma$. Then $\Psi \in C^1(X,\mathbb{R})$ and*
$$\Psi'(u)(v) = \int_{\partial\Omega} f(x,u(x))v(x)d\sigma$$
for every $v \in X$. Moreover, the operator $\Psi' : X \to X^*$ is compact.

We say that a function $u \in X$ is a *weak solution* of problem (1.1) if
$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,dx + \int_\Omega a(x)|u|^{p(x)-2}uv\,dx = \lambda\int_{\partial\Omega} f(x,u)v\,d\sigma$$
for all $v \in X$.

We cite the very recent monograph by Kristály et al. [21] as a general reference for the basic notions used in the paper.

3. Main results

In this section we present our main results. First, we establish the existence of one non-trivial solution for the problem (1.1).

Theorem 3.1. *Let $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying* (F0). *Assume that there exist $d \ge 1$ and $c \ge m$ with $d^{p^+}\|a\|_1 < \frac{p^-}{p^+}(\frac{c}{m})^{p^-}$, such that*
$$\frac{\int_{\partial\Omega} \max_{|t|\le c} F(x,t)d\sigma}{\left(\frac{c}{m}\right)^{p^-}} < \frac{p^-\int_{\partial\Omega} F(x,d)d\sigma}{p^+ d^{p^+}\|a\|_1}. \tag{3.1}$$

Then, for each
$$\lambda \in \Lambda := \Big]\frac{d^{p^+}\|a\|_1}{p^-\int_{\partial\Omega} F(x,d)d\sigma}, \frac{\left(\frac{c}{m}\right)^{p^-}}{p^+\int_{\partial\Omega}\max_{|t|\le c} F(x,t)d\sigma}\Big[, \tag{3.2}$$

problem (1.1) *admits at least one non-trivial weak solution* $\bar{u}_1 \in X$ *such that*

$$\max_{x\in\Omega} |\bar{u}_1(x)| < c.$$

Proof. Our aim is to apply Theorem 2.1 to (1.1). To this end, for each $u \in X$, let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \int_\Omega \frac{1}{p(x)} \Big(|\nabla u|^{p(x)} + a(x)|u|^{p(x)}\Big)\, dx,$$

$$\Psi(u) := \int_{\partial\Omega} F(x, u(x)) d\sigma,$$

and put

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u), \quad u \in X.$$

Note that the weak solutions of (1.1) are exactly the critical points of I_λ. The functionals Φ and Ψ satisfy the regularity assumptions of Theorem 2.1. Indeed, by standard arguments, we have that Φ is Gâteaux differentiable and its Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla v + a(x)|u|^{p(x)-2}uv\Big)dx$$

for every $v \in X$. Moreover, Φ is sequentially weakly lower semicontinuous and its inverse derivative is continuous (since it is a continuous convex functional) and, thanks to Lemma 2.3, the functional Φ turns out to be coercive. On the other hand, by Theorem 2.4, the functional Ψ is well defined, continuously Gâteaux differentiable and with compact derivative, whose Gâteaux derivative at the point $u \in X$ is given by

$$\Psi'(u)(v) = \int_{\partial\Omega} f(x, u(x))v(x)d\sigma$$

for every $v \in X$. So, owing to [2, Proposition 2.1], the functional I_λ satisfies the $(\mathrm{PS})^{[r]}$-condition for all $r \in \mathbb{R}$.

We will verify condition (2.1) of Theorem 2.1. Let w be the function defined by $w(x) := d$ for all $x \in \bar{\Omega}$ and put

$$r := \frac{1}{p^+}\Big(\frac{c}{m}\Big)^{p^-}.$$

Clearly, $w \in X$ and from our assumption one has

$$0 < \Phi(w) = \int_\Omega \frac{1}{p(x)} a(x) d^{p(x)} dx \le \frac{1}{p^-}\|a\|_1 d^{p^+} < r.$$

For all $u \in X$ with $\Phi(u) < r$, owing to Lemma 2.3, definitively one has

$$\min\big\{\|u\|_a^{p^+}, \|u\|_a^{p^-}\big\} < rp^+.$$

Then

$$\|u\|_a < \max\big\{(p^+r)^{\frac{1}{p^+}}, (p^+r)^{\frac{1}{p^-}}\big\} = \frac{c}{m},$$

and so, by (2.3),

$$\max_{x\in\Omega} |u(x)| \le m\|u\|_a < c.$$

Therefore,

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} \le \frac{\int_{\partial\Omega} \max_{|t|\le c} F(x,t) d\sigma}{\frac{1}{p^+}\big(\frac{c}{m}\big)^{p^-}}$$

On the other hand, taking into account that

$$\Phi(w) \leq \frac{1}{p^-} d^{p^+} \|a\|_1,$$

we have

$$\frac{\Psi(w)}{\Phi(w)} \geq \frac{\int_{\partial\Omega} F(x,d)d\sigma}{\frac{1}{p^-} d^{p^+} \|a\|_1}.$$

Therefore, by the assumption (3.1), condition (2.1) of Theorem 2.1 is verified.

Therefore, all the assumptions of Theorem 2.1 are satisfied. So, for each

$$\lambda \in \Lambda \subseteq \Big]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\Big[,$$

the functional I_λ has at least one non-zero critical point $\bar{u}_1 \in X$ such that $\max_{x\in\Omega} |\bar{u}_1(x)| < c$ that is the weak solution of the problem (1.1). □

The following result, in which the global Ambrosetti-Rabinowitz condition is also used, ensures the existence at least two weak solutions.

Theorem 3.2. *Assume that all the assumptions of Theorem 3.1 hold. Furthermore, suppose that $f(\cdot,0) \neq 0$ in $\partial\Omega$, and*

(AR) *there exist two constants $\mu > p^+$ and $R > 0$ such that for all $x \in \partial\Omega$ and $|s| \geq R$,*

$$0 < \mu F(x,s) \leq sf(x,s).$$

Then, for each $\lambda \in \Lambda$, where Λ is given by (3.2)*, the problem* (1.1) *has at least two non-trivial weak solutions $\bar{u}_1,\ \bar{u}_2 \in X$ such that*

$$\max_{x\in\Omega} |\bar{u}_1(x)| < c.$$

Proof. Fix λ as in the conclusion. So, Theorem 3.1 ensures that the problem (1.1) admits at least one non-trivial weak solution $\bar{u}_1$ which is a local minimum of the functional I_λ.

Now, we prove the existence of the second local minimum distinct from the first one. To this end, we must show that the functional I_λ satisfies the hypotheses of the mountain pass theorem.

Clearly, the functional I_λ is of class C^1 and $I_\lambda(0) = 0$.

We can assume that $\bar{u}_1$ is a strict local minimum for I_λ in X. Therefore, there is $\rho > 0$ such that $\inf_{\|u-\bar{u}_1\|=\rho} I_\lambda(u) > I_\lambda(\bar{u}_1)$, so condition [23, (I_1), Theorem 2.2] is verified.

From (AR), by standard computations, there is a positive constant C such that

$$F(x,s) \geq C|s|^\mu \tag{3.3}$$

for all $x \in \partial\Omega$ and $|s| > R$. In fact, setting $\gamma(x) = \min_{|\xi|=R} F(x,\xi)$ and

$$\varphi_s(t) = F(x,ts) \quad \forall t > 0, \tag{3.4}$$

by (AR), for every $x \in \partial\Omega$ and $|s| > R$ one has

$$0 < \mu\varphi_s(t) = \mu F(x,ts) \leq tsf(x,ts) = t\varphi_s'(t) \quad \forall t > 0.$$

Therefore,

$$\int_{R/|s|}^1 \frac{\varphi_s'(t)}{\varphi_s(t)} dt \geq \int_{R/|s|}^1 \frac{\mu}{t} dt.$$

Then

$$\varphi_s(1) \geq \varphi_s\Big(\frac{R}{|s|}\Big)|s|^\mu.$$

Taking into account (3.4), we obtain

$$F(x,s) \geq F\Big(x, \frac{R}{|s|}s\Big)|s|^\mu \geq \gamma(x)|s|^\mu \geq C|s|^\mu,$$

and (3.3) is proved. Now, by choosing any $u \in X \setminus \{0\}$ and $t > 1$, one has

$$\begin{aligned}
I_\lambda(tu) &= (\Phi - \lambda\Psi)(tu) \\
&= \int_\Omega \frac{t^{p(x)}}{p(x)} \Big(|\nabla u|^{p(x)} + a(x)|u|^{p(x)}\Big)\,dx - \lambda \int_{\partial\Omega} F(x, tu(x))d\sigma \\
&\leq t^{p^+} \int_\Omega \frac{1}{p(x)} \Big(|\nabla u|^{p(x)} + a(x)|u|^{p(x)}\Big)\,dx - Ct^\mu\lambda \int_{\partial\Omega} |u(x)|^\mu d\sigma.
\end{aligned}$$

Since $\mu > p^+$, the functional I_λ is unbounded from below. So, condition [23, (I_2), Theorem 2.2] is verified. Therefore, I_λ satisfies the geometry of mountain pass.

Now, to verify the (PS)-condition it is sufficient to prove that any (PS)-sequence is bounded. To this end, suppose that $\{u_n\} \subset X$ is a (PS)-sequence; i.e., there is $M > 0$ such that

$$\sup |I_\lambda(u_n)| \leq M, \quad I'_\lambda(u_n) \to 0 \quad \text{as } n \to +\infty.$$

Let us show that $\{u_n\}$ is bounded in X. Using hypothesis (AR), since $I_\lambda(u_n)$ is bounded, we have for n large enough:

$$\begin{aligned}
M + 1 &\geq I_\lambda(u_n) - \frac{1}{\mu}\langle I'_\lambda(u_n), u_n\rangle + \frac{1}{\mu}\langle I'_\lambda(u_n), u_n\rangle \\
&= \int_\Omega \frac{1}{p(x)} \Big(|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}\Big)\,dx - \lambda \int_{\partial\Omega} F(x, u_n(x))d\sigma \\
&\quad - \frac{1}{\mu}\Big[\int_\Omega \Big(|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}\Big)\,dx - \lambda \int_{\partial\Omega} f(x, u_n(x))u_n(x)d\sigma\Big] \\
&\quad + \frac{1}{\mu}\langle I'_\lambda(u_n), u_n\rangle \\
&\geq \big(\frac{1}{p^+} - \frac{1}{\mu}\big)\|u_n\|_a^{p^-} - \frac{1}{\mu}\|I'_\lambda(u_n)\|_{X^*}\|u_n\|_a - c_1 \\
&\geq \big(\frac{1}{p^+} - \frac{1}{\mu}\big)\|u_n\|_a^{p^-} - \frac{c_2}{\mu}\|u_n\|_a - c_1,
\end{aligned}$$

where c_1 and c_2 are two positive constants. Since $\mu > p^+$, from the above inequality we know that $\{u_n\}$ is bounded in X. Hence, the classical theorem of Ambrosetti and Rabinowitz ensures a critical point $\bar{u}_2$ of I_λ such that $I_\lambda(\bar{u}_2) > I_\lambda(\bar{u}_1)$. So, $\bar{u}_1$ and $\bar{u}_2$ are two distinct weak solutions of (1.1) and the proof is complete. □

Here we give the following result as a direct consequence of Theorem 3.2 in the autonomous case.

Theorem 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(0) \neq 0$ and $|f(s)| \leq \alpha + b|s|^{\beta-1}$ for all $s \in \mathbb{R}$, where $\alpha > 0$, $b \geq 0$ and $1 < \beta < p^-$ are three constants. Put $F(t) := \int_0^t f(\xi)d\xi$ for all $t \in \mathbb{R}$. Under the following conditions*

(i) *there exist* $d \geq 1$ *and* $c \geq m$ *with* $d^{p^+}\|a\|_1 < \frac{p^-}{p^+}\left(\frac{c}{m}\right)^{p^-}$, *such that*

$$\frac{\max_{|t|\leq c} F(t)}{\left(\frac{c}{m}\right)^{p^-}} < \frac{p^- F(d)}{p^+ d^{p^+}\|a\|_1};$$

(ii) *there exist two constants* $\mu > p^+$ *and* $R > 0$ *such that for all* $|s| \geq R$,

$$0 < \mu F(s) \leq s f(s),$$

and for each

$$\lambda \in \Big]\frac{d^{p^+}\|a\|_1}{p^-|\partial\Omega| F(d)}, \frac{\left(\frac{c}{m}\right)^{p^-}}{p^+|\partial\Omega| \max_{|t|\leq c} F(t)}\Big[,$$

the problem

$$\begin{gathered}\Delta_{p(x)} u = a(x)|u|^{p(x)-2}u \quad \text{in } \Omega,\\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \lambda f(u) \quad \text{on } \partial\Omega,\end{gathered}$$

admits at least two non-trivial weak solutions $\bar{u}_1, \bar{u}_2 \in X$ *such that*

$$\max_{x\in\Omega} |\bar{u}_1(x)| < c.$$

Now, we point out the following result of three weak solutions.

Theorem 3.4. *Let* $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ *be a Carathéodory function satisfying* (F0). *Assume that there exist* $d \geq 1$ *and* $c \geq m$ *with* $d^{p^-}\|a\|_1 > \left(\frac{c}{m}\right)^{p^-}$, *such that the assumption* (3.1) *in Theorem 3.1 holds. Then, for each* $\lambda \in \Lambda$, *where* Λ *is given by* (3.2), *the problem* (1.1) *has at least three weak solutions.*

Proof. Our goal is to apply Theorem 2.2. The functionals Φ and Ψ defined in the proof of Theorem 3.1 satisfy all regularity assumptions requested in Theorem 2.2. So, our aim is to verify (i) and (ii). Arguing as in the proof of Theorem 3.1, put $r := \frac{1}{p^+}\left(\frac{c}{m}\right)^{p^-}$ and $w(x) := d$ for all $x \in \bar{\Omega}$, bearing in mind that $d^{p^-}\|a\|_1 > (\frac{c}{m})^{p^-}$, we have

$$\Phi(w) = \int_\Omega \frac{1}{p(x)} a(x) d^{p(x)} dx \geq \frac{1}{p^+} d^{p^-}\|a\|_1 > r > 0.$$

Therefore, the assumption (i) of Theorem 2.2 is satisfied.

We prove that the functional I_λ is coercive for all $\lambda > 0$. If $u \in X$, then by condition (2.4) and the embedding theorem (see [12, Theorem 2.1]) we have $u \in L^{\beta(x)}(\partial\Omega)$. Then there is some constant $C > 0$ such that

$$\|u\|_{L^{\beta(x)}(\partial\Omega)} \leq C\|u\|_a, \quad \forall u \in X.$$

Now, by using Hölder inequality (see [17]) and condition (F0), for all $u \in X$ such that $\|u\|_a \geq 1$, we have

$$\begin{aligned}\Psi(u) &= \int_{\partial\Omega} F(x, u(x))d\sigma = \int_{\partial\Omega}\Big(\int_0^{u(x)} f(x,t)dt\Big)d\sigma\\ &\leq \int_{\partial\Omega}\Big(\alpha(x)|u(x)| + \frac{b}{\beta(x)}|u(x)|^{\beta(x)}\Big)d\sigma\\ &\leq 2\|\alpha\|_{L^{\frac{\beta(x)}{\beta(x)-1}}(\partial\Omega)}\|u\|_{L^{\beta(x)}(\partial\Omega)} + \frac{b}{\beta^-}\int_{\partial\Omega}|u(x)|^{\beta(x)}d\sigma\end{aligned}$$

$$\leq 2C\|\alpha\|_{L^{\frac{\beta(x)}{\beta(x)-1}}(\partial\Omega)}\|u\|_a + \frac{b}{\beta^-}\int_{\partial\Omega}|u(x)|^{\beta(x)}d\sigma.$$

On the other hand, there is a constant $C' > 0$ such that

$$\int_{\partial\Omega}|u(x)|^{\beta(x)}d\sigma \leq \max\left\{\|u\|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}, \|u\|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}\right\} \leq C'\|u\|_a^{\beta^+}.$$

Then,

$$\Psi(u) \leq 2C\|\alpha\|_{L^{\frac{\beta(x)}{\beta(x)-1}}(\partial\Omega)}\|u\|_a + \frac{b}{\beta^-}C'\|u\|_a^{\beta^+}.$$

Since

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + a(x)|u|^{p(x)}\right)dx \geq \frac{1}{p^+}\|u\|_a^{p^-},$$

for every $\lambda > 0$ we have

$$I_\lambda(u) \geq \frac{1}{p^+}\|u\|_a^{p^-} - 2\lambda C\|\alpha\|_{L^{\frac{\beta(x)}{\beta(x)-1}}(\partial\Omega)}\|u\|_a - \frac{\lambda bC'}{\beta^-}\|u\|_a^{\beta^+}.$$

Since $p^- > \beta^+$, the functional I_λ is coercive. Then also condition (ii) holds. So, for each $\lambda \in \Lambda$, the functional I_λ admits at least three distinct critical points that are weak solutions of problem (1.1). $\square$

Remark 3.5. If we assume that $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a non-negative Carathéodory function satisfying (F0), then the previous theorems guarantee the existence of non-negative weak solutions. In fact, let $\bar{u}$ be a weak solution of the problem (1.1). We claim that it is non-negative. Arguing by contradiction and setting $A := \{x \in \bar{\Omega} : \bar{u}(x) < 0\}$, one has $A \neq \emptyset$. Put $\bar{v} := \min\{\bar{u}, 0\}$, one has $\bar{v} \in X$. So, taking into account that $\bar{u}$ is a weak solution and by choosing $v = \bar{v}$, one has

$$\int_A |\nabla\bar{u}|^{p(x)}dx + \int_A a(x)|\bar{u}|^{p(x)}dx = \lambda\int_{\partial\Omega} f(x,\bar{u}(x))\bar{u}(x)d\sigma \leq 0,$$

that is, $\|\bar{u}\|_{W^{1,p(x)}(A)} = 0$ which is absurd. Hence, our claim is proved.

Also, when f is a non-negative function, condition (3.1) becomes

$$\frac{\int_{\partial\Omega}F(x,c)d\sigma}{\left(\frac{c}{m}\right)^{p^-}} < \frac{p^-\int_{\partial\Omega}F(x,d)d\sigma}{p^+d^{p^+}\|a\|_1}.$$

In this case, the previous theorems ensure the existence of non-negative solutions to the problem (1.1) for each

$$\lambda \in \left]\frac{d^{p^+}\|a\|_1}{p^-\int_{\partial\Omega}F(x,d)d\sigma}, \frac{\left(\frac{c}{m}\right)^{p^-}}{p^+\int_{\partial\Omega}F(x,c)d\sigma}\right[.$$

Remark 3.6. Theorems 3.1 and 3.4 ensure more precise conclusions rather than [1, Theorems 1.1 and 1.3]. In fact, Theorem 1.1 of [1] proves that for any $\lambda \in]0,+\infty[$, the problem (1.1), when $a \equiv 1$, has at least a non-trivial weak solution. Also, Theorem 3.1 of [1] establishes that there exists an open interval $\Lambda \subset]0,+\infty[$ such that, for every $\lambda \in \Lambda$, the problem (1.1), when $a \equiv 1$, admits at least three solutions. Hence, a location of the interval Λ in $]0,+\infty[$ is not established.

Proof of Theorem 1.1. Fix $\lambda > \lambda^\star := \frac{d^p\|a\|_1}{p|\partial\Omega|F(d)}$ for some $d \geq 1$ such that $F(d) > 0$. Since

$$\liminf_{\xi\to 0}\frac{F(\xi)}{\xi^p} = 0,$$

there is a sequence $\{c_n\} \subset]0, +\infty[$ such that $\lim_{n\to+\infty} c_n = 0$ and

$$\lim_{n\to+\infty} \frac{F(c_n)}{c_n^p} = 0.$$

Therefore, there exists $\overline{c} \geq m$ such that

$$\frac{F(\overline{c})}{\overline{c}^p} < \min\Big\{\frac{F(d)}{(md)^p\|a\|_1}, \frac{1}{p|\partial\Omega|m^p\lambda}\Big\}$$

and $\overline{c} < md\|a\|_1^{1/p}$. Also, by the assumption

$$\limsup_{|\xi|\to+\infty} \frac{F(\xi)}{\xi^p} = 0,$$

the functional I_λ is coercive. Hence, by taking Remark 3.5 into account, the conclusion follows from Theorem 3.4. □

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Ghasem A. Afrouzi

Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

E-mail address: `afrouzi@umz.ac.ir`

Armin Hadjian

Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, P.O. Box 1339, Bojnord 94531, Iran

E-mail address: `hadjian83@gmail.com`

Shapour Heidarkhani

Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran

E-mail address: `s.heidarkhani@razi.ac.ir`