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EXISTENCE AND STABILITY OF MILD SOLUTIONS TO IMPULSIVE STOCHASTIC NEUTRAL PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study a class of impulsive stochastic neutral partial functional differential equations in a real separable Hilbert space. By using Banach fixed point theorem, we give sufficient conditions for the existence and uniqueness of a mild solution. Also the exponential p-stability of a mild solution and its sample paths are obtained.

1. Introduction

Since deterministic neutral functional differential equations were originally proposed by Hale and Meyer [3], many researchers including Hale and Verduyn Lunel [4], Kolmanovskii and Nosov [6] have done extensive works on this subject and their applications. However, a system is usually affected by external perturbations which in many cases are of great random uncertainties such as stochastic forces on the physical systems and noisy measurements caused by environmental uncertainties [17, 18, 19], a stochastic neutral functional differential equations should be produced instead of a deterministic one. Correspondingly, a number of interesting results on the stochastic neutral functional differential equations have been reported [1, 5, 7, 9, 10, 12]. Sakthivel, Ren and Kim [16] studied the existence and asymptotic stability in pth moment of mild solutions to second-order nonlinear neutral stochastic differential equations. Ren and Sakthivel [13] studied the existence, uniqueness, and stability of mild solutions for second-order neutral stochastic evolution equations with infinite delay and Poisson jumps. More recently, the existence, uniqueness and exponential stability of a mild solution of the stochastic neutral partial functional differential equations have been considered by Luo [8].

However, in addition to stochastic effects, impulsive effects exist in many evolution processes in which states are changed abruptly at certain moments of time, involved in such fields as medicine and biology, economics, mechanics, electronics. Impulsive effects often make the system properties decline or even cause instability. Therefore, impulsive effects should be taken into account in researching the exponential stability of the stochastic systems. Some significant progress has

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been made in the techniques and methods of studying the existence and stability for impulsive stochastic difference equations, impulsive stochastic differential equations with delays and impulsive stochastic partial functional differential equations [14, 15, 20, 21, 22, 23, 24]. However, so far no work has been reported on the corresponding problems for impulsive stochastic neutral partial functional differential equations and the aim of this paper is to close this gap.

Motivated by the above discussions, we will study the existence, uniqueness and exponential p-stability of a mild solution of the impulsive stochastic neutral partial functional differential equations. By using Banach fixed point theorem, we give some sufficient conditions for the existence and uniqueness of a mild solution of this class of equations. Also the exponential p-stability of a mild solution as well as its sample paths are obtained.

2. Preliminaries

Throughout this paper, unless otherwise specified, X and Y are two separable Hilbert spaces. with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively. Let $\mathcal{L}(Y,X)$ be the space of all bounded linear operators from Y into X equipped with the usual norm $\|\cdot\|$.

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions of complete sub-σ-fields of $\mathcal{F}$. Let $w(t)$, $t\geq 0$, be a Y-valued, Q-Wiener process which is assumed to be adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and for every $t>s$ the increments $w(t)-w(s)$ are independent of $\mathcal{F}_s$. Hence, $w(t), t\geq 0$, is a continuous martingale relative to $\{\mathcal{F}_t\}_{t\geq 0}$ and we have the following representation of $w(t)$:

$$w(t)=\sum_{i=1}^{\infty}B_t^i e_i,$$

where $\{e_i\}$ is an orthonormal set of eigenvectors of Q, $\{B_t^i\}$, $t\geq 0$, is a family of mutually independent real Wiener processes with incremental covariance $\lambda_i>0$, $Qe_i=\lambda_i e_i$ and $trQ=\sum_{i=1}^{\infty}\lambda_i<\infty$. Let $h(t)$ be an $\mathcal{L}(Y,X)$-valued function and λ be an arbitrary sequence $\{\lambda_1,\lambda_2,\dots\}$ of positive numbers, we may often use the following notation

$$\|h(t)\|_\lambda:=\Big\{\sum_{n=1}^{\infty}|\sqrt{\lambda_n}h(t)e_n|^2\Big\}^{1/2}$$

whenever this series is convergent.

$$\begin{aligned}PC[\mathbb{J},X]=\Big\{&\psi:\mathbb{J}\to X:\psi(s)\text{ is continuous for all but at most countably}\\&\text{many points } s\in\mathbb{J}\text{ and at these points }\psi(s^+)\text{ and }\psi(s^-)\\&\text{exist and }\psi(s)=\psi(s^+)\Big\},\end{aligned}$$

where $\mathbb{J}\subset\mathbb{R}$ is an interval, $\psi(s^+)$ and $\psi(s^-)$ denote the right-hand and left-hand limits of the function $\psi(s)$ at time s, respectively. Let $PC:=PC[[-\tau,0],X]$ with the norm $\|\varphi\|_{PC}=\sup_{-\tau\leq\theta\leq 0}\|\varphi(\theta)\|_X$. Let $PC^b_{\mathscr{F}_0}[[-\tau,0],X]$ be the Banach space of all bounded $\mathscr{F}_0$-measurable, $PC[[-\tau,0],X]$-valued random variables ϕ, satisfying $\sup_{-\tau\leq\theta\leq 0}E\|\phi(\theta)\|_X^p<\infty$ for $p>0$, where E denote the expectation of stochastic process.

Assume that $\{S(t), t \geq 0\}$ is an analytic semigroup with its infinitesimal generator A, then it is possible under some circumstances (see [11]) to define the fractional power $(-A)^\alpha$ for any $\alpha \in [0,1]$ which is a closed linear operator with its domain $\mathcal{D}((-A)^\alpha)$.

Consider the following impulsive stochastic neutral partial functional differential equations

$$
\begin{gathered}
d[x(t) + G(t, x_t)] = [Ax(t) + f(t, x_t)]dt + g(t, x_t)dw(t), \quad t \geq 0,\ t \neq t_k, \\
\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad t = t_k,\ k = 1, 2, \dots, N, \\
x_i(t) = \phi_i(t), \quad -\tau \leq t \leq 0,
\end{gathered} \tag{2.1}
$$

where $x_t(\theta) = x(t+\theta)$ for $\theta \in [-\tau, 0]$, $G, f : \mathbb{R}_+ \times PC \to X$, and $g : \mathbb{R}_+ \times PC \to \mathcal{L}(Y, X)$ are all Borel measurable. I_k shows the jump in the state x at time t_k, and t_k satisfies $0 < t_1 < \dots < t_N < \lim_{k\to\infty} t_k = \infty$.

Definition 2.1. A process $\{x(t), t \in [0, T]\}$, $0 \leq T < \infty$, is called a mild solution of (2.1) if

(i) $x(t)$ is $\mathcal{F}_t$-adapted;
(ii) $x(t) \in X$ has càdlàg paths on $t \in [0, T]$ a.s and for each $t \in [0, T]$, $x(t)$ satisfies the integral equation

$$
\begin{aligned}
x(t) = {} & S(t)[\phi(0) + G(0, \phi)] - G(t, x_t) - \int_0^t AS(t-s)G(s, x_s)ds \\
& + \int_0^t S(t-s)f(s, x_s)ds + \int_0^t S(t-s)g(s, x_s)dw(s) \\
& + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)),
\end{aligned} \tag{2.2}
$$

for any $\phi \in PC^b_{\mathscr{F}_0}[[-\tau, 0], X]$ almost surely.

3. Main results

We assume the following hyptheses:

(H1) A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in X satisfying $\|S(t)\|_X \leq \gamma e^{vt}$, $t \geq 0$, for some constants $\gamma \geq 1$ and $v \in \mathbb{R}$.

(H2) There exist constants $C_1 > 0$ and $\alpha \in [0, 1]$ such that for any $x, y \in PC$ and $t \geq 0$, $G(t, x) \in \mathcal{D}((-A)^\alpha)$,
$$
\|(-A)^\alpha G(t, x) - (-A)^\alpha G(t, y)\|_X \leq C_1 \|x - y\|_{PC}.
$$

(H3) The functions f and g satisfy the Lipschitz and linear growth conditions; that is, there exist positive constants C_2, C_3, C_4 such that
$$
\begin{gathered}
\|f(t, x) - f(t, y)\|_X \leq C_2 \|x - y\|_{PC}, \\
\|g(t, x) - g(t, y)\|_\lambda \leq C_3 \|x - y\|_{PC}, \\
\|f(t, x)\|_X + \|g(t, x)\|_\lambda \leq C_4(1 + \|x\|_{PC}),
\end{gathered}
$$
for any $x, y \in PC$, $t \geq 0$.

(H4) There exist nonnegative constants q_k such that for any $x, y \in PC$,
$$
\|I_k(x) - I_k(y)\|_X \leq q_k \|x - y\|_{PC}, \quad k = 1, 2, \dots, N. \tag{3.1}
$$

Lemma 3.1 ([8, Lemma 2.1]). *Suppose that the assumption (H1) holds, then for any $\beta \geq 0$,*

(i) *for each $x \in \mathcal{D}((-A)^\beta)$,*
$$S(t)(-A)^\beta x = (-A)^\beta S(t)x;$$

(ii) *There exist constants $M_\beta > 0$ and $v \in \mathbb{R}$ such that*
$$\|(-A)^\beta S(t)\|_X \leq M_\beta t^{-\beta} e^{vt}, \quad t > 0.$$

Theorem 3.2. *Under assumptions* (H1)–(H4)*, system* (2.1) *has a unique mild solution $x(t)$, $0 \leq t \leq T$, if the following condition holds*
$$\begin{aligned} L := 5^{p-1}\Big[&\|(-A)^{-\alpha}\|^p C_1^p + M_{1-\alpha}^p \tau^p(T) T^{p-1} + \gamma^p e^{p|v|T} C_2^p T^{p-1} \\ &+ \gamma^p e^{p|v|T} C_3^p T^{\frac{p-2}{2}} + N^{p-1}\gamma^p e^{p|v|T}\big(\sum_{k=1}^N q_k^p\big)\Big] < 1, \end{aligned}$$
where $M_{1-\alpha}$ and v are the constants in Lemma 3.1 (ii), and
$$p \geq 2, \quad \tau(T) = \int_0^T t^{-(1-\alpha)} e^{vt} dt < \infty.$$

Proof. Define a nonlinear operator $\Psi : PC^b_{\mathscr{F}_0}[[-\tau, 0], X] \to PC^b_{\mathscr{F}_0}[[-\tau, 0], X]$ by
$$\begin{aligned} \Psi x(t) &= S(t)\phi(0) + S(t)(-A)^{-\alpha}(-A)^\alpha G(0,\phi) - (-A)^{-\alpha}(-A)^\alpha G(t, x_t) \\ &\quad + \int_0^t (-A)^{1-\alpha} S(t-s)(-A)^\alpha G(s, x_s) ds + \int_0^t S(t-s) f(s, x_s) ds \\ &\quad + \int_0^t S(t-s) g(s, x_s) dw(s) + \sum_{0<t_k<t} S(t-t_k) I_k(x(t_k^-)), \quad 0 \leq t \leq T. \end{aligned} \tag{3.2}$$
From Lemma 3.1, (H1), (H2) and (H4), we obtain that for any $0 \leq t \leq T$,
$$\begin{aligned} \|x(t)\|_X &\leq \gamma e^{|v|T}\|\phi(0)\|_X + \gamma e^{|v|T} C\|(-A)^{-\alpha}\|(1 + \|\phi\|_{PC}) \\ &\quad + C\|(-A)^{-\alpha}\|(1 + \|x_t\|_{PC}) \\ &\quad + \int_0^t C M_{1-\alpha}(t-s)^{-(1-\alpha)} e^{v(t-s)}(1 + \|x_s\|_{PC}) ds \\ &\quad + \|\int_0^t S(t-s) f(s, x_s) ds\|_X + \|\int_0^t S(t-s) g(s, x_s) dw(s)\|_X \\ &\quad + \|\sum_{0<t_k<t} S(t-t_k) I_k(x(t_k^-))\|_X \\ &\leq \gamma e^{|v|T}\|\phi(0)\|_X + \gamma e^{|v|T} C\|(-A)^{-\alpha}\|(1 + \|\phi\|_{PC}) \\ &\quad + C\|(-A)^{-\alpha}\|(1 + \|x_t\|_{PC}) + C M_{1-\alpha}\tau(T)(1 + \sup_{0\leq s\leq t}\|x_s\|_{PC}) \\ &\quad + \gamma e^{|v|T}\sum_{k=1}^N q_k\|x(t_k^-)\|_{PC} + \|\int_0^t S(t-s) f(s, x_s) ds\|_X \\ &\quad + \|\int_0^t S(t-s) g(s, x_s) dw(s)\|_X, \end{aligned} \tag{3.3}$$

where C is a positive constant. Since

$$\sup_{0\le t\le T}\|x_t\|_{PC}\le \sup_{0\le t\le T}\|x(t)\|_X+\|\phi\|_{PC},$$

using (3.3) we have

$$\begin{aligned}
&\sup_{0\le t\le T}\|x(t)\|_X\\
&\le \gamma e^{|v|T}\|\phi(0)\|_X+\gamma e^{|v|T}C\|(-A)^{-\alpha}\|(1+\|\phi\|_{PC})\\
&\quad + C\|(-A)^{-\alpha}\|(1+\sup_{0\le t\le T}\|x(t)\|_X+\|\phi\|_{PC})\\
&\quad + CM_{1-\alpha}\tau(T)(1+\sup_{0\le t\le T}\|x(t)\|_X+\|\phi\|_{PC})+\gamma e^{|v|T}\sum_{k=1}^N q_k\sup_{0\le t\le T}\|x(t)\|_X\\
&\quad + \sup_{0\le t\le T}\|\int_0^t S(t-s)f(s,x_s)ds\|_X+\sup_{0\le t\le T}\|\int_0^t S(t-s)g(s,x_s)dw(s)\|_X;
\end{aligned} \tag{3.4}$$

that is,

$$\begin{aligned}
&[1-C\|(-A)^{-\alpha}\|-CM_{1-\alpha}\tau(T)-\gamma e^{|v|T}\sum_{k=1}^N q_k]\sup_{0\le t\le T}\|x(t)\|_X\\
&\le \gamma e^{|v|T}\|\phi(0)\|_X+C(1+\|\phi\|_{PC})[(\gamma e^{|v|T}+1)\|(-A)^{-\alpha}\|+M_{1-\alpha}\tau(T)]\\
&\quad + \sup_{0\le t\le T}\|\int_0^t S(t-s)f(s,x_s)ds\|_X+\sup_{0\le t\le T}\|\int_0^t S(t-s)g(s,x_s)dw(s)\|_X.
\end{aligned} \tag{3.5}$$

So, by the linear growth conditions in (H3), Hölder's inequality and Burkholder-Davis-Gundy type of inequality for stochastic convolutions [2], for any $p\ge 2$, there exist a number $c(p,T)>0$ such that

$$\begin{aligned}
&\sup_{0\le t\le T}E\|\Psi x(t)\|_X^p\\
&\le 3^{p-1}[1-C\|(-A)^{-\alpha}\|-CM_{1-\alpha}\tau(T)-\gamma e^{|v|T}\sum_{k=1}^N q_k]^{-p}\\
&\quad\times\Big\{(\gamma e^{|v|T}\|\phi(0)\|_X+C(1+\|\phi\|_{PC})[(\gamma e^{|v|T}+1)\|(-A)^{-\alpha}\|+M_{1-\alpha}\tau(T)])^p\\
&\quad + 2^p\gamma^p e^{p|v|T}(T^{p-1}+c(p,T))C_4^p\int_0^T(1+E\|x_s\|_{PC}^p)ds\Big\}.
\end{aligned} \tag{3.6}$$

This means that $\Psi x(t)\in PC^b_{\mathscr{F}_0}[[-\tau,0],X]$, if $x(t)\in PC^b_{\mathscr{F}_0}[[-\tau,0],X]$. Further, for $x(t),y(t)\in PC$, $0\le t\le T$, we have

$$\begin{aligned}
&\sup_{t\in[0,T]}E\|\Psi x(t)-\Psi y(t)\|_X^p\\
&\le 5^{p-1}\sup_{t\in[0,T]}E\|G(t,x_t)-G(t,y_t)\|_X^p\\
&\quad + 5^{p-1}\sup_{t\in[0,T]}E\|\int_0^t AS(t-s)G(s,x_s)ds-\int_0^t AS(t-s)G(s,y_s)ds\|_X^p
\end{aligned}$$

$$
\begin{aligned}
&+ 5^{p-1} \sup_{t\in[0,T]} E\Big\| \int_0^t S(t-s)f(s,x_s)ds - \int_0^t S(t-s)f(s,y_s)ds\Big\|_X^p \\
&+ 5^{p-1} \sup_{t\in[0,T]} E\Big\| \int_0^t S(t-s)g(s,x_s)dw(s) - \int_0^t S(t-s)g(s,y_s)dw(s)\Big\|_X^p \\
&+ 5^{p-1} \sup_{t\in[0,T]} E\Big\| \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)) - \sum_{0<t_k<t} S(t-t_k)I_k(y(t_k^-))\Big\|_X^p \\
\le{}& 5^{p-1}\|(-A)^{-\alpha}\|^p C_1^p \sup_{t\in[0,T]} E\|x_t - y_t\|_{PC}^p + 5^{p-1}(M_{1-\alpha}^p \tau^p(T)T^{p-1} \\
&+ \gamma^p e^{p|v|T} C_2^p T^{p-1} + \gamma^p e^{p|v|T} C_3^p T^{\frac{p-2}{2}}) \sup_{t\in[0,T]} E\int_0^t \|x_t - y_t\|_{PC}^p ds \\
&+ 5^{p-1} N^{p-1}\gamma^p e^{p|v|T} \sup_{t\in[0,T]} \sum_{k=1}^N E\|I_k(x(t_k^-)) - I_k(y(t_k^-))\|_X^p \\
\le{}& 5^{p-1}\|(-A)^{-\alpha}\|^p C_1^p \sup_{t\in[0,T]} E\|x_t - y_t\|_{PC}^p + 5^{p-1}(M_{1-\alpha}^p \tau^p(T)T^{p-1} \\
&+ \gamma^p e^{p|v|T} C_2^p T^{p-1} + \gamma^p e^{p|v|T} C_3^p T^{\frac{p-2}{2}}) \sup_{t\in[0,T]} E\int_0^t \|x_t - y_t\|_{PC}^p ds \\
&+ 5^{p-1} N^{p-1}\gamma^p e^{p|v|T} (\sum_{k=1}^N q_k^p) \sup_{t\in[0,T]} E\|x(t) - y(t)\|_{PC}^p \\
\le{}& 5^{p-1}\Big[\|(-A)^{-\alpha}\|^p C_1^p + M_{1-\alpha}^p \tau^p(T)T^{p-1} + \gamma^p e^{p|v|T} C_2^p T^{p-1} \\
&+ \gamma^p e^{p|v|T} C_3^p T^{\frac{p-2}{2}} + N^{p-1}\gamma^p e^{p|v|T}(\sum_{k=1}^N q_k^p)\Big] \sup_{t\in[0,T]} E\|x_t - y_t\|_{PC}^p.
\end{aligned}
$$

Hence, we obtain

$$
\sup_{t\in[0,T]} E\|\Psi x_t - \Psi y_t\|_{PC}^p \le L \sup_{t\in[0,T]} E\|x_t - y_t\|_{PC}^p.
$$

For $L < 1$, the mapping Ψ is a contraction mapping. By Banach fixed point theorem, there exists a unique fixed point, which implies system (2.1) has a unique mild solution. The proof is complete. □

Theorem 3.3. *Under assumptions* (H1)–(H4)*, the mild solution of* (2.1) *is exponentially p-stable* $(p \ge 2)$ *provided*

$$
\|I_k(x(t_k^-))\|_X^p \le d_k, \quad d_k \ge 0,\ k = 1,2,\dots,N,
$$

and

$$
\frac{[6^{p-1}C_1^p M_{1-\alpha}^p \Gamma^{p/q}(q\alpha - q + 1)/r^{p\alpha-1} + 6^{p-1}C_2^p\gamma^p/r^{p-1} + 6^{p-1}\gamma^p C_3^p/r^{\frac{p}{2}-1}]}{(1 - 6^{p-1}C_1^p\|(-A)^{-\alpha}\|^p)} > 0, \tag{3.7}
$$

where $M_{1-\alpha} > 0$*,* $r = -v > 0$ *are the constants in Lemma 3.1 (ii) and* $\Gamma(\cdot)$ *is the gamma function.*

Proof. Note that in this situation, for any $\beta \ge 0$, there exist constants $M_\beta > 0$ and $r > 0$ such that

$$
\|(-A)^\beta S(t)\|_X \le M_\beta t^{-\beta} e^{-rt}, \quad t > 0.
$$

It is known that

$$\begin{aligned} x(t) &= S(t)[\phi(0)+G(0,\phi)] - G(t,x_t) - \int_0^t AS(t-s)G(s,x_s)ds \\ &\quad + \int_0^t S(t-s)f(s,x_s)ds + \int_0^t S(t-s)g(s,x_s)dw(s) \\ &\quad + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)). \end{aligned}$$

By Lemma 3.1, (H2), and the fact that $f(t,0)\equiv 0$ almost surely in t, we obtain that for any $t\ge\tau$, $-\tau\le\theta\le 0$,

$$\begin{aligned} &\|x(t+\theta)\|_X \\ &\le \gamma e^{-r(t+\theta)}(\|\phi(0)\| + C_1\|\phi\|_{PC}\|(-A)^{-\alpha}\|) + C_1\|(-A)^{-\alpha}\|\|x_{t+\theta}\|_{PC} \\ &\quad + \int_0^{t+\theta} C_1\frac{M_{1-\alpha}e^{-r(t+\theta-s)}}{(t+\theta-s)^{(1-\alpha)}}\|x_s\|_{PC}ds + \|\int_0^{t+\theta} S(t+\theta-s)f(s,x_s)ds\|_X \\ &\quad + \|\int_0^{t+\theta} S(t+\theta-s)g(s,x_s)dw(s)\|_X + \sum_{k=1}^N \gamma e^{-r(t+\theta-t_k)}\|I_k(x(t_k^-))\|_X. \end{aligned} \tag{3.8}$$

Thus, for any $t\ge\tau$ and $-\tau\le\theta\le 0$, we have

$$\begin{aligned} E\|x(t+\theta)\|_X^p &\le 6^{p-1}\gamma^p e^{-pr(t+\theta)}(\|\phi(0)\| + C_1\|(-A)^{-\alpha}\|\|\phi\|_{PC})^p \\ &\quad + 6^{p-1}C_1^p\|(-A)^{-\alpha}\|^p E\|x_{t+\theta}\|_{PC}^p \\ &\quad + 6^{p-1}E(\int_0^{t+\theta} C_1\frac{M_{1-\alpha}e^{-r(t+\theta-s)}}{(t+\theta-s)^{(1-\alpha)}}\|x_s\|_{PC}ds)^p \\ &\quad + 6^{p-1}E\|\int_0^{t+\theta} S(t+\theta-s)f(s,x_s)ds\|_X^p \\ &\quad + 6^{p-1}E\|\int_0^{t+\theta} S(t+\theta-s)g(s,x_s)dw(s)\|_X^p \\ &\quad + 6^{p-1}N^{p-1}\gamma^p\sum_{k=1}^N e^{-pr(t+\theta-t_k)}E\|I_k(x(t_k^-))\|_X^p. \end{aligned} \tag{3.9}$$

By using Hölder's inequality and Burkholder-Davis-Gundy type of inequality, we have that for any $t\ge\tau$, $-\tau\le\theta\le 0$,

$$\begin{aligned} &E\Big(\int_0^{t+\theta} C_1\frac{M_{1-\alpha}e^{-r(t+\theta-s)}}{(t+\theta-s)^{(1-\alpha)}}\|x_s\|_{PC}ds\Big)^p \\ &= E\Big(\int_0^{t+\theta} C_1\frac{M_{1-\alpha}e^{-(\frac{1}{q}+\frac{1}{p})r(t+\theta-s)}}{(t+\theta-s)^{(1-\alpha)}}\|x_s\|_{PC}ds\Big)^p \\ &\le \Big[\Big(\int_0^{t+\theta}(C_1\frac{M_{1-\alpha}e^{-\frac{r}{q}(t+\theta-s)}}{(t+\theta-s)^{(1-\alpha)}})^q ds\Big)^{1/q}\Big]^p \\ &\quad\times E\Big[\Big(\int_0^{t+\theta}(e^{-\frac{r}{p}(t+\theta-s)}\|x_s\|_{PC})^p ds\Big)^{\frac{1}{p}}\Big]^p \\ &= C_1^p M_{1-\alpha}^p\Big(\int_0^{t+\theta}\frac{e^{-rs}}{s^{q(1-\alpha)}}ds\Big)^{p/q}\int_0^{t+\theta} e^{-r(t+\theta-s)}E\|x_s\|_{PC}^p ds \end{aligned}$$

$$
= C_1^p M_{1-\alpha}^p \Gamma^{p/q}(q\alpha - q + 1)/r^{p\alpha-1}\int_0^{t+\theta} e^{-r(t+\theta-s)}E\|x_s\|_{PC}^p ds,
$$

where $p, q > 1$, $\frac{1}{q} + \frac{1}{p} = 1$.

$$
\begin{aligned}
E\|\int_0^{t+\theta} S(t+\theta-s)f(s,x_s)ds\|_X^p &\le \gamma^p E\|\int_0^{t+\theta} e^{-r(t+\theta-s)}f(s,x_s)ds\|_X^p \\
&= \gamma^p E\|\int_0^{t+\theta} e^{-(\frac{1}{q}+\frac{1}{p})r(t+\theta-s)}f(s,x_s)ds\|_X^p \\
&\le C_2^p\gamma^p/r^{p-1}\int_0^{t+\theta} e^{-r(t+\theta-s)}E\|x_s\|_{PC}^p ds,
\end{aligned} \tag{3.10}
$$

and

$$
\begin{aligned}
&E\|\int_0^{t+\theta} S(t+\theta-s)g(s,x_s)dw(s)\|_X^p \\
&= E[\|\int_0^{t+\theta} S(t+\theta-s)g(s,x_s)dw(s)\|_X^2]^{\frac{p}{2}} \\
&\le \gamma^p E[\int_0^{t+\theta} e^{-r(t+\theta-s)}\|g(s,x_s)\|_\lambda^2 ds]^{\frac{p}{2}} \\
&= \gamma^p E[\int_0^{t+\theta} e^{-(1-\frac{2}{p})r(t+\theta-s)}e^{-\frac{2}{p}r(t+\theta-s)}\|g(s,x_s)\|_\lambda^2 ds]^{\frac{p}{2}} \\
&\le \gamma^p C_3^p/r^{\frac{p}{2}-1}\int_0^{t+\theta} e^{-r(t+\theta-s)}E\|x_s\|_{PC}^p ds,
\end{aligned} \tag{3.11}
$$

which immediately implies

$$
\begin{aligned}
E\|x(t+\theta)\|_X^p &\le 6^{p-1}\gamma^p e^{-pr(t+\theta)}(\|\phi(0)\| + C_1\|(-A)^{-\alpha}\|\|\phi\|_{PC})^p \\
&\quad + 6^{p-1}C_1^p\|(-A)^{-\alpha}\|^p E\|x_{t+\theta}\|_{PC}^p \\
&\quad + [6^{p-1}C_1^p M_{1-\alpha}^p\Gamma^{p/q}(q\alpha-q+1)/r^{p\alpha-1} + 6^{p-1}C_2^p\gamma^p/r^{p-1} \\
&\quad + 6^{p-1}\gamma^p C_3^p/r^{\frac{p}{2}-1}]\int_0^{t+\theta} e^{-r(t+\theta-s)}E\|x_s\|_{PC}^p ds \\
&\quad + 6^{p-1}N^{p-1}\gamma^p e^{-pr(t+\theta)}\sum_{k=1}^N e^{prt_k}d_k;
\end{aligned} \tag{3.12}
$$

that is,

$$
\begin{aligned}
E\|x(t+\theta)\|_X^p &\le 6^{p-1}\gamma^p e^{-pr(t+\theta)}[(\|\phi(0)\| + C_1\|(-A)^{-\alpha}\|\|\phi\|_{PC})^p + d] \\
&\quad + 6^{p-1}C_1^p\|(-A)^{-\alpha}\|^p E\|x_{t+\theta}\|_{PC}^p \\
&\quad + [6^{p-1}C_1^p M_{1-\alpha}^p\Gamma^{p/q}(q\alpha-q+1)/r^{p\alpha-1} + 6^{p-1}C_2^p\gamma^p/r^{p-1} \\
&\quad + 6^{p-1}\gamma^p C_3^p/r^{\frac{p}{2}-1}]\int_0^{t+\theta} e^{-r(t+\theta-s)}E\|x_s\|_{PC}^p ds,
\end{aligned} \tag{3.13}
$$

where $d = N^{p-1}\sum_{k=1}^N e^{prt_k}d_k$.

Therefore, for arbitrary $0 < \epsilon < r$ and $T > \tau$ large enough, we have

$$\begin{aligned}
&\int_\tau^T e^{\epsilon t} E\|x(t+\theta)\|_X^p dt \\
&\le 6^{p-1}\gamma^p[(\|\phi(0)\| + C_1\|(-A)^{-\alpha}\|\|\phi\|_{PC})^p + d]\int_\tau^T e^{-pr(t+\theta)+\epsilon t}dt \\
&\quad + 6^{p-1}C_1^p\|(-A)^{-\alpha}\|^p \int_\tau^T e^{\epsilon t}E\|x_{t+\theta}\|_{PC}^p dt \\
&\quad + [6^{p-1}C_1^p M_{1-\alpha}^p \Gamma^{p/q}(q\alpha - q + 1)/r^{p\alpha-1} + 6^{p-1}C_2^p\gamma^p/r^{p-1} \\
&\quad + 6^{p-1}\gamma^p C_3^p/r^{\frac{p}{2}-1}]\int_\tau^T \int_0^{t+\theta} e^{-r(t+\theta-s)+\epsilon t}E\|x_s\|_{PC}^p \,ds\,dt.
\end{aligned} \tag{3.14}$$

On the other hand, note that for any $-\tau \le \theta \le 0$ and $t \ge \tau$,

$$\begin{aligned}
&\int_\tau^T \int_0^{t+\theta} e^{-r(t+\theta-s)+\epsilon t}E\|x_s\|_{PC}^p \,ds\,dt \\
&= \int_0^{\tau+\theta}\int_\tau^T e^{-r(t+\theta-s)+\epsilon t}E\|x_s\|_{PC}^p \,dt\,ds \\
&\quad + \int_{\tau+\theta}^{T+\theta}\int_{s-\theta}^T e^{-r(t+\theta-s)+\epsilon t}E\|x_s\|_{PC}^p \,dt\,ds \\
&\le \frac{1}{r-\epsilon}\int_0^{\tau+\theta} e^{r(s-\theta)}E\|x_s\|_{PC}^p ds + \frac{1}{r-\epsilon}\int_{\tau+\theta}^{T+\theta} e^{\epsilon(s-\theta)}E\|x_s\|_{PC}^p ds \\
&\le \frac{1}{r-\epsilon}\int_0^{\tau} e^{r(s-\theta)}E\|x_s\|_{PC}^p ds + \frac{1}{r-\epsilon}\int_0^{T} e^{\epsilon(s-\theta)}E\|x_s\|_{PC}^p ds.
\end{aligned} \tag{3.15}$$

Therefore, substituting (3.15) into (3.14) yields

$$\begin{aligned}
&\int_\tau^T e^{\epsilon t} E\|x(t+\theta)\|_X^p dt \\
&\le 6^{p-1}\gamma^p[(\|\phi(0)\| + C_1\|(-A)^{-\alpha}\|\|\phi\|_{PC})^p + d]\int_\tau^T e^{-pr(t+\theta)+\epsilon t}dt \\
&\quad + 6^{p-1}C_1^p\|(-A)^{-\alpha}\|^p \int_\tau^T e^{\epsilon t}E\|x_{t+\theta}\|_{PC}^p dt \\
&\quad + [6^{p-1}C_1^p M_{1-\alpha}^p \Gamma^{p/q}(q\alpha - q + 1)/r^{p\alpha-1} + 6^{p-1}C_2^p\gamma^p/r^{p-1} \\
&\quad + 6^{p-1}\gamma^p C_3^p/r^{\frac{p}{2}-1}]\frac{1}{(r-\epsilon)}\int_0^\tau e^{r(s-\theta)}E\|x_s\|_{PC}^p ds \\
&\quad + [6^{p-1}C_1^p M_{1-\alpha}^p \Gamma^{p/q}(q\alpha - q + 1)/r^{p\alpha-1} + 6^{p-1}C_2^p\gamma^p/r^{p-1} \\
&\quad + 6^{p-1}\gamma^p C_3^p/r^{\frac{p}{2}-1}]\frac{1}{(r-\epsilon)}\int_0^T e^{\epsilon(s-\theta)}E\|x_s\|_{PC}^p ds \\
&\le 6^{p-1}\gamma^p[(\|\phi(0)\| + C_1\|(-A)^{-\alpha}\|\|\phi\|_{PC})^p + d]\int_\tau^T e^{-pr(t-\tau)+\epsilon t}dt \\
&\quad + 6^{p-1}C_1^p\|(-A)^{-\alpha}\|^p e^{\epsilon\tau}\int_\tau^T e^{\epsilon s}E\|x_s\|_{PC}^p ds
\end{aligned}$$

$$
\begin{aligned}
&+ 6^{p-1}C_1^p\|(-A)^{-\alpha}\|^p e^{\epsilon\tau}\int_0^r e^{\epsilon s}E\|x_s\|_{PC}^p ds\\
&+ [6^{p-1}C_1^pM_{1-\alpha}^p\Gamma^{p/q}(q\alpha-q+1)/r^{p\alpha-1} + 6^{p-1}C_2^p\gamma^p/r^{p-1}\\
&+ 6^{p-1}\gamma^pC_3^p/r^{\frac{p}{2}-1}]\frac{1}{(r-\epsilon)}e^{r\tau}\int_0^\tau e^{rs}E\|x_s\|_{PC}^p ds\\
&+ [6^{p-1}C_1^pM_{1-\alpha}^p\Gamma^{p/q}(q\alpha-q+1)/r^{p\alpha-1} + 6^{p-1}C_2^p\gamma^p/r^{p-1}\\
&+ 6^{p-1}\gamma^pC_3^p/r^{\frac{p}{2}-1}]\frac{1}{(r-\epsilon)}e^{\epsilon\tau}\int_0^T e^{\epsilon s}E\|x_s\|_{PC}^p ds;
\end{aligned}
$$

i. e.,

$$
\int_\tau^T e^{\epsilon t}E\|x_t\|_{PC}^p dt \le L_1(\epsilon) + L_2(\epsilon)\int_0^T e^{\epsilon s}E\|x_s\|_{PC}^p ds, \tag{3.16}
$$

where

$$
\begin{aligned}
&L_1(\epsilon)\\
&= (1-6^{p-1}C_1^p\|(-A)^{-\alpha}\|^p e^{\epsilon\tau})^{-1}\Big\{6^{p-1}\gamma^p[(\|\phi(0)\| + C_1\|(-A)^{-\alpha}\|\|\phi\|_{PC})^p + d]\\
&\quad\times\int_\tau^T e^{-pr(t-\tau)+\epsilon t}dt + 6^{p-1}C_1^p\|(-A)^{-\alpha}\|^p e^{\epsilon\tau}\int_0^\tau e^{\epsilon s}E\|x_s\|_{PC}^p ds\\
&\quad+ [6^{p-1}C_1^pM_{1-\alpha}^p\Gamma^{p/q}(q\alpha-q+1)/r^{p\alpha-1} + 6^{p-1}C_2^p\gamma^p/r^{p-1}\\
&\quad+ 6^{p-1}\gamma^pC_3^p/r^{\frac{p}{2}-1}]\frac{1}{(r-\epsilon)}e^{r\tau}\int_0^\tau e^{rs}E\|x_s\|_{PC}^p ds\Big\},
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
L_2(\epsilon) &= (1-6^{p-1}C_1^p\|(-A)^{-\alpha}\|^p e^{\epsilon\tau})^{-1}\Big\{[6^{p-1}C_1^pM_{1-\alpha}^p\Gamma^{p/q}(q\alpha-q+1)/r^{p\alpha-1}\\
&\quad+ 6^{p-1}C_2^p\gamma^p/r^{p-1} + 6^{p-1}\gamma^pC_3^p/r^{\frac{p}{2}-1}]\frac{1}{(r-\epsilon)}\Big\}e^{\epsilon r}.
\end{aligned}
\tag{3.18}
$$

Hence, from (3.16) we have

$$
\int_0^T e^{\epsilon t}E\|x_t\|_{PC}^p dt \le L_1(\epsilon) + \int_0^r e^{\epsilon t}E\|x_t\|_{PC}^p dt + L_2(\epsilon)\int_0^T e^{\epsilon t}E\|x_t\|_X^p dt. \tag{3.19}
$$

On the other hand, by virtue of (3.7), it is possible to choose a suitable $0<\epsilon<r$ small enough such that $L_2(\epsilon)<1$, for such an $\epsilon>0$, we may deduce that there exists a real number $L_3(\epsilon)>0$ such that

$$
\int_0^T e^{\epsilon t}E\|x_t\|_{PC}^p dt \le (1-L_2(\epsilon))^{-1}(L_1(\epsilon) + \int_0^r e^{\epsilon t}E\|x_t\|_{PC}^p dt) := L_3(\epsilon) < \infty. \tag{3.20}
$$

From (3.13), for any $-\tau\le\theta\le 0$, $t\ge\tau$, we have

$$
\begin{aligned}
&E\|x(t+\theta)\|_X^p\\
&\le 6^{p-1}\gamma^p e^{-pr(t+\theta)}[(\|\phi(0)\| + C_1\|(-A)^{-\alpha}\|\|\phi\|_{PC})^p + d]e^{-\epsilon(t-\tau)}\\
&\quad+ 6^{p-1}C_1^p\|(-A)^{-\alpha}\|^p E\|x_{t+\theta}\|_{PC}^p\\
&\quad+ [6^{p-1}C_1^pM_{1-\alpha}^p\Gamma^{p/q}(q\alpha-q+1)/r^{p\alpha-1} + 6^{p-1}C_2^p\gamma^p/r^{p-1}\\
&\quad+ 6^{p-1}\gamma^pC_3^p/r^{\frac{p}{2}-1}]L_3(\epsilon)e^{-\epsilon(t-\tau)}.
\end{aligned}
\tag{3.21}
$$

Now, we proceed with our arguments by considering two possible situations for any fixed $t \geq 0$.

Firstly, suppose that $\sup_{-\tau\leq\theta\leq 0} E\|x_{t+\theta}\|_{PC}^p = E\|x_t\|_{PC}^p$, then from (3.21) we have

$$\begin{aligned} E\|x_t\|_X^p &\leq 6^{p-1}\gamma^p e^{-pr(t+\theta)}[(\|\phi(0)\| + C_1\|(-A)^{-\alpha}\|\|\phi\|_{PC})^p + d]e^{-\epsilon(t-\tau)} \\ &\quad + 6^{p-1}C_1^p\|(-A)^{-\alpha}\|^p E\|x_t\|_{PC}^p \\ &\quad + [6^{p-1}C_1^p M_{1-\alpha}^p \Gamma^{p/q}(q\alpha - q + 1)/r^{p\alpha-1} + 6^{p-1}C_2^p\gamma^p/r^{p-1} \\ &\quad + 6^{p-1}\gamma^p C_3^p/r^{\frac{p}{2}-1}]L_3(\epsilon)e^{-\epsilon(t-\tau)}, \end{aligned} \tag{3.22}$$

which immediately yields

$$\begin{aligned} E\|x_t\|_{PC}^p &\leq (1 - 6^{p-1}C_1^p\|(-A)^{-\alpha}\|^p)^{-1}[6^{p-1}\gamma^p e^{-pr(t+\theta)} \\ &\quad \times [(\|\phi(0)\| + C_1\|(-A)^{-\alpha}\|\|\phi\|_{PC})^p + d]e^{\epsilon\tau} \\ &\quad + [6^{p-1}C_1^p M_{1-\alpha}^p \Gamma^{p/q}(q\alpha - q + 1)/r^{p\alpha-1} + 6^{p-1}C_2^p\gamma^p/r^{p-1} \\ &\quad + 6^{p-1}\gamma^p C_3^p/r^{\frac{p}{2}-1}]L_3(\epsilon)e^{\epsilon\tau}]e^{-\epsilon t} := L_4(\epsilon)e^{-\epsilon t}. \end{aligned} \tag{3.23}$$

On the other hand, for this fixed $t \geq 0$, if $\sup_{-\tau\leq\theta\leq 0} E\|x_{t+\theta}\|_{PC}^p = E\|x_{t-\tau}\|_{PC}^p$, then from (3.21) we have

$$\begin{aligned} E\|x_t\|_{PC}^p &\leq 6^{p-1}\gamma^p e^{-pr(t+\theta)}[(\|\phi(0)\| + C_1\|(-A)^{-\alpha}\|\|\phi\|_{PC})^p + d]e^{-\epsilon(t-\tau)} \\ &\quad + 6^{p-1}C_1^p\|(-A)^{-\alpha}\|^p E\|x_{t-r}\|_{PC}^p \\ &\quad + [6^{p-1}C_1^p M_{1-\alpha}^p \Gamma^{p/q}(q\alpha - q + 1)/r^{p\alpha-1} + 6^{p-1}C_2^p\gamma^p/r^{p-1} \\ &\quad + 6^{p-1}\gamma^p C_3^p/r^{\frac{p}{2}-1}]L_3(\epsilon)e^{-\epsilon(t-\tau)}. \end{aligned} \tag{3.24}$$

Hence, in this situation we have

$$E\|x_t\|_{PC}^p \leq L_5(\epsilon)e^{-\epsilon t} + L_6(\epsilon)E\|x_{t-\tau}\|_{PC}^p, \tag{3.25}$$

where

$$\begin{aligned} L_5(\epsilon) &= 6^{p-1}\gamma^p e^{-pr(t+\theta)}[(\|\phi(0)\| + C_1\|(-A)^{-\alpha}\|\|\phi\|_{PC})^p + d]e^{\epsilon\tau} \\ &\quad + [6^{p-1}C_1^p M_{1-\alpha}^p \Gamma^{p/q}(q\alpha - q + 1)/r^{p\alpha-1} + 6^{p-1}C_2^p\gamma^p/r^{p-1} \\ &\quad + 6^{p-1}\gamma^p C_3^p/r^{\frac{p}{2}-1}]L_3(\epsilon)e^{\epsilon\tau}, \end{aligned} \tag{3.26}$$

and

$$L_6(\epsilon) = 6^{p-1}C_1^p\|(-A)^{-\alpha}\|^p. \tag{3.27}$$

Combining (3.23) with (3.25), for any $t \geq \tau$, we have

$$E\|x_t\|_{PC}^p \leq \max\{L_4(\epsilon)e^{-\epsilon t}, L_5(\epsilon)e^{-\epsilon t} + L_6(\epsilon)E\|x_{t-\tau}\|_{PC}^p\}. \tag{3.28}$$

In other words, for any $t \geq 2\tau$, we have

$$E\|x_{t-\tau}\|_{PC}^p \leq \max\{L_4(\epsilon)e^{-\epsilon(t-\tau)}, L_5(\epsilon)e^{-\epsilon(t-\tau)} + L_6(\epsilon)E\|x_{t-2\tau}\|_{PC}^p\}. \tag{3.29}$$

In view of (3.28) and (3.29), we obtain that for any $t \geq 2\tau$,

$$\begin{aligned} &E\|x_t\|_{PC}^p \\ &\leq \max\Big\{L_4(\epsilon)e^{-\epsilon t}, [L_5(\epsilon)+L_6(\epsilon)L_4(\epsilon)e^{-\epsilon\tau}]e^{-\epsilon t}, \\ &\quad \sum_{i=1}^{2}(L_5(\epsilon)L_6^{i-1}(\epsilon)e^{\epsilon(i-1)\tau})e^{-\epsilon t} + L_6^2(\epsilon)E\|x_{t-2\tau}\|_{PC}^p\Big\} \\ &\quad := \max\{\hat{L}_2(\epsilon)e^{-\epsilon t}, \sum_{i=1}^{2}(L_5(\epsilon)L_6^{i-1}(\epsilon)e^{\epsilon(i-1)\tau})e^{-\epsilon t} + L_6^2(\epsilon)E\|x_{t-2\tau}\|_{PC}^p\}, \end{aligned} \tag{3.30}$$

where

$$\hat{L}_2(\epsilon) = \max\{L_4(\epsilon), L_5(\epsilon) + L_6(\epsilon)L_4(\epsilon)e^{-\epsilon\tau}\}.$$

By induction, there exists a positive number $\hat{L}_m(\epsilon)$ such that for any $t \geq \tau$,

$$\begin{aligned} &E\|x_t\|_{PC}^p \\ &\leq \max\big\{\hat{L}_m(\epsilon)e^{-\epsilon t}, \sum_{i=1}^{m}(L_5(\epsilon)L_6^{i-1}(\epsilon)e^{\epsilon(i-1)\tau})e^{-\epsilon t} + L_6^m(\epsilon)E\|x_{t-m\tau}\|_{PC}^p\big\}, \end{aligned} \tag{3.31}$$

where m is the positive integer such that $0 \leq t - m\tau < \tau$. Obviously, to obtain the desired exponential stability of (2.1), we need to consider only the term

$$E\|x_t\|_{PC}^p \leq \sum_{i=1}^{m}(L_5(\epsilon)L_6^{i-1}(\epsilon)e^{\epsilon(i-1)\tau})e^{-\epsilon t} + L_6^m(\epsilon)E\|x_{t-m\tau}\|_{PC}^p. \tag{3.32}$$

In fact, choose a suitable $\epsilon > 0$ small enough such that $L_6(\epsilon)e^{\epsilon\tau} < 1$, then we may deduce from (3.32) that

$$\begin{aligned} &E\|x_t\|_{PC}^p \\ &\leq \Big(L_5(\epsilon)\sum_{i=1}^{m}(L_6^{i-1}(\epsilon)e^{\epsilon(i-1)\tau})\Big)e^{-\epsilon t} + L_6^m(\epsilon)(\|\phi\|_X^p + E\|x_\tau\|_{PC}^p) \\ &\leq \Big(L_5(\epsilon)\sum_{i=1}^{m}(L_6^{i-1}(\epsilon)e^{\epsilon(i-1)\tau})\Big)e^{-\epsilon t} + L_6^m(\epsilon)(L_5(\epsilon)e^{-\epsilon\tau} + (1+L_6(\epsilon))\|\phi\|_{PC}^p) \\ &\leq \Big(L_5(\epsilon)\sum_{i=1}^{m}(L_6^{i-1}(\epsilon)e^{\epsilon(i-1)\tau})\Big)e^{-\epsilon t} + (L_5(\epsilon)e^{-\epsilon\tau} + (1+L_6(\epsilon))\|\phi\|_{PC}^p)(L_6(\epsilon))^{t/r} \\ &\leq \Big(L_5(\epsilon)\sum_{i=1}^{\infty}(L_6^{i-1}(\epsilon)e^{\epsilon(i-1)\tau})\Big)e^{-\epsilon t} + (L_5(\epsilon)e^{-\epsilon\tau} + (1+L_6(\epsilon))\|\phi\|_{PC}^p)(L_6(\epsilon))^{t/r} \\ &= \frac{L_5(\epsilon)}{1-L_6(\epsilon)e^{\epsilon\tau}}e^{-\epsilon t} + (L_5(\epsilon)e^{-\epsilon\tau} + (1+L_6(\epsilon))\|\phi\|_{PC}^p)(L_6(\epsilon))^{t/r} \\ &= \frac{L_5(\epsilon)}{1-L_6(\epsilon)e^{\epsilon\tau}}e^{-\epsilon t} + (L_5(\epsilon)e^{-\epsilon\tau} + (1+L_6(\epsilon))\|\phi\|_{PC}^p)e^{-\epsilon_1 t}, \end{aligned} \tag{3.33}$$

where $0 < \epsilon_1 = -\ln L_6(\epsilon)/\tau$.

Hence, from (3.31) and (3.33), ones may deduce that there exist numbers $C(\phi) > 0$ and $\epsilon > 0$ such that

$$E\|x(t;\phi)\|_X^p \leq E\|x_t\|_{PC}^p \leq C(\phi)e^{-\epsilon t}, \quad \text{for any } t \geq \tau.$$

The proof is complete. □

4. An illustrative example

In this section, we illustrate the obtained result. Let $X = L^2([0,\pi])$ and A be defined as $Af = f''$ with domain $D(A) = \{f(\cdot) \in L^2([0,\pi]) : f'' \in L^2([0,\pi]),\ f(0) = f(\pi) = 0\}$.

It is well known that the analytic semigroup $S(t)(t \geq 0)$, generated by the operator A on X which is a separable Hilbert spaces. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2, n \in N$, and corresponding normalized eigenfunctions given by $e_n(\xi) := (2/\pi)^{1/2}\sin(n\xi)$.

In addition to the above, the following conditions hold:

(a) $\{e_n : n \in N\}$ is an orthonormal basis of X.

(b) If $f \in X$, then $S(t)f = \sum_{n=1}^{\infty} e^{-n^2t}\langle f, e_n\rangle e_n$ and $Af = -\sum_{n=1}^{\infty} n^2\langle f, e_n\rangle e_n$ for every $f \in D(A)$.

(c) For $f \in X$, $(-A)^{-\frac{1}{2}}f = \sum_{n=1}^{\infty}\frac{1}{n}\langle f, e_n\rangle e_n$.

(d) The operator $(-A)^{1/2} : D((-A)^{1/2}) \subseteqq X \to X$ is given by
$$(-A)^{1/2}f = \sum_{n=1}^{\infty} n\langle f, e_n\rangle e_n, \quad \forall f \in D((-A)^{1/2}),$$
where $D((-A)^{1/2}) = \{f(\cdot) \in X : \sum_{n=1}^{\infty} n\langle f, e_n\rangle e_n \in X\}$.

Consider the stochastic neutral partial functional differential equation
$$\begin{gathered}
\frac{\partial}{\partial t}\Big[x(t,\xi) + \int_{-\tau}^{0}\int_0^{\pi} b(s,\eta,\xi)x(t+s,\eta)d\eta ds\Big] \\
= \frac{\partial^2}{\partial\xi^2}x(t,\xi) + p_0(\xi)x(t,\xi) + \int_{-\tau}^{0} p(s)x(t+s,\xi)ds + p_1\cos x(t-\tau,\xi)dB_t, \\
t \neq t_k,\ \xi \in I = [0,\pi], \\
\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad t = t_k,\ k = 1,2,\dots,N, \\
x(t,0) = x(t,\pi) = 0, \quad t \in R, \\
x(s,\xi) = \phi(s,\xi), \quad \phi(\cdot,\xi) \in PC^b_{\mathscr{F}_0}[[-\tau,0],R], \quad \phi(s,\cdot) \in L^2([0,\pi]),
\end{gathered} \tag{4.1}$$
where B_t is the standard one-dimensional Wiener process, the functions p_0, p are continuous, $p_1 \geq 0$. And $I_k(x(t_k)) = e^{-k}x(t_k)$. We also assume that

(i) The function $b(\cdot)$ is (Lebesgue) measurable and
$$\int_0^{\pi}\int_{-\tau}^{0}\int_0^{\pi} b^2(\theta,\eta,\xi)\,d\eta d\theta\,d\xi < \infty.$$

(ii) The function $\frac{\partial^i}{\partial\xi^i}b(\theta,\eta,\xi)$, $i = 1,2$, are measurable, $b(\theta,\eta,0) = (\theta,\eta,\pi) = 0$ for every (θ,η) and
$$L_1 := \max\Big\{\int_0^{\pi}\int_{-\tau}^{0}\int_0^{\pi}\Big(\frac{\partial^i}{\partial\xi^i}b(\theta,\eta,\xi)\Big)^2 d\eta d\theta d\xi : i = 0,1,2\Big\} < \infty.$$

Define $G, f : C([-\tau,0];X)$ by setting
$$G(t,x)(\xi) := B(x)(\xi) := \int_{-\tau}^{0}\int_0^{\pi} b(s,\eta,\xi)x(s,\eta)\,d\eta\,ds,$$

$$f(t,x)(\xi) := p_0(\xi)x(t,\xi) + \int_{-\tau}^{0} p(s)x(s,\xi)ds.$$

From (i), it is clear that B is a bounded linear operator on X. Furthermore, from the definition of B and (ii), we obtain

$$\langle B(x), e_n\rangle = \int_0^\pi \Big[\int_{-\tau}^0\int_0^\pi b(s,\eta,\xi)x(s,\eta)d\eta ds\Big](\frac{2}{\pi})^{1/2}\sin(n\xi)d\xi \tag{4.2}$$

$$= \frac{1}{n}(\frac{2}{\pi})^{1/2}\langle\int_{-\tau}^0\int_0^\pi \frac{\partial}{\partial\xi}b(s,\eta,\xi)x(s,\eta)d\eta ds, \cos(n\xi)\rangle \tag{4.3}$$

$$= \frac{1}{n}(\frac{2}{\pi})^{1/2}\langle B_1(x), \cos(n\xi)\rangle, \tag{4.4}$$

where $B_1(x) = \int_{-\tau}^0\int_0^\pi \frac{\partial}{\partial\xi}b(s,\eta,\xi)x(s,\eta)d\eta\, ds$. By using (ii) again, we obtain that $B_1 : X \to X$ is a bounded linear operator with $\|B_1\| \le L_1$, so $\|(-A)^{1/2}B(x)\| = \|B_1(x)\|$. Hence $B(x) \in D[(-A)^{1/2}]$, and $\|(-A)^{1/2}B\| \le L_1$. Similarly, we can prove that f is a bounded linear operator on X,

$$\|f(t,\cdot)\| \le \sup_{t\in\mathbb{R}}\|p_0(t)\| + \sqrt{\tau(\int_{-\tau}^0 p^2(s)ds)}, \quad t\in\mathbb{R}.$$

By simple parameters computation, we can easily verify that all conditions of Theorems 3.2 and 3.3 are satisfied. Therefore, (4.1) has a unique mild solution, which is exponential p-stable.

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