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EXISTENCE AND REGULARITY OF ENTROPY SOLUTIONS FOR STRONGLY NONLINEAR $p(x)$-ELLIPTIC EQUATIONS

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Abstract. This article is devoted to study the existence of solutions for the strongly nonlinear $p(x)$-elliptic problem

$$\begin{gathered} -\operatorname{div} a(x,u,\nabla u) + g(x,u,\nabla u) = f - \operatorname{div}\phi(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

with $f \in L^1(\Omega)$ and $\phi \in C^0(\mathbb{R}^N)$, also we will give some regularity results for these solutions.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ with $N \geq 2$. For $2 - \frac{1}{N} < p < N$, Boccardo and Gallouët [6] studied the problem

$$\begin{gathered} Au = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where $Au = -\operatorname{div} a(x,u,\nabla u)$ is a Leray-Lions operator from $W_0^{1,p}(\Omega)$ into its dual, and f is a bounded Radon measure on Ω. They proved the existence of solutions $u \in W_0^{1,q}(\Omega)$ for all $1 < q < \bar{q} = \frac{N(p-1)}{N-1}$. Moreover, they showed the critical regularity $u \in W_0^{1,\bar{q}}(\Omega)$ under the assumption $f \log(1+|f|) \in L^1(\Omega)$. Boccardo [5] studied the existence of entropy solutions for the problem

$$\begin{gathered} -\operatorname{div} a(x,u,\nabla u) = f - \operatorname{div}\phi(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $f \in L^1(\Omega)$ and $\phi \in C^0(\mathbb{R}^N)$, he proved the solutions existence and some regularity results, under the above assumptions. Aharouch and Azroul [1] studied the problem (1.1) in Oricz-sobolev spaces. They proved the existence of entropy solutions $u \in W_0^{1,q}(\Omega)$. In the case of $p = N$, they assume in addition that there exists an N-function H such that $H(t^N)$ is equivalent to $M(t)$. Kbiri Alaoui, Meskine and Souissi [12] proved the critical regularity $W_0^{1,\bar{q}}(\Omega)$ of solutions for nonlinear elliptic problems with right-hand side in $L\log^\alpha L(\Omega)$ and $\alpha \geq \frac{N-1}{N}$. Also they proved some regularity results when $\alpha < \frac{N-1}{N}$.

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In this article, we consider the problem

$$
\begin{gathered}
-\operatorname{div} a(x,u,\nabla u)+g(x,u,\nabla u)=f-\operatorname{div}\phi(u) \quad \text{in } \Omega,\\
u=0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.2}
$$

where the right hand side is assumed to satisfy

$$
f\in L^1(\Omega) \quad \text{and} \quad \phi\in C^0(\mathbb{R}^N). \tag{1.3}
$$

We will study the strongly nonlinear boundary-value problem (1.2) in the framework of variable exponent Sobolev spaces, we will prove the existence of entropy solutions and some $\bar{q}(x)$-regularity results.

Recall that, since no growth hypothesis is assumed on ϕ, the term $\operatorname{div}\phi(v)$ may be meaningless, even as a distribution for a function $v\in W_0^{1,r(x)}(\Omega)$, $r(x)>1$ (see [5] and [7] for the case of constant exponent).

Definition 1.1. For $k>0$ and $s\in\mathbb{R}$, the truncation function $T_k(.)$ is defined by

$$
T_k(s)=\begin{cases} s & \text{if } |s|\le k,\\ k\frac{s}{|s|} & \text{if } |s|>k.\end{cases}
$$

This article is organized as follows. In the section 2 we recall some important definitions and results of variable exponent Lebesgue and Sobolev spaces. We introduce in the section 3 some assumptions on $a(x,s,\xi)$ and $g(x,s,\xi)$ for which our problem has a solutions. The section 4 contains some important lemmas useful to prove our main results. The section 5 will be devoted to show the existence of entropy solutions for the problem (1.2), also we will give some important $L^{\bar{q}(x)}$-regularity results for these solutions (the case $p=2-1/N$ and $p=N$ are excluded).

2. Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N\ge 2$), we say that a real-valued continuous function $p(.)$ is log-Hölder continuous in Ω if

$$
|p(x)-p(y)|\le \frac{C}{|\log|x-y||} \quad \forall x,y\in\overline{\Omega} \text{ such that } |x-y|<\frac{1}{2},
$$

with possible different constant C. We denote

$$
C_+(\overline{\Omega})=\{\text{log-Hölder continuous function } p:\overline{\Omega}\to\mathbb{R} \text{ with } 1<p_-\le p_+<N\},
$$

where

$$
p_-=\min\{p(x):x\in\overline{\Omega}\} \quad p_+=\max\{p(x):x\in\overline{\Omega}\}.
$$

We define the variable exponent Lebesgue space for $p\in C_+(\overline{\Omega})$ by

$$
L^{p(x)}(\Omega)=\{u:\Omega\to\mathbb{R} \text{ measurable}: \int_\Omega |u(x)|^{p(x)}dx<\infty\},
$$

the space $L^{p(x)}(\Omega)$ under the norm

$$
\|u\|_{p(x)}=\inf\big\{\lambda>0:\int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx\le 1\big\}
$$

is a uniformly convex Banach space, and therefore reflexive. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$ (see [10, 14]).

Proposition 2.1 (Generalized Hölder inequality [10, 14]). *(i) For any functions $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$|\int_\Omega uv dx| \leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\|u\|_{p(x)}\|v\|_{p'(x)}.$$

(ii) For all $p_1, p_2 \in C_+(\overline{\Omega})$ such that $p_1(x) \leq p_2(x)$ a.e. in Ω, we have $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.

Proposition 2.2 ([10, 14]). *If we denote*

$$\rho(u) = \int_\Omega |u|^{p(x)} dx \quad \forall u \in L^{p(x)}(\Omega),$$

then, the following assertions hold

(i) $\|u\|_{p(x)} < 1$ *(resp, $= 1$, > 1) if and only if* $\rho(u) < 1$ *(resp, $= 1$, > 1);*
(ii) $\|u\|_{p(x)} > 1$ *implies* $\|u\|_{p(x)}^{p_-} \leq \rho(u) \leq \|u\|_{p(x)}^{p_+}$*, and* $\|u\|_{p(x)} < 1$ *implies* $\|u\|_{p(x)}^{p_+} \leq \rho(u) \leq \|u\|_{p(x)}^{p_-}$*;*
(iii) $\|u\|_{p(x)} \to 0$ *if and only if* $\rho(u) \to 0$*, and* $\|u\|_{p(x)} \to \infty$ *if and only if* $\rho(u) \to \infty$*.*

Now, we define the variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \text{ and } |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)} \quad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, and we define the Sobolev exponent by $p^*(x) = \frac{Np(x)}{N-p(x)}$ for $p(x) < N$.

Proposition 2.3 ([10, 11]). (i) *Assuming $1 < p_- \leq p_+ < \infty$, the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*
(ii) *If $q \in C_+(\bar{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \Omega$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact.*
(iii) *Poincaré inequality: there exists a constant $C > 0$, such that*

$$\|u\|_{p(x)} \leq C\|\nabla u\|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

(vi) *Sobolev-Poincaré inequality : there exists an other constant $C > 0$, such that*

$$\|u\|_{p*(x)} \leq C\|\nabla u\|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

Remark 2.4. By (iii) of Proposition 2.3, we deduce that $\|\nabla u\|_{p(x)}$ and $\|u\|_{1,p(x)}$ are equivalent norms in $W_0^{1,p(x)}(\Omega)$.

Definition 2.5 ([8]). We denote the dual of the Sobolev space $W_0^{1,p(x)}(\Omega)$ by $W^{-1,p'(x)}(\Omega)$, and for each $F \in W^{-1,p'(x)}(\Omega)$ there exists $f_0, f_1, \dots, f_N \in L^{p'(x)}(\Omega)$ such that $F = f_0 + \sum_{i=1}^N \frac{\partial f_i}{\partial x_i}$. Moreover, for all $u \in W_0^{1,p(x)}(\Omega)$ we have

$$\langle F, u\rangle = \int_\Omega f_0 u dx - \sum_{i=1}^N \int_\Omega f_i \frac{\partial u}{\partial x_i} dx.$$

and we define a norm on the dual space by

$$\|F\|_{-1,p'(x)} \simeq \sum_{i=0}^{N} \|f_i\|_{p'(x)}.$$

Now, we define

$$T_0^{1,p(x)}(\Omega) := \{\text{measurable function } u \text{ such that } T_k(u) \in W_0^{1,p(x)}(\Omega) \quad \forall k > 0\}.$$

Proposition 2.6. *Let $u \in T_0^{1,p(x)}(\Omega)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that*

$$v.\chi_{\{|u|\le k\}} = \nabla T_k(u) \quad \text{for a.e. } x \in \Omega \text{ and for all } k > 0.$$

We will define the gradient of u as the function v, and we will denote it by $v = \nabla u$.

Definition 2.7. A measurable function u is an entropy solution of the Dirichlet problem (1.2) if

$$\begin{gathered} T_k(u) \in W_0^{1,p(x)}(\Omega) \quad \forall k > 0, \\ \int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)dx + \int_\Omega g(x,u,\nabla u)T_k(u-\varphi)dx \\ \le \int_\Omega f T_k(u-\varphi)dx + \int_\Omega \phi(u)\nabla T_k(u-\varphi)dx \end{gathered}$$

for all $\varphi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$.

Lemma 2.8. *Let $\lambda \in \mathbb{R}$ and let u and v be two functions which are finite almost everywhere, and which belong to $\mathcal{T}_0^{1,p(x)}(\Omega)$, then*

$$\nabla(u+\lambda v) = \nabla u + \lambda \nabla v \qquad \text{a.e. in } \Omega,$$

where ∇u, ∇v and $\nabla(u+\lambda v)$ are the gradients of u, v and $u+\lambda v$ introduced in the Definition 2.7.

Proof. Let $E_n = \{|u| \le n\} \cap \{|v| \le n\}$. We have $T_n(u) = u$ and $T_n(v) = v$ in E_n, then for every $k > 0$

$$T_k(T_n(u) + \lambda T_n(v)) = T_k(u + \lambda v) \quad \text{a.e. in } E_n,$$

and therefore, since both functions belong to $W_0^{1,p(x)}(\Omega)$,

$$\nabla T_k(T_n(u) + \lambda T_n(v)) = \nabla T_k(u+\lambda v) \quad \text{a.e. in } E_n. \tag{2.1}$$

Since $T_n(u)$ and $T_n(v)$ belong to $W_0^{1,p(x)}(\Omega)$, we have by using a classical property of the truncates functions in $W_0^{1,p(x)}(\Omega)$, and the definition of ∇u and ∇v,

$$\begin{aligned} \nabla T_k(T_n(u) + \lambda T_n(v)) &= \chi_{\{|T_n(u)+\lambda T_n(v)|\le k\}}(\nabla T_n(u) + \lambda \nabla T_n(v)) \\ &= \chi_{\{|T_n(u)+\lambda T_n(v)|\le k\}}(\nabla u.\chi_{\{|u|\le n\}} + \lambda \nabla v.\chi_{\{|v|\le n\}}) \end{aligned}$$

a.e. in Ω. Therefore,

$$\nabla T_k(T_n(u) + \lambda T_n(v)) = \chi_{\{|u+\lambda v|\le k\}}(\nabla u + \lambda \nabla v) \quad \text{a.e. in } E_n. \tag{2.2}$$

On the other hand, by definition of $\nabla(u+\lambda v)$,

$$\nabla T_k(u+\lambda v) = \chi_{\{|u+\lambda v|\le k\}}\nabla(u+\lambda v) \quad \text{a.e. in } E_n. \tag{2.3}$$

Putting together (2.1), (2.2) and (2.3), we obtain

$$\chi_{\{|u+\lambda v|\le k\}}\nabla(u+\lambda v) = \chi_{\{|u+\lambda v|\le k\}}(\nabla u + \lambda \nabla v) \quad \text{a.e. in } E_n. \tag{2.4}$$

We have $\cup_{n\in\mathbb{N}} E_n$ (resp. $\cup_{k\in\mathbb{N}}\{|u+\lambda v|\le k\}$) differs at most from Ω by a set of zero Lebesgue measure, since u and v are almost everywhere finite, then (2.4) holds almost everywhere in Ω. which conclude the proved of Lemma 2.8.

3. Essential assumption

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N\ge 2)$ and $p\in C_+(\bar{\Omega})$, we consider a Leray-Lions operator from $W_0^{1,p(x)}(\Omega)$ into its dual $W^{-1,p'(x)}(\Omega)$, defined by the formula

$$Au = -\operatorname{div}\ a(x,u,\nabla u) \tag{3.1}$$

where $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function (measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$, and continuous with respect to (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$ for almost every x in Ω) which satisfies the following conditions

$$|a(x,s,\xi)|\le \beta(K(x)+|s|^{p(x)-1}+|\xi|^{p(x)-1}), \tag{3.2}$$

$$a(x,s,\xi)\xi\ge \alpha|\xi|^{p(x)}, \tag{3.3}$$

$$[a(x,s,\xi)-a(x,s,\overline{\xi})](\xi-\overline{\xi})>0 \quad \text{for all } \xi\ne\overline{\xi} \text{ in } \mathbb{R}^N, \tag{3.4}$$

for a.e. $x\in\Omega$, all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, where $K(x)$ is a positive function lying in $L^{p'(x)}(\Omega)$ and $\alpha,\beta>0$.

The nonlinear term $g(x,s,\xi)$ is a Carathéodory function which satisfies

$$g(x,s,\xi)s\ge 0, \tag{3.5}$$

$$|g(x,s,\xi)|\le b(|s|)(c(x)+|\xi|^{p(x)}), \tag{3.6}$$

where $b:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous, nondecreasing function, and $c:\Omega\to\mathbb{R}^+$ with $c\in L^1(\Omega)$. We consider the problem

$$\begin{gathered} -\operatorname{div} a(x,u,\nabla u)+g(x,u,\nabla u)=f-\operatorname{div}\phi(u) \quad \text{in } \Omega,\\ u=0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.7}$$

with

$$f\in L^1(\Omega) \quad \text{and} \quad \phi\in C^0(\mathbb{R}^N). \tag{3.8}$$

The symbol $\rightharpoonup$ will denote the weak convergence, and the constants C_i, $i=1,2,\dots$ used in each steps of proof are independent. □

4. Some technical Lemmas

Lemma 4.1 ([2]). *Let $g\in L^{r(x)}(\Omega)$ and $g_n\in L^{r(x)}(\Omega)$ with $\|g_n\|_{r(x)}\le C$ for $1<r(x)<\infty$. If $g_n(x)\to g(x)$ a.e. on Ω, then $g_n\rightharpoonup g$ in $L^{r(x)}(\Omega)$.*

Lemma 4.2. *Let $u\in W_0^{1,p(x)}(\Omega)$ then $T_k(u)\in W_0^{1,p(x)}(\Omega)$ with $k>0$. Moreover, we have $T_k(u)\to u$ in $W_0^{1,p(x)}(\Omega)$ as $k\to\infty$.*

Proof. Let $k>0$ and $T_k:\mathbb{R}\to\mathbb{R}$,

$$T_k(s)=\begin{cases} s & \text{if } |s|\le k,\\ k.\text{sign}(s) & \text{if } |s|>k,\end{cases}$$

then for all $u\in W_0^{1,p(x)}(\Omega)$ we have $T_k(u)\in W_0^{1,p(x)}(\Omega)$, and

$$\int_\Omega |T_k(u)-u|^{p(x)}dx+\int_\Omega |\nabla T_k(u)-\nabla u|^{p(x)}dx$$

$$
\begin{aligned}
&= \int_{\{|u|\le k\}} |T_k(u)-u|^{p(x)}dx + \int_{\{|u|>k\}} |T_k(u)-u|^{p(x)}dx \\
&\quad + \int_{\{|u|\le k\}} |\nabla T_k(u)-\nabla u|^{p(x)} + \int_{\{|u|>k\}} |\nabla T_k(u)-\nabla u|^{p(x)}dx \\
&= \int_{\{|u|>k\}} |T_k(u)-u|^{p(x)}dx + \int_{\{|u|>k\}} |\nabla u|^{p(x)}dx.
\end{aligned}
$$

Since $T_k(u)\to u$ as $k\to\infty$ and by using the dominated convergence theorem, we have

$$
\int_{\{|u|>k\}} |T_k(u)-u|^{p(x)}dx + \int_{\{|u|>k\}} |\nabla u|^{p(x)}dx \to 0 \quad \text{as } k\to\infty.
$$

Finally $\|T_k(u)-u\|_{W_0^{1,p(x)}(\Omega)} \to 0$ as $k\to\infty$. □

Lemma 4.3 ([3]). *Let $p(\cdot)$ be a continuous function in $C_+(\overline{\Omega})$ and u a function in $W_0^{1,p(x)}(\Omega)$. Suppose $2-\frac{1}{N}<p_-\le p_+<N$, and that there exists a constant c_1 such that*

$$
\int_{\{k\le|u|\le k+1\}} |\nabla u|^{p(x)}dx \le c_1 \quad \forall k>0.
$$

Then there exists a constant $c_2>0$, depending on c_1, such that

$$
\|u\|_{1,q(x)} \le c_2,
$$

for all continuous functions $q(\cdot)$ on $\overline{\Omega}$ satisfying

$$
1\le q(x) < \frac{N(p(x)-1)}{N-1} \quad \text{for all } x\in\overline{\Omega}.
$$

Lemma 4.4. *Assume (3.2)-(3.4), and let $(u_n)_n$ be a sequence in $W_0^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and*

$$
\int_\Omega [a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u)]\nabla(u_n-u)dx \to 0, \tag{4.1}
$$

then $u_n\to u$ in $W_0^{1,p(x)}(\Omega)$ for a subsequence.

Proof. Let $D_n=[a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u)]\nabla(u_n-u)$, thanks to (3.4) we have D_n is a positive function, and by (4.1), $D_n\to 0$ in $L^1(\Omega)$ as $n\to\infty$.

Since $u_n\rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ then $u_n\to u$ a.e. in Ω, and since $D_n\to 0$ a.e. in Ω, there exists a subset B in Ω with measure zero such that for all $x\in\Omega\backslash B$,

$$
|u(x)|<\infty, \quad |\nabla u(x)|<\infty, \quad K(x)<\infty, \quad u_n\to u, \quad D_n\to 0.
$$

Taking $\xi_n=\nabla u_n$ and $\xi=\nabla u$, we have

$$
\begin{aligned}
D_n(x) &= [a(x,u_n,\xi_n)-a(x,u_n,\xi)](\xi_n-\xi) \\
&= a(x,u_n,\xi_n)\xi_n + a(x,u_n,\xi)\xi - a(x,u_n,\xi_n)\xi - a(x,u_n,\xi)\xi_n \\
&\ge \alpha|\xi_n|^{p(x)} + \alpha|\xi|^{p(x)} - \beta(K(x)+|u_n|^{p(x)-1}+|\xi_n|^{p(x)-1})|\xi| \\
&\quad - \beta(K(x)+|u_n|^{p(x)-1}+|\xi|^{p(x)-1})|\xi_n| \\
&\ge \alpha|\xi_n|^{p(x)} - C_x(1+|\xi_n|^{p(x)-1}+|\xi_n|),
\end{aligned}
$$

where C_x depending on x, without dependence on n. (since $u_n(x) \to u(x)$ then $(u_n)_n$ is bounded), we obtain

$$D_n(x) \geq |\xi_n|^{p(x)}\big(\alpha - \frac{C_x}{|\xi_n|^{p(x)}} - \frac{C_x}{|\xi_n|} - \frac{C_x}{|\xi_n|^{p(x)-1}}\big),$$

by the standard argument $(\xi_n)_n$ is bounded almost everywhere in Ω, (Indeed, if $|\xi_n| \to \infty$ in a measurable subset $E \in \Omega$ then

$$\lim_{n\to\infty}\int_\Omega D_n(x)dx \geq \lim_{n\to\infty}\int_E |\xi_n|^{p(x)}\big(\alpha - \frac{C_x}{|\xi_n|^{p(x)}} - \frac{C_x}{|\xi_n|} - \frac{C_x}{|\xi_n|^{p(x)-1}}\big)dx = \infty,$$

which is absurd since $D_n \to 0$ in $L^1(\Omega)$).

Let ξ^* an accumulation point of $(\xi_n)_n$, we have $|\xi^*| < \infty$ and by the continuity of $a(.,.,.)$ we obtain,

$$[a(x,u(x),\xi^*) - a(x,u(x),\xi)](\xi^* - \xi) = 0,$$

thanks to (3.4) we have $\xi^* = \xi$, the uniqueness of the accumulation point implies that $\nabla u_n \to \nabla u$ a.e. in Ω. since $(a(x,u_n,\nabla u_n))_n$ is bounded in $(L^{p'(x)}(\Omega))^N$ and $a(x,u_n,\nabla u_n) \to a(x,u,\nabla u)$ a.e. in Ω, by the Lemma 4.1, we can establish that

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u) \quad \text{in } (L^{p'(x)}(\Omega))^N.$$

Let us taking $\bar{y}_n = a(x,u_n,\nabla u_n)\nabla u_n$ and $\bar{y} = a(x,u,\nabla u)\nabla u$, then $\bar{y}_n \to \bar{y}$ in $L^1(\Omega)$, according to the condition (3.3) we have

$$\alpha|\nabla u_n|^{p(x)} \leq a(x,u_n,\nabla u_n)\nabla u_n,$$

Let $z_n = \nabla u_n$, $z = \nabla u$ and $y_n = \frac{\bar{y}_n}{\alpha}$, $y = \frac{\bar{y}}{\alpha}$, in view of the Fatou Lemma, we obtain

$$\int_\Omega 2.y dx \leq \liminf_{n\to\infty}\int_\Omega (y_n + y - |z_n - z|^{p(x)})dx,$$

then $0 \leq -\limsup_{n\to\infty}\int_\Omega |z_n - z|^{p(x)}dx$, and since

$$0 \leq \liminf_{n\to\infty}\int_\Omega |z_n - z|^{p(x)}dx \leq \limsup_{n\to\infty}\int_\Omega |z_n - z|^{p(x)}dx \leq 0,$$

it follows that $\int_\Omega |\nabla u_n - \nabla u|^{p(x)}dx \to 0$ as $n \to \infty$, and we get

$$\nabla u_n \to \nabla u \quad in \quad (L^{p(x)}(\Omega))^N$$

we deduce that

$$u_n \to u \quad \text{in } W_0^{1,p(x)}(\Omega),$$

which completes our proof. □

Now, we consider $\phi_n(s) = \phi(T_n(s))$ with $\phi \in C^0(\mathbb{R}^N)$ and

$$g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1 + \frac{1}{n}|g(x,s,\xi)|}$$

such that $g(x,s,\xi)$ satisfies (3.5) – (3.6), note that

$$g_n(x,s,\xi)s \geq 0, \quad |g_n(x,s,\xi)| \leq |g(x,s,\xi)|, \quad |g_n(x,s,\xi)| \leq n \quad \forall n \in \mathbb{N}^*.$$

We define the operator $G_n : W_0^{1,p(x)}(\Omega) \to W^{-1,,p'(x)}(\Omega)$, by

$$\langle G_n u, v\rangle = \int_\Omega g_n(x,u,\nabla u)v dx \quad \forall v \in W_0^{1,p(x)}(\Omega).$$

Thanks to the Hölder inequality, we have that for all $u, v \in W_0^{1,p(x)}(\Omega)$,

$$
\begin{aligned}
&|\int_\Omega g_n(x,u,\nabla u)v dx| \\
&\le (\frac{1}{p_-} + \frac{1}{p'_-})\|g_n(x,u,\nabla u)\|_{p'(x)}\|v\|_{p(x)} \\
&\le (\frac{1}{p_-} + \frac{1}{p'_-})\Big(\int_\Omega |g_n(x,u,\nabla u)|^{p'(x)} dx + 1\Big)^{\frac{1}{p'_-}}\|v\|_{1,p(x)} \\
&\le (\frac{1}{p_-} + \frac{1}{p'_-})\Big(\int_\Omega n^{p'(x)} dx + 1\Big)^{\frac{1}{p'_-}}\|v\|_{1,p(x)} \\
&\le (\frac{1}{p_-} + \frac{1}{p'_-})\big(n^{p'_+}.\operatorname{meas}(\Omega) + 1\big)^{\frac{1}{p'_-}}\|v\|_{1,p(x)} \\
&\le C_0\|v\|_{1,p(x)},
\end{aligned}
\tag{4.2}
$$

and we define the operator $R_n = \operatorname{div}\phi_n : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$, such that

$$
\langle R_n(u), v\rangle = \langle \operatorname{div}\phi_n(u), v\rangle = -\int_\Omega \phi_n(u)\nabla v dx \quad \forall u, v \in W_0^{1,p(x)}(\Omega),
$$

we have

$$
\begin{aligned}
|\int_\Omega \phi_n(u)\nabla v dx| &\le (\frac{1}{p_-} + \frac{1}{p'_-})\|\phi_n(u)\|_{p'(x)}\|\nabla v\|_{p(x)} \\
&\le (\frac{1}{p_-} + \frac{1}{p'_-})\Big(\int_\Omega |\phi_n(u)|^{p'(x)} dx + 1\Big)^{\frac{1}{p'_-}}\|v\|_{1,p(x)} \\
&\le (\frac{1}{p_-} + \frac{1}{p'_-})\Big(\sup_{|s|\le n}(|\phi(s)|+1)^{p'_+}\operatorname{meas}(\Omega) + 1\Big)^{1/p'_-}\|v\|_{1,p(x)} \\
&\le C_1\|v\|_{1,p(x)}.
\end{aligned}
\tag{4.3}
$$

Lemma 4.5. *The operator $B_n = A + G_n + R_n$ is pseudo-monotone from $W_0^{1,p(x)}(\Omega)$ into $W^{-1,p'(x)}(\Omega)$. Moreover, B_n is coercive in the following sense*

$$
\frac{\langle B_n v, v\rangle}{\|v\|_{1,p(x)}} \to +\infty \quad \textit{as} \quad \|v\|_{1,p(x)} \to +\infty \quad \textit{for} \quad v \in W_0^{1,p(x)}(\Omega).
$$

Proof. Using Hölder's inequality and the growth condition (3.2), we can show that the operator A is bounded, and by using (4.2) and (4.3) we conclude that B_n bounded. For the coercivity, we have for any $u \in W_0^{1,p(x)}(\Omega)$,

$$
\begin{aligned}
\langle B_n u, u\rangle &= \langle Au, u\rangle + \langle G_n u, u\rangle + \langle R_n u, u\rangle \\
&= \int_\Omega a(x,u,\nabla u)\nabla u dx + \int_\Omega g_n(x,u,\nabla u)u dx - \int_\Omega \phi_n(u)\nabla u dx \\
&\ge \alpha\int_\Omega |\nabla u|^{p(x)} dx - (\frac{1}{p_-} + \frac{1}{p'_-})\|\phi_n(u)\|_{p'(x)}\|\nabla u\|_{p(x)} \\
&\ge \alpha\|\nabla u\|_{p(x)}^{\delta} - C_1.\|u\|_{1,p(x)} \quad (\text{using}(4.3)) \\
&\ge \alpha'\|u\|_{1,p(x)}^{\delta} - C_1.\|u\|_{1,p(x)}, \quad (\text{using the Poincaré inequality})
\end{aligned}
$$

with

$$\delta = \begin{cases} p_- & \text{if } \|\nabla u\|_{p(x)} > 1, \\ p_+ & \text{if } \|\nabla u\|_{p(x)} \le 1, \end{cases}$$

then, we obtain

$$\frac{\langle B_n u, u\rangle}{\|u\|_{1,p(x)}} \to +\infty \quad \text{as } \|u\|_{1,p(x)} \to +\infty.$$

It remains to show that B_n is pseudo-monotone. Let $(u_k)_k$ a sequence in $W_0^{1,p(x)}(\Omega)$ such that

$$\begin{gathered} u_k \rightharpoonup u \quad \text{in } W_0^{1,p(x)}(\Omega), \\ B_n u_k \rightharpoonup \chi \quad \text{in } W^{-1,p'(x)}(\Omega), \\ \limsup_{k\to\infty} \langle B_n u_k, u_k\rangle \le \langle \chi, u\rangle. \end{gathered} \tag{4.4}$$

We will prove that

$$\chi = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \to \langle \chi, u\rangle \quad \text{as } k \to +\infty.$$

Firstly, since $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$, then $u_k \to u$ in $L^{p(x)}(\Omega)$ for a subsequence still denoted $(u_k)_k$.

We have $(u_k)_k$ is a bounded sequence in $W_0^{1,p(x)}(\Omega)$, then by the growth condition $(a(x, u_k, \nabla u_k))_k$ is bounded in $(L^{p'(x)}(\Omega))^N$, therefore, there exists a function $\varphi \in (L^{p'(x)}(\Omega))^N$ such that

$$a(x, u_k, \nabla u_k) \rightharpoonup \varphi \quad \text{in } (L^{p'(x)}(\Omega))^N \text{ as } k \to \infty. \tag{4.5}$$

Similarly, since $(g_n(x, u_k, \nabla u_k))_k$ is bounded in $L^{p'(x)}(\Omega)$, then there exists a function $\psi_n \in L^{p'(x)}(\Omega)$ such that

$$g_n(x, u_k, \nabla u_k) \rightharpoonup \psi_n \quad \text{in } L^{p'(x)}(\Omega) \text{ as } k \to \infty, \tag{4.6}$$

and since $\phi_n = \phi \circ T_n$ is a bounded continuous function and $u_k \to u$ in $L^{p(x)}(\Omega)$, it follows

$$\phi_n(u_k) \to \phi_n(u) \quad \text{in } (L^{p'(x)}(\Omega))^N \text{ as } k \to \infty. \tag{4.7}$$

For all $v \in W_0^{1,p(x)}(\Omega)$, we have

$$\begin{aligned} \langle \chi, v\rangle &= \lim_{k\to\infty} \langle B_n u_k, v\rangle \\ &= \lim_{k\to\infty} \int_\Omega a(x, u_k, \nabla u_k)\nabla v dx + \lim_{k\to\infty} \int_\Omega g_n(x, u_k, \nabla u_k) v dx \\ &\quad - \lim_{k\to\infty} \int_\Omega \phi_n(u_k)\nabla v dx \\ &= \int_\Omega \varphi\nabla v dx + \int_\Omega \psi_n v dx - \int_\Omega \phi_n(u)\nabla v dx. \end{aligned} \tag{4.8}$$

Using (4.4) and (4.8), we obtain

$$\begin{aligned} &\limsup_{k\to\infty} \langle B_n(u_k), u_k\rangle \\ &= \limsup_{k\to\infty} \Big\{ \int_\Omega a(x, u_k, \nabla u_k)\nabla u_k dx + \int_\Omega g_n(x, u_k, \nabla u_k) u_k dx - \int_\Omega \phi_n(u_k)\nabla u_k dx \Big\} \\ &\le \int_\Omega \varphi\nabla u dx + \int_\Omega \psi_n u dx - \int_\Omega \phi_n(u)\nabla u dx, \end{aligned} \tag{4.9}$$

thanks to (4.6) and (4.7), we have

$$\int_\Omega g_n(x,u_k,\nabla u_k)u_k dx \to \int_\Omega \psi_n u dx, \quad \int_\Omega \phi_n(u_k)\nabla u_k dx \to \int_\Omega \phi_n(u)\nabla u dx; \tag{4.10}$$

therefore,

$$\limsup_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k dx \le \int_\Omega \varphi\nabla u dx. \tag{4.11}$$

On the other hand, using (3.4), we have

$$\int_\Omega (a(x,u_k,\nabla u_k)-a(x,u_k,\nabla u))(\nabla u_k-\nabla u)dx\,.\ge 0, \tag{4.12}$$

Then

$$\begin{aligned}&\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k dx\\ &\ge -\int_\Omega a(x,u_k,\nabla u)\nabla u dx + \int_\Omega a(x,u_k,\nabla u_k)\nabla u dx + \int_\Omega a(x,u_k,\nabla u)\nabla u_k dx,\end{aligned}$$

and by (4.5), we get

$$\liminf_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k dx \ge \int_\Omega \varphi\nabla u dx,$$

this implies, thanks to (4.11), that

$$\lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k dx = \int_\Omega \varphi\nabla u dx. \tag{4.13}$$

By combining of (4.8), (4.10) and (4.13), we deduce that

$$\langle B_n u_k, u_k\rangle \to \langle \chi, u\rangle \quad \text{as } k\to+\infty.$$

Now, by (4.13) we can obtain

$$\lim_{k\to+\infty}\int_\Omega (a(x,u_k,\nabla u_k)-a(x,u_k,\nabla u))(\nabla u_k-\nabla u)dx = 0,$$

in view of the Lemma 4.4, we obtain

$$u_k \to u, \quad W_0^{1,p(x)}(\Omega), \quad \nabla u_k \to \nabla u \quad \text{a.e. in } \Omega,$$

then

$$a(x,u_k,\nabla u_k) \rightharpoonup a(x,u,\nabla u), \quad \phi_n(u_k)\to\phi_n(u) \quad \text{in } (L^{p'(x)}(\Omega))^N,$$

and

$$g_n(x,u_k,\nabla u_k) \rightharpoonup g_n(x,u,\nabla u) \quad \text{in } L^{p'(x)}(\Omega),$$

we deduce that $\chi = B_n u$, which completes the proof. □

5. Main results

In the sequel we assume that Ω is an open bounded subset of $\mathbb{R}^N$ ($N \geq 2$), and let $p(.) \in C_+(\overline{\Omega})$. We will prove the following existence results

Theorem 5.1. *Assuming that* (3.2)-(3.6) *hold,* $p(.) \in C_+(\overline{\Omega})$*,* $f \in L^1(\Omega)$ *and* $\phi \in C^0(\mathbb{R}^N)$*, then the problem*

$$\begin{gathered} T_k(u) \in W_0^{1,p(x)}(\Omega) \quad \forall k > 0, \\ \int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)dx + \int_\Omega g(x,u,\nabla u)T_k(u-\varphi)dx \\ \leq \int_\Omega fT_k(u-\varphi)dx + \int_\Omega \phi(u)\nabla T_k(u-\varphi)dx, \quad \forall \varphi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega), \end{gathered} \tag{5.1}$$

has at least one solution.

The above theorem is prove in the following 5 steps.

Step 1: Approximate problems. Let $(f_n)_n$ be a sequence in $W^{-1,p'(x)}(\Omega) \cap L^1(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$ with $\|f_n\|_1 \leq \|f\|_1$ and we consider the approximate problem

$$\begin{gathered} Au_n + g_n(x,u_n,\nabla u_n) = f_n - \operatorname{div}\phi_n(u_n) \\ u_n \in W_0^{1,p(x)}(\Omega), \end{gathered} \tag{5.2}$$

with $\phi_n(s) = \phi(T_n(s))$ and $g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}$. In view of the Lemma 4.5, there exists at least one weak solution $u_n \in W_0^{1,p(x)}(\Omega)$ of the problem (5.2), (cf. [13]).

Step 2: A priori estimates. Taking $T_k(u_n)$ as a test function in (5.2), we obtain

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n)dx \\ &= \int_\Omega f_nT_k(u_n)dx + \int_\Omega \phi_n(u_n)\nabla T_k(u_n)dx. \end{aligned} \tag{5.3}$$

Thanks to (3.3) and Young's inequality, we obtain

$$\begin{aligned} &\alpha\int_\Omega |\nabla T_k(u_n)|^{p(x)}dx \\ &\leq \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n)dx \\ &= \int_\Omega f_nT_k(u_n)dx + \int_\Omega \phi_n(T_k(u_n))\nabla T_k(u_n)dx \\ &\leq k\int_\Omega |f_n|dx + \int_\Omega \frac{|\phi_n(T_k(u_n))|}{(\frac{\alpha}{2}p(x))^{\frac{1}{p(x)}}}(\frac{\alpha}{2}p(x))^{\frac{1}{p(x)}}|\nabla T_k(u_n)|dx \\ &\leq k\|f_n\|_1 + \int_\Omega \frac{|\phi_n(T_k(u_n))|^{p'(x)}}{p'(x)(\frac{\alpha}{2}p(x))^{\frac{p'(x)}{p(x)}}}dx + \int_\Omega \frac{\frac{\alpha}{2}p(x)|\nabla T_k(u_n)|^{p(x)}}{p(x)}dx \\ &\leq k\|f\|_1 + C_2\int_\Omega |\phi_n(T_k(u_n))|^{p'(x)}dx + \frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n)|^{p(x)}dx, \end{aligned} \tag{5.4}$$

and since

$$\begin{aligned}\int_\Omega |\phi_n(T_k(u_n))|^{p'(x)}dx &\leq \int_\Omega \sup_{|s|\leq k} |\phi_n(s)|^{p'(x)}dx\\ &\leq \int_\Omega \sup_{|s|\leq n} |\phi(s)|^{p'(x)}dx\\ &\leq \big(\sup_{|s|\leq n} |\phi(s)|+1\big)^{p'_+}.\operatorname{meas}(\Omega),\end{aligned}$$

by (5.4), we obtain

$$\frac{\alpha}{2}\|\nabla T_k(u_n)\|^{\gamma}_{p(x)} \leq \frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n)|^{p(x)}dx \leq k\|f\|_1 + C_3,$$

with

$$\gamma = \begin{cases} p_+ & \text{if } \|\nabla T_k(u_n)\|_{p(x)} \leq 1,\\ p_- & \text{if } \|\nabla T_k(u_n)\|_{p(x)} > 1,\end{cases}$$

we deduce that

$$\|\nabla T_k(u_n)\|_{p(x)} \leq C_4 k^{\frac{1}{\gamma}} \quad \text{for all } k \geq 1, \tag{5.5}$$

where C_4 is a constant that does not depend on k.

Now, we show that $(u_n)_n$ is a Cauchy sequence in measure. Indeed, we have

$$\begin{aligned} k\operatorname{meas}\{|u_n|>k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|dx \leq \int_\Omega |T_k(u_n)|dx\\ &\leq \big(\frac{1}{p_-}+\frac{1}{p'_-}\big)\|1\|_{p'(x)}\|T_k(u_n)\|_{p(x)}\\ &\leq \big(\frac{1}{p_-}+\frac{1}{p'_-}\big)(\operatorname{meas}(\Omega)+1)^{\frac{1}{p'_-}}\|T_k(u_n)\|_{p(x)}\\ &\leq C_5 k^{\frac{1}{\gamma}},\end{aligned}$$

according to the Poincaré inequality and (5.5). Therefore,

$$\operatorname{meas}\{|u_n|>k\} \leq C_5\frac{1}{k^{1-\frac{1}{\gamma}}} \to 0 \quad \text{as } k\to\infty. \tag{5.6}$$

Since for all $\delta>0$,

$$\begin{aligned}&\operatorname{meas}\{|u_n-u_m|>\delta\}\\ &\leq \operatorname{meas}\{|u_n|>k\}+\operatorname{meas}\{|u_m|>k\}+\operatorname{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\},\end{aligned}$$

using (5.6), we get that for all $\varepsilon>0$, there exists $k_0>0$ such that

$$\operatorname{meas}\{|u_n|>k\} \leq \frac{\varepsilon}{3}, \quad \operatorname{meas}\{|u_m|>k\} \leq \frac{\varepsilon}{3} \quad \forall k\geq k_0(\varepsilon), \tag{5.7}$$

On the other hand, by (5.5), the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,p(x)}(\Omega)$, then there exists a subsequence still denoted $(T_k(u_n))_n$ such that

$$T_k(u_n) \rightharpoonup \eta_k \quad \text{in } W_0^{1,p(x)}(\Omega) \quad \text{as } n\to\infty.$$

and by the compact embedding, we obtain

$$T_k(u_n) \to \eta_k \quad \text{in } L^{p(x)}(\Omega) \text{ and a.e. in } \Omega.$$

Therefore, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω, then for all $k>0$ and $\delta,\varepsilon>0$ there exists $n_0=n_0(k,\delta,\varepsilon)$ such that

$$\operatorname{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\}\leq\frac{\varepsilon}{3}\quad\forall m,n\geq n_0. \tag{5.8}$$

Combining (5.7) and (5.8), we obtain that for all $\delta,\varepsilon>0$, there exists $n_0=n_0(\delta,\varepsilon)$ such that

$$\operatorname{meas}\{|u_n-u_m|>\delta\}\leq\varepsilon\quad\forall n,m\geq n_0,$$

it follows that $(u_n)_n$ is a Cauchy sequence in measure, then there exists a subsequence still denoted $(u_n)_n$ such that

$$u_n\to u\quad\text{a.e. in }\Omega.$$

We obtain

$$\begin{gathered}T_k(u_n)\rightharpoonup T_k(u)\quad\text{in } W_0^{1,p(x)}(\Omega)\\ T_k(u_n)\to T_k(u)\quad\text{in } L^{p(x)}(\Omega)\text{ and a.e. in }\Omega.\end{gathered}\tag{5.9}$$

Step 3: Convergence of the gradient. In the sequel, we denote by $\varepsilon_i(n)$ $i=1,2,\dots$ various functions of real numbers which converge to 0 as n tends to infinity. Let $\varphi_k(s)=s\exp(\gamma s^2)$ where $\gamma=\big(\frac{b(k)}{2\alpha}\big)^2$, it is obvious that

$$\varphi_k'(s)-\frac{b(k)}{\alpha}|\varphi_k(s)|\geq\frac{1}{2}\qquad\forall s\in\mathbb{R},$$

we consider $h>k>0$ and $M=4k+h$, we set

$$\omega_n=T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u)).$$

Taking $\varphi_k(\omega_n)$ as a test function in (5.2), we obtain

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\varphi_k'(\omega_n)\nabla\omega_n dx+\int_\Omega g_n(x,u_n,\nabla u_n)\varphi_k(\omega_n)dx\\ &=\int_\Omega f_n\varphi_k(\omega_n)dx+\int_\Omega\phi_n(u_n)\varphi_k'(\omega_n)\nabla\omega_n dx,\end{aligned}$$

it is easy to see that $\nabla\omega_n=0$ on $\{|u_n|>M\}$, and since $g_n(x,u_n,\nabla u_n)\varphi_k(\omega_n)\geq 0$ on $\{|u_n|>k\}$, we have

$$\begin{aligned}&\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\nabla\omega_n dx+\int_{\{|u_n|\leq k\}}g_n(x,u_n,\nabla u_n)\varphi_k(\omega_n)dx\\ &\leq\int_\Omega f_n\varphi_k(\omega_n)dx+\int_{\{|u_n|\leq M\}}\phi_n(T_M(u_n))\varphi_k'(\omega_n)\nabla\omega_n dx.\end{aligned}\tag{5.10}$$

We have

$$\begin{aligned}&\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\nabla\omega_n dx\\ &=\int_{\{|u_n|\leq k\}}a(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)\nabla T_{2k}(u_n-T_k(u))dx\\ &\quad+\int_{\{|u_n|>k\}}a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\nabla T_{2k}(u_n-T_h(u_n)\\ &\quad+T_k(u_n)-T_k(u))dx.\end{aligned}\tag{5.11}$$

On the one hand, since $|u_n - T_k(u)| \le 2k$ on $\{|u_n| \le k\}$, we have

$$\begin{aligned}
&\int_{\{|u_n|\le k\}} a(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)\nabla T_{2k}(u_n - T_k(u))dx \\
&= \int_{\{|u_n|\le k\}} a(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)\nabla (T_k(u_n) - T_k(u))dx \\
&= \int_{\Omega} a(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)\nabla (T_k(u_n) - T_k(u))dx \\
&\quad - \int_{\{|u_n|> k\}} a(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)\nabla (T_k(u_n) - T_k(u))dx.
\end{aligned} \tag{5.12}$$

Since $1 \le \varphi_k'(\omega_n) \le \varphi_k'(2k)$, it follows that

$$\begin{aligned}
&- \int_{\{|u_n|> k\}} a(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)\nabla (T_k(u_n) - T_k(u))dx \\
&= \int_{\{|u_n|> k\}} a(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)\nabla T_k(u)dx \\
&\le \varphi_k'(2k)\int_{\{|u_n|> k\}} |a(x,T_k(u_n),\nabla T_k(u_n))||\nabla T_k(u)|dx,
\end{aligned}$$

and since $(|a(x,T_k(u_n),\nabla T_k(u_n))|)_n$ is bounded in $L^{p'(x)}(\Omega)$, then there exists $\vartheta \in L^{p'(x)}(\Omega)$ such that

$$|a(x,T_k(u_n),\nabla T_k(u_n))| \rightharpoonup \vartheta \quad \text{in} \quad L^{p'(x)}(\Omega),$$

then

$$\int_{\{|u_n|> k\}} |a(x,T_k(u_n),\nabla T_k(u_n))||\nabla T_k(u)|dx \to \int_{\{|u|> k\}} \vartheta|\nabla T_k(u)|dx = 0,$$

and we obtain

$$\int_{\{|u_n|> k\}} a(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)\nabla (T_k(u_n) - T_k(u))dx = \varepsilon_0(n), \tag{5.13}$$

with $\varepsilon_0(n)$ tend to 0 as $n \to \infty$.

On the other hand, for the second term on the right hand side of (5.11), taking $z_n = u_n - T_h(u_n) + T_k(u_n) - T_k(u)$,

$$\begin{aligned}
&\int_{\{|u_n|> k\}} a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))dx \\
&= \int_{\{|u_n|> k\}\cap\{|z_n|\le 2k\}} a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\nabla (u_n - T_h(u_n) + T_k(u_n) \\
&\quad - T_k(u))dx \\
&= \int_{\{|u_n|> k\}\cap\{|z_n|\le 2k\}} a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\nabla (u_n - T_k(u)).\chi_{\{|u_n|>h\}}dx \\
&\quad - \int_{\{|u_n|> k\}\cap\{|z_n|\le 2k\}} a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\nabla T_k(u).\chi_{\{|u_n|\le h\}}dx \\
&\ge - \int_{\{|u_n|> k\}} |a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|\varphi_k'(\omega_n)dx.
\end{aligned} \tag{5.14}$$

By combining (5.11)-(5.13) and (5.14), we obtain

$$\begin{aligned}&\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\nabla\omega_n dx\\ &\geq \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)\nabla(T_k(u_n)-T_k(u))dx\\ &\quad -\int_{\{|u_n|>k\}}|a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|\varphi_k'(\omega_n)dx-\varepsilon_0(n),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))\varphi_k'(\omega_n)dx\\ &\leq \int_{\{|u_n|>k\}}|a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|\varphi_k'(\omega_n)dx\\ &\quad+\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\nabla\omega_n dx\\ &\quad-\int_\Omega a(x,T_k(u_n),\nabla T_k(u))(\nabla T_k(u_n)-\nabla T_k(u))\varphi_k'(\omega_n)dx+\varepsilon_0(n).\end{aligned}$$

We obtain

$$\begin{aligned}&\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\varphi_k'(\omega_n)(\nabla T_k(u_n)-\nabla T_k(u))dx\\ &\leq \varphi_k'(2k)\int_{\{|u_n|>k\}}|a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|dx\\ &+\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\nabla\omega_n dx\\ &+\varphi_k'(2k)\int_\Omega |a(x,T_k(u_n),\nabla T_k(u))||\nabla T_k(u_n)-\nabla T_k(u)|dx+\varepsilon_0(n).\end{aligned} \tag{5.15}$$

Now, we study each terms on the right hand side of the above inequality. For the first term, we have $(|a(x,T_M(u_n),\nabla T_M(u_n))|)_n$ is bounded in $L^{p'(x)}(\Omega)$, and since

$$|\nabla T_k(u)|^{p(x)}\chi_{\{|u_n|>k\}}\leq |\nabla T_k(u)|^{p(x)},$$

and

$$|\nabla T_k(u)|^{p(x)}\chi_{\{|u_n|>k\}}\to 0,\quad \text{a.e. in } \Omega \text{ as } n\to\infty,$$

by the Lebesgue dominated convergence theorem, we deduce that

$$|\nabla T_k(u)|\chi_{\{|u_n|>k\}}\to 0,\quad \text{in } L^{p(x)}(\Omega) \text{ as } n\to\infty,$$

which implies that the first term in the right hand side of (5.15) tends to 0 as n tends to ∞, and we can write

$$\varphi_k'(2k)\int_{\{|u_n|>k\}}|a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|dx=\varepsilon_1(n). \tag{5.16}$$

For the third term on the right-hand side of (5.15), we have

$$|a(x,T_k(u_n),\nabla T_k(u))|\to |a(x,T_k(u),\nabla T_k(u))|\quad \text{in } L^{p'(x)}(\Omega) \text{ as } n\to\infty,$$

and since $\nabla T_k(u_n)$ tends weakly to $\nabla T_k(u)$ in $(L^{p(x)}(\Omega))^N$, we obtain

$$\varphi_k'(2k)\int_\Omega |a(x,T_k(u_n),\nabla T_k(u))||\nabla T_k(u_n)-\nabla T_k(u)|dx\to 0\quad \text{as } n\to\infty,$$

then

$$\varphi_k'(2k)\int_\Omega |a(x,T_k(u_n),\nabla T_k(u))||\nabla T_k(u_n)-\nabla T_k(u)|dx=\varepsilon_2(n). \tag{5.17}$$

By (5.15) we conclude that

$$\begin{aligned}&\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\varphi_k'(\omega_n)(\nabla T_k(u_n)-\nabla T_k(u))dx\\&\le \int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\nabla\omega_n dx+\varepsilon_3(n).\end{aligned} \tag{5.18}$$

Now, we turn to the second term on the left-hand side of (5.10); by (3.6) we have

$$\begin{aligned}&\Big|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\varphi_k(\omega_n)dx\Big|\\
&\le \int_{\{|u_n|\le k\}} b(|u_n|)(c(x)+|\nabla T_k(u_n)|^{p(x)})|\varphi_k(\omega_n)|dx\\
&\le b(k)\int_{\{|u_n|\le k\}} c(x)|\varphi_k(\omega_n)|dx\\
&\quad+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\varphi_k(\omega_n)|dx\\
&\le b(k)\int_{\{|u_n|\le k\}} c(x)|\varphi_k(\omega_n)|dx+\frac{b(k)}{\alpha}\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))\\
&\quad-a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))|\varphi_k(\omega_n)|dx\\
&\quad+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u))(\nabla T_k(u_n)-\nabla T_k(u))|\varphi_k(\omega_n)|dx\\
&\quad+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)|\varphi_k(\omega_n)|dx.\end{aligned}$$

Then

$$\begin{aligned}&\frac{b(k)}{\alpha}\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)\\
&-\nabla T_k(u))|\varphi_k(\omega_n)|dx\\
&\ge \Big|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\varphi_k(\omega_n)dx\Big|-b(k)\int_{\{|u_n|\le k\}} c(x)|\varphi_k(\omega_n)|dx\\
&\quad-\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u))(\nabla T_k(u_n)-\nabla T_k(u))|\varphi_k(\omega_n)|dx\\
&\quad-\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)|\varphi_k(\omega_n)|dx.\end{aligned} \tag{5.19}$$

We have

$$\int_{\{|u_n|\le k\}} c(x)|\varphi_k(\omega_n)|dx\to\int_{\{|u|\le k\}} c(x)|\varphi_k(T_{2k}(u-T_h(u)))|dx=0\quad\text{as } n\to\infty. \tag{5.20}$$

Concerning the third term on the right hand side of (5.19), we have

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u))(\nabla T_k(u_n)-\nabla T_k(u))|\varphi_k(\omega_n)|dx$$

$$\le \varphi_k(2k)\int_\Omega |a(x,T_k(u_n),\nabla T_k(u))||\nabla T_k(u_n)-\nabla T_k(u)|dx,$$

and by (5.17), we deduce that

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u))(\nabla T_k(u_n)-\nabla T_k(u))|\varphi_k(\omega_n)|dx \to 0 \quad \text{as } n\to\infty. \tag{5.21}$$

For the last term of right hand side of (5.19), we have $(a(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $(L^{p'(x)}(\Omega))^N$, then there exists $\varphi \in (L^{p'(x)}(\Omega))^N$ such that

$$a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varphi$$

in $(L^{p'(x)}(\Omega))^N$, and since

$$\nabla T_k(u)|\varphi_k(\omega_n)| \to \nabla T_k(u)|\varphi_k(T_{2k}(u-T_h(u)))| \quad \text{in } (L^{p(x)}(\Omega))^N,$$

it follows that

$$\begin{aligned} &\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)|\varphi_k(\omega_n)|dx \\ &\to \int_\Omega \varphi\nabla T_k(u)|\varphi_k(T_{2k}(u-T_h(u)))|dx = 0. \end{aligned} \tag{5.22}$$

Combining (5.19), (5.21) and (5.22), we obtain

$$\begin{aligned} &\frac{b(k)}{\alpha}\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) \\ &- a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))|\varphi_k(\omega_n)|dx \\ &\ge \Big|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\varphi_k(\omega_n)dx\Big| + \varepsilon_4(n). \end{aligned} \tag{5.23}$$

Thanks to (5.18) and (5.23), we obtain

$$\begin{aligned} &\int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\big) \\ &\quad\times \big(\nabla T_k(u_n)-\nabla T_k(u)\big)\Big(\varphi_k'(\omega_n) - \frac{b(k)}{\alpha}.|\varphi_k(\omega_n)|\Big)dx \\ &\le \int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\nabla\omega_n dx \\ &\quad - \big|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\varphi_k(\omega_n)dx\big| + \varepsilon_5(n) \\ &\le \int_\Omega f_n\varphi_k(\omega_n)dx + \int_{\{|u_n|\le M\}} \phi_n(T_M(u_n))\varphi_k'(\omega_n)\nabla\omega_n dx + \varepsilon_5(n). \end{aligned} \tag{5.24}$$

We have $\omega_n \rightharpoonup T_{2k}(u-T_h(u))$ weak-$*$ in $L^\infty(\Omega)$ then

$$\int_\Omega f_n\varphi_k(\omega_n)dx \to \int_\Omega f\varphi_k(T_{2k}(u-T_h(u)))dx \quad \text{as } n\to\infty, \tag{5.25}$$

and for n large enough (for example $n\ge M$), we can write

$$\int_\Omega \phi_n(T_M(u_n))\varphi_k'(\omega_n)\nabla\omega_n dx = \int_{\{|u_n|\le M\}} \phi(T_M(u_n))\varphi_k'(\omega_n)\nabla\omega_n dx,$$

it follows that

$$\begin{aligned} &\int_\Omega \phi_n(T_M(u_n))\varphi_k'(\omega_n)\nabla\omega_n dx \\ &\to \int_\Omega \phi(T_M(u))\varphi_k'(T_{2k}(u-T_h(u)))\nabla T_{2k}(u-T_h(u))dx \quad \text{as } n\to\infty. \end{aligned} \tag{5.26}$$

Combining (5.24) and (5.26), we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))dx \\ &\le \int_\Omega f\varphi_k(T_{2k}(u-T_h(u)))dx \\ &\quad + \int_\Omega \phi(T_M(u))\varphi_k'(T_{2k}(u-T_h(u)))\nabla T_{2k}(u-T_h(u))dx + \varepsilon_6(n). \end{aligned} \tag{5.27}$$

Taking $\Psi(t)=\int_0^t \phi(\tau)\varphi_k'(\tau-T_h(\tau))d\tau$, then $\Psi(0)=0_{\mathbb{R}^N}$ and $\Psi\in C^1(\mathbb{R}^N)$. By the Divergence Theorem (see also [7]), we obtain

$$\begin{aligned} &\int_\Omega \phi(T_M(u))\varphi_k'(T_{2k}(u-T_h(u)))\nabla T_{2k}(u-T_h(u))dx \\ &= \int_{\{h<|u|\le 2k+h\}} \phi(u)\varphi_k'(u-T_h(u))\nabla u dx \\ &= \int_{\{|u|\le 2k+h\}} \phi(T_{2k+h}(u))\varphi_k'(T_{2k+h}(u)-T_h(u))\nabla T_{2k+h}(u)dx \\ &\quad - \int_{\{|u|\le h\}} \phi(T_h(u))\varphi_k'(T_h(u)-T_h(u))\nabla T_h(u)dx \\ &= \int_\Omega \operatorname{div}\Psi(T_{2k+h}(u))dx - \int_\Omega \operatorname{div}\Psi(T_h(u))dx \\ &= \int_{\partial\Omega} \Psi(T_{2k+h}(u)).\overrightarrow{n}\,dx - \int_{\partial\Omega}\Psi(T_h(u)).\overrightarrow{n}\,dx \\ &= \sum_{i=1}^N \Big(\int_{\partial\Omega}\Psi_i(T_{2k+h}(u)).n_i dx - \int_{\partial\Omega}\Psi_i(T_h(u)).n_i dx\Big) = 0, \end{aligned}$$

since $u=0$ on $\partial\Omega$, with $\Psi=(\Psi_1,\dots,\Psi_N)$ and $\overrightarrow{n}=(n_1,n_2,\dots,n_N)$ the normal vector on $\partial\Omega$. Then, by letting h tend to infinity in (5.27), we obtain

$$\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))dx \to 0 \tag{5.28}$$

as $n\to\infty$. Using Lemma 4.4, we deduce that

$$T_k(u_n)\to T_k(u) \quad \text{in } W_0^{1,p(x)}(\Omega); \tag{5.29}$$

then

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega.$$

Step 4: Equi-integrability of $g_n(x,u_n,\nabla u_n)$. To prove that

$$g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u) \quad \text{strongly in } L^1(\Omega),$$

using Vitalis theorem, it is sufficient to prove that $g_n(x,u_n,\nabla u_n)$ is uniformly equi-integrable. Indeed, taking $T_1(u_n - T_h(u_n))$ as a test function in (5.2), we obtain

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla T_1(u_n-T_h(u_n))dx+\int_\Omega g_n(x,u_n,\nabla u_n)T_1(u_n-T_h(u_n))dx\\&=\int_\Omega f_nT_1(u_n-T_h(u_n))dx+\int_\Omega \phi_n(u_n)\nabla T_1(u_n-T_h(u_n))dx,\end{aligned}\tag{5.30}$$

which is equivalent to

$$\begin{aligned}&\int_{\{h<|u_n|\le h+1\}} a(x,u_n,\nabla u_n)\nabla u_n dx+\int_{\{h\le|u_n|\}} g_n(x,u_n,\nabla u_n)T_1(u_n-T_h(u_n))dx\\&=\int_{\{h\le|u_n|\}} f_nT_1(u_n-T_h(u_n))dx+\int_{\{h<|u_n|\le h+1\}} \phi_n(u_n)\nabla u_n dx.\end{aligned}\tag{5.31}$$

Taking $\Phi_n(t)=\int_0^t\phi_n(\tau)d\tau$, we have $\Phi_n(0)=0_{\mathbb{R}^N}$ and $\Phi_n\in C^1(\mathbb{R}^N)$. In view of the Divergence theorem,

$$\begin{aligned}&\int_{\{h<|u_n|\le h+1\}}\phi_n(u_n)\nabla u_n dx\\&=\int_{\{|u_n|\le h+1\}}\phi_n(u_n)\nabla u_n dx-\int_{\{|u_n|\le h\}}\phi_n(u_n)\nabla u_n dx\\&=\int_\Omega \phi_n(T_{h+1}(u_n))\nabla T_{h+1}(u_n)dx-\int_\Omega\phi_n(T_h(u_n))\nabla T_h(u_n)dx\\&=\int_\Omega \operatorname{div}\Phi_n(T_{h+1}(u_n))dx-\int_\Omega \operatorname{div}\Phi_n(T_h(u_n))dx\\&=\int_{\partial\Omega}\Phi_n(T_{h+1}(u_n)).\overrightarrow{n}d\sigma-\int_{\partial\Omega}\Phi_n(T_h(u_n)).\overrightarrow{n}d\sigma=0.\end{aligned}$$

Since $u_n=0$ on $\partial\Omega$, with $\Phi_n=(\Phi_{n,1},\dots,\Phi_{n,N})$, and since

$$\int_{\{h<|u_n|\le h+1\}} a(x,u_n,\nabla u_n)\nabla u_n dx\ge 0,$$

it follows that

$$\begin{aligned}\int_{\{h+1\le|u_n|\}}|g_n(x,u_n,\nabla u_n)|dx&=\int_{\{h+1\le|u_n|\}} g_n(x,u_n,\nabla u_n)T_1(u_n-T_h(u_n))dx\\&\le\int_{\{h\le|u_n|\}} g_n(x,u_n,\nabla u_n)T_1(u_n-T_h(u_n))dx\\&\le\int_{\{h\le|u_n|\}} f_nT_1(u_n-T_h(u_n))dx\\&\le\int_{\{h\le|u_n|\}}|f_n|dx,\end{aligned}$$

thus, for all $\eta>0$, there exists $h(\eta)>0$ such that

$$\int_{\{h(\eta)\le|u_n|\}}|g_n(x,u_n,\nabla u_n)|dx\le\frac{\eta}{2}.\tag{5.32}$$

On the other hand, for any measurable subset $E \subset \Omega$, we have

$$\begin{aligned}\int_E |g_n(x,u_n,\nabla u_n)|dx \le \int_{E\cap\{|u_n|<h(\eta)\}} & b(h(\eta))(c(x)+|\nabla u_n|^{p(x)})dx \\ & + \int_{\{|u_n|\ge h(\eta)\}} |g_n(x,u_n,\nabla u_n)|dx,\end{aligned} \tag{5.33}$$

thanks to (5.29), there exists $\beta(\eta) > 0$ such that

$$\int_{E\cap\{|u_n|<h(\eta)\}} b(h(\eta))(c(x)+|\nabla u_n|^{p(x)})dx \le \frac{\eta}{2} \quad \text{for } \operatorname{meas}(E) \le \beta(\eta). \tag{5.34}$$

Finally, by combining (5.32), (5.33) and (5.34), we obtain

$$\int_E |g_n(x,u_n,\nabla u_n)|dx \le \eta, \quad \text{with } \operatorname{meas}(E) \le \beta(\eta). \tag{5.35}$$

Then $(g_n(x,u_n,\nabla u_n))_n$ is equi-integrable, and by the Vitali's Theorem we deduce that

$$g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \quad \text{in } L^1(\Omega). \tag{5.36}$$

Step 5: Passage to the limit. Let $\varphi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ and $M = k + \|\varphi\|_\infty$ with $k > 0$, we will show that

$$\liminf_{n\to\infty} \int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-\varphi)dx \ge \int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)dx.$$

If $|u_n| > M$ then $|u_n-\varphi| \ge |u_n| - \|\varphi\|_\infty > k$; therefore $\{|u_n-\varphi| \le k\} \subseteq \{|u_n| \le M\}$, which implies that

$$\begin{aligned} & a(x,u_n,\nabla u_n)\nabla T_k(u_n-\varphi) \\ & = a(x,u_n,\nabla u_n)\nabla(u_n-\varphi)\chi_{\{|u_n-\varphi|\le k\}} \\ & = a(x,T_M(u_n),\nabla T_M(u_n))(\nabla T_M(u_n)-\nabla\varphi)\chi_{\{|u_n-\varphi|\le k\}}. \end{aligned} \tag{5.37}$$

Then

$$\begin{aligned} & \int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-\varphi)dx \\ & = \int_\Omega a(x,T_M(u_n)\nabla T_M(u_n))(\nabla T_M(u_n)-\nabla\varphi)\chi_{\{|u_n-\varphi|\le k\}}dx \\ & = \int_\Omega (a(x,T_M(u_n),\nabla T_M(u_n)) - a(x,T_M(u_n),\nabla\varphi)) \\ & \quad \times (\nabla T_M(u_n)-\nabla\varphi)\chi_{\{|u_n-\varphi|\le k\}}dx \\ & \quad + \int_\Omega a(x,T_M(u_n),\nabla\varphi)(\nabla T_M(u_n)-\nabla\varphi)\chi_{\{|u_n-\varphi|\le k\}}dx, \end{aligned} \tag{5.38}$$

we obtain

$$\begin{aligned} & \liminf_{n\to+\infty} \int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-\varphi)dx \\ & \ge \int_\Omega (a(x,T_M(u),\nabla T_M(u)) - a(x,T_M(u),\nabla\varphi))(\nabla T_M(u)-\nabla\varphi)\chi_{\{|u-\varphi|\le k\}}dx \\ & \quad + \lim_{n\to+\infty} \int_\Omega a(x,T_M(u_n),\nabla\varphi)(\nabla T_M(u_n)-\nabla\varphi)\chi_{\{|u_n-\varphi|\le k\}}dx. \end{aligned} \tag{5.39}$$

Note that the second term in the right hand side of (5.39) is equal to

$$\int_\Omega a(x, T_M(u), \nabla\varphi)(\nabla T_M(u) - \nabla\varphi)\chi_{\{|u-\varphi|\le k\}}dx.$$

Finally, we have

$$\begin{aligned}
&\liminf_{n\to+\infty}\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - \varphi)dx\\
&\ge \int_\Omega a(x, T_M(u), \nabla T_M(u))(\nabla T_M(u) - \nabla\varphi)\chi_{\{|u-\varphi|\le k\}}dx,\\
&= \int_\Omega a(x, u, \nabla u)(\nabla u - \nabla\varphi)\chi_{\{|u-\varphi|\le k\}}dx\\
&= \int_\Omega a(x, u, \nabla u)\nabla T_k(u - \varphi)dx.
\end{aligned}$$

Now, taking $T_k(u_n - \varphi)$ as a test function in (5.2) and passing to the limit, we conclude the desired statement. This completes the 5 steps for the proof of Theorem 5.1.

Theorem 5.2. *Assume that* (3.2)-(3.6) *and* (3.8) *hold,* $p(.) \in C_+(\bar\Omega)$ *such that* $2-\frac{1}{N} < p_- \le p_+ < N$. *Then problem* (5.1) *has at least one solution* $u \in W_0^{1,q(x)}(\Omega)$ *for all continuous functions* $q(.) \in C_+(\bar\Omega)$ *such that* $1 < q(x) < \bar q(x) = \frac{N(p(x)-1)}{N-1}$.

Proof. Let $(f_n)_n$ be a sequence in $W^{-1,p'(x)}(\Omega)\cap L^1(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$ and $\|f_n\|_1 \le \|f\|_1$. we consider the approximate problem

$$\begin{gathered}
Au_n + g_n(x, u_n, \nabla u_n) = f_n - \operatorname{div}\phi_n(u_n)\\
u_n \in W_0^{1,p(x)}(\Omega),
\end{gathered}\tag{5.40}$$

where $\phi_n(s) = \phi(T_n(s))$ and $g_n(x, s, \xi) = \frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}$.

Thanks to the first step in the proof of Theorem 5.1, there exists at least one weak solution $u_n \in W_0^{1,p(x)}(\Omega)$ for this approximate problem. Let $\psi_k(t)$ be a real valued function

$$\psi_k(t) = \begin{cases}
0 & \text{if } 0 \le t \le k,\\
t - k & \text{if } k < t \le k + 1,\\
1 & \text{if } k + 1 < t,\\
-\psi_k(-t) & \text{otherwise },
\end{cases}\tag{5.41}$$

and we define the sets

$$B_0 = \{x \in \Omega : |u_n| \le 1\}, \quad B_k = \{x \in \Omega : k < |u_n| \le k + 1\} \quad \text{for } k \in \mathbb{N}^*.$$

Taking $\psi_k(u_n)$ as a test function in the approximate problem (5.40), we obtain

$$\begin{aligned}
&\int_\Omega a(x, u_n, \nabla u_n)\nabla\psi_k(u_n)dx + \int_\Omega g_n(x, u_n, \nabla u_n)\psi_k(u_n)dx\\
&= \int_\Omega f_n\psi_k(u_n)dx + \int_\Omega \phi_n(u_n)\nabla\psi_k(u_n)dx.
\end{aligned}$$

Then

$$\int_{B_k} a(x, u_n, \nabla u_n)\nabla u_n dx + \int_{\{|u_n|>k\}} g_n(x, u_n, \nabla u_n)\psi_k(u_n)dx$$

$$= \int_{\{|u_n|>k\}} f_n \psi_k(u_n) dx + \int_{B_k} \phi_n(u_n) \nabla u_n dx \,.$$

By the Divergence theorem,

$$\begin{aligned} \int_{B_k} \phi_n(u_n)\nabla u_n dx &= \int_{\{|u_n|\le k+1\}} \phi_n(u_n)\nabla u_n dx - \int_{\{|u_n|\le k\}} \phi_n(u_n)\nabla u_n dx \\ &= \int_\Omega \phi_n(T_{k+1}(u_n))\nabla T_{k+1}(u_n) dx - \int_\Omega \phi_n(T_k(u_n))\nabla T_k(u_n) dx \\ &= \int_\Omega \operatorname{div} \Phi_n(T_{k+1}(u_n)) dx - \int_\Omega \operatorname{div} \Phi_n(T_k(u_n)) dx = 0 \,. \end{aligned} \tag{5.42}$$

Since $\psi_k(u_n)$ has the same sign as u_n, $g_n(x,u_n,\nabla u_n)\psi_k(u_n) \ge 0$ and we obtain

$$\int_{B_k} a(x,u_n,\nabla u_n)\nabla u_n dx \le \int_{\{|u_n|>k\}} f_n\psi_k(u_n) dx \le \int_\Omega |f_n| dx,$$

using (3.3), we deduce that

$$\alpha \int_{B_k} |\nabla u_n|^{p(x)} dx \le \|f\|_1 \quad \text{for all } k \ge 0. \tag{5.43}$$

In view of the Lemma 4.3, there exists a constant C that does not depend on n such that

$$\|u_n\|_{1,q(x)} \le C,$$

for any continuous exponent $q(\cdot) \in C_+(\overline{\Omega})$ with $1 < q(x) < \bar q(x) = \frac{N(p(x)-1)}{N-1}$. By using the same steps in the proof of Theorem 5.1, we can show that there exists a subsequence still denoted $(u_n)_n$ which converge to u, then

$$\|u\|_{1,q(x)} \le C,$$

where u is solution of 5.1. □

Theorem 5.3. *Assume that* (3.2)–(3.6) *and* (3.8) *hold,* $p(.) \in C_+(\overline{\Omega})$ *such that* $2 - \frac{1}{N} < p_- \le p_+ < N$. *If* $f \log(1+|f|) \in L^1(\Omega)$ *then* (5.1) *has at least one solution* $u \in W_0^{1,\bar q(x)}(\Omega)$ *with* $\bar q(x) = \frac{N(p(x)-1)}{N-1}$.

Proof. Let $(f_n)_n$ be a sequence in $W^{-1,p'(x)}(\Omega) \cap L^1(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$, with $\|f_n\|_1 \le \|f\|_1$ and $\|f_n \log(1+|f_n|)\|_1 \le \|f\log(1+|f|)\|_1$ (for example $f_n = T_n(f)$). We consider the approximate problem

$$\begin{gathered} Au_n + g_n(x,u_n,\nabla u_n) = f_n - \operatorname{div} \phi_n(u_n) \\ u_n \in W_0^{1,p(x)}(\Omega), \end{gathered} \tag{5.44}$$

where $\phi_n(s) = \phi(T_n(s))$ and $g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}$, there exists at least one weak solution $u_n \in W_0^{1,p(x)}(\Omega)$ for this approximate problem.

Let $\psi_k(t)$ be defined by (5.41), and

$$S_k = \Big\{x \in \Omega, \quad k < |u_n|\Big\} = \cup_{r=k}^\infty B_r \qquad \forall k \in \mathbb{N}.$$

By using $\psi_k(u_n)$ as a test function in the approximate problem (5.44), we obtain

$$\alpha \int_{B_k} |\nabla u_n|^{p(x)} dx \le \int_{S_k} |f_n| dx \quad \text{for all } k \in \mathbb{N}. \tag{5.45}$$

Let $\bar{q}(x) = \frac{N(p(x)-1)}{N-1}$, we have

$$\begin{aligned}
\int_\Omega \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)} dx &= \sum_{k=0}^{\infty} \int_{B_k} \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)} dx \\
&\le \sum_{k=0}^{\infty} \frac{1}{k+1} \int_{B_k} |\nabla u_n|^{p(x)} dx \\
&\le \frac{1}{\alpha} \sum_{k=0}^{\infty} \frac{1}{k+1} \int_{S_k} |f_n| dx \\
&\le \frac{1}{\alpha} \sum_{k=0}^{\infty} \frac{1}{k+1} \sum_{s=k}^{\infty} \int_{B_s} |f_n| dx \\
&= \frac{1}{\alpha} \sum_{k=0}^{\infty} \sum_{s=k}^{\infty} \int_{B_s} |f_n| \frac{1}{k+1} dx \\
&= \frac{1}{\alpha} \sum_{s=0}^{\infty} \sum_{k=0}^{s} \int_{B_s} |f_n| \frac{1}{k+1} dx \,.
\end{aligned}$$

Since $\sum_{k=0}^{\infty} \sum_{s=k}^{\infty} v_{s,k} = \sum_{s=0}^{\infty} \sum_{k=0}^{s} v_{s,k}$, the above expression equals

$$\begin{aligned}
\frac{1}{\alpha} \sum_{s=0}^{\infty} \int_{B_s} |f_n| (\sum_{k=0}^{s} \frac{1}{k+1}) dx &\le \frac{1}{\alpha} \sum_{s=0}^{\infty} \int_{B_s} |f_n| [1 + \log(1+s)] dx \\
&\le \frac{1}{\alpha} \sum_{s=0}^{\infty} \int_{B_s} |f_n| [1 + \log(1+|u_n|)] dx \\
&\le \frac{1}{\alpha} \int_\Omega |f_n| [1 + \log(1+|u_n|)] dx,
\end{aligned}$$

and since $ab \le a\log(1+a) + e^b$ for all $a, b \ge 0$, we obtain

$$\begin{aligned}
&\frac{1}{\alpha} \int_\Omega |f_n| [1 + \log(1+|u_n|)] dx \\
&= \frac{1}{\alpha} \int_\Omega |f_n| dx + \frac{1}{\alpha} \int_\Omega |f_n| \log(1+|u_n|) dx \\
&\le \frac{1}{\alpha} \int_\Omega |f_n| dx + \frac{1}{\alpha} \int_\Omega |f_n| \log(1+|f_n|) dx + \frac{1}{\alpha} \int_\Omega (1+|u_n|) dx \\
&\le \frac{1}{\alpha} \|f\|_1 + \frac{1}{\alpha} \|f \log(1+|f|)\|_1 + \frac{1}{\alpha} \int_\Omega (1+|u_n|) dx \,.
\end{aligned}$$

In view of the Theorem 5.2 we have $u_n \in W_0^{1,q(x)}(\Omega)$; then $\int_\Omega |u_n| dx$ is bounded. It follows that

$$\int_\Omega \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)} dx \le C_1, \tag{5.46}$$

with C_1 is a constant that does not depend on n.

Now, observe that $\overline{\Omega}$ is compact, therefore, we can cover it with a finite number of balls $(B_i)_{i=1,\dots,m}$, with $B_i = B(x_i, \delta)$. we denote

$$p_{i-} = \min\{p(x) : x \in \overline{B_i \cap \Omega}\} \quad \text{and} \quad p_{i+} = \max\{p(x) : x \in \overline{B_i \cap \Omega}\},$$

since $p(\cdot)$ is a real-valued continuous function on $\overline{\Omega}$, then, by taking $\delta > 0$ small enough such that

$$\frac{(N-p_{i-})(p_{i-}-1)^2}{N+p_{i-}^2-2p_{i-}} + p_{i-} > p_{i+} \quad \text{in } B_i\cap\Omega \text{ for } i=1,\dots,m, \tag{5.47}$$

and there exists a constant $a>0$ such that

$$\operatorname{meas}(B_i\cap\Omega) > a \quad \text{for } i=1,\dots,m.$$

By the Generalized Hölder inequality, we have

$$\begin{aligned}
&\int_{B_i\cap\Omega}|\nabla u_n|^{\bar q(x)}dx\\
&= \int_{B_i\cap\Omega}\frac{|\nabla u_n|^{\bar q(x)}}{(1+|u_n|)^{\frac{\bar q(x)}{p(x)}}}(1+|u_n|)^{\frac{\bar q(x)}{p(x)}}dx\\
&= \int_{B_i\cap\Omega}\big(\frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)}\big)^{\frac{\bar q(x)}{p(x)}}(1+|u_n|)^{\frac{\bar q(x)}{p(x)}}dx\\
&\le \big(\frac{N(p_{i+}-1)}{(N-1)p_{i-}}+\frac{N-p_{i-}}{(N-1)p_{i-}}\big)\big\|\big(\frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)}\big)^{\frac{\bar q(x)}{p(x)}}\big\|_{L^{\frac{p(x)}{\bar q(x)}}(B_i\cap\Omega)}\\
&\quad\times\big\|(1+|u_n|)^{\frac{\bar q(x)}{p(x)}}\big\|_{L^{\frac{p(x)}{p(x)-\bar q(x)}}(B_i\cap\Omega)}
\end{aligned}\tag{5.48}$$

On the one hand, using (5.46) we have

$$\begin{aligned}
\big\|\big(\frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)}\big)^{\frac{\bar q(x)}{p(x)}}\big\|_{L^{\frac{p(x)}{\bar q(x)}}(B_i\cap\Omega)} &\le \Big(\int_{B_i\cap\Omega}\frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)}dx+1\Big)^{\frac{(N-1)p_{i+}}{N(p_{i-}-1)}}\\
&\le \Big(\int_{\Omega}\frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)}dx+1\Big)^{\frac{(N-1)p_{+}}{N(p_{-}-1)}}\\
&\le (C_1+1)^{\frac{(N-1)p_{+}}{N(p_{-}-1)}}
\end{aligned}\tag{5.49}$$

On the other hand, thanks to the Sobolev-Poincaré inequality, we have

$$\begin{aligned}
\|u_n\|_{L^{\bar q^*(x)}(B_i\cap\Omega)} &\le \|u_n-\overline{u}_{n,i}\|_{L^{\bar q^*(x)}(B_i\cap\Omega)} + \|\overline{u}_{n,i}\|_{L^{\bar q^*(x)}(B_i\cap\Omega)}\\
&\le c\|\nabla u_n\|_{L^{\bar q(x)}(B_i\cap\Omega)} + \|\overline{u}_{n,i}\|_{L^{\bar q^*(x)}(B_i\cap\Omega)}
\end{aligned}$$

with $\overline{u}_{n,i} = \frac{1}{|B_i\cap\Omega|}\int_{B_i\cap\Omega}u_n dx$, and since

$$\bar q^*(x) = \frac{\bar q(x)}{p(x)-\bar q(x)} = \frac{N(p(x)-1)}{N-p(x)},$$

we obtain

$$\begin{aligned}
&\int_{B_i\cap\Omega}(1+|u_n|)^{\frac{\bar q(x)}{p(x)-\bar q(x)}}dx\\
&\le C_2\int_{B_i\cap\Omega}(1+|u_n|^{\frac{\bar q(x)}{p(x)-\bar q(x)}})dx\\
&= C_2\big(\operatorname{meas}(B_i\cap\Omega)+\int_{B_i\cap\Omega}|u_n|^{\bar q^*(x)}dx\big)\\
&\le C_2\big(\operatorname{meas}(B_i\cap\Omega)+\|u_n\|^{\sigma_1}_{L^{\bar q^*(x)}(B_i\cap\Omega)}\big)\\
&\le C_3\big(\operatorname{meas}(B_i\cap\Omega)+\|\nabla u_n\|^{\sigma_1}_{L^{\bar q(x)}(B_i\cap\Omega)}+\|\overline{u}_{n,i}\|^{\sigma_1}_{L^{\bar q^*(x)}(B_i\cap\Omega)}\big),
\end{aligned}$$

with

$$\sigma_1 = \begin{cases} \frac{N(p_{i+}-1)}{N-p_{i+}} & \text{if } \|u_n\|_{L^{\bar{q}^*(x)}(B_i\cap\Omega)} > 1, \\ \frac{N(p_{i-}-1)}{N-p_{i-}} & \text{if } \|u_n\|_{L^{\bar{q}^*(x)}(B_i\cap\Omega)} \le 1, \end{cases}$$

since $|\overline{u}_{n,i}| \le \frac{1}{a}\int_\Omega |u_n|dx$, it follows that $\|\overline{u}_{n,i}\|_{L^{\bar{q}^*(x)}(B_i\cap\Omega)}$ is bounded and

$$\begin{aligned} &\big\|(1+|u_n|)^{\frac{\bar{q}(x)}{p(x)}}\big\|_{L^{\frac{p(x)}{p(x)-\bar{q}(x)}}(B_i\cap\Omega)} \\ &\le \Big(\int_{B_i\cap\Omega}(1+|u_n|)^{\frac{\bar{q}(x)}{p(x)-\bar{q}(x)}}dx\Big)^{\sigma_2} \\ &\le \Big(C_3\big(\operatorname{meas}(B_i\cap\Omega)+\|\nabla u_n\|^{\sigma_1}_{L^{\bar{q}(x)}(B_i\cap\Omega)}+\|\overline{u}_{n,i}\|^{\sigma_1}_{L^{\bar{q}^*(x)}(B_i\cap\Omega)}\big)\Big)^{\sigma_2} \\ &\le C_4(1+\|\nabla u_n\|^{\sigma_1\sigma_2}_{L^{\bar{q}(x)}(B_i\cap\Omega)}), \end{aligned} \tag{5.50}$$

and

$$\sigma_2 = \begin{cases} \frac{N-p_{i-}}{(N-1)p_{i-}} & \text{if } \big\|(1+|u_n|)^{\frac{\bar{q}(x)}{p(x)}}\big\|_{L^{\frac{p(x)}{p(x)-\bar{q}(x)}}(B_i\cap\Omega)} > 1, \\ \frac{N-p_{i+}}{(N-1)p_{i+}} & \text{if } \big\|(1+|u_n|)^{\frac{\bar{q}(x)}{p(x)}}\big\|_{L^{\frac{p(x)}{p(x)-\bar{q}(x)}}(B_i\cap\Omega)} \le 1. \end{cases}$$

By combining (5.48), (5.49) and (5.50), we obtain

$$\int_{B_i\cap\Omega}|\nabla u_n|^{\bar{q}(x)}dx \le C_5 + C_5\|\nabla u_n\|^{\sigma_1\sigma_2}_{L^{\bar{q}(x)}(B_i\cap\Omega)}.$$

Then

$$\begin{aligned} &\|\nabla u_n\|^{\pi}_{L^{\bar{q}(x)}(B_i\cap\Omega)} - C_5\|\nabla u_n\|^{\sigma_1\sigma_2}_{L^{\bar{q}(x)}(B_i\cap\Omega)} \\ &\le \int_{B_i\cap\Omega}|\nabla u_n|^{\bar{q}(x)}dx - C_5\|\nabla u_n\|^{\sigma_1\sigma_2}_{L^{\bar{q}(x)}(B_i\cap\Omega)} \le C_5, \end{aligned} \tag{5.51}$$

with

$$\pi = \begin{cases} \bar{q}_{i-} & \text{if } \|\nabla u_n\|_{L^{\bar{q}(x)}(B_i\cap\Omega)} > 1, \\ \bar{q}_{i+} & \text{if } \|\nabla u_n\|_{L^{\bar{q}(x)}(B_i\cap\Omega)} \le 1, \end{cases}$$

and since $\sigma_1\sigma_2 < \bar{q}_{i-} \le \pi$ in $B_i\cap\Omega$, it follows that $\|\nabla u_n\|_{L^{\bar{q}(x)}(B_i\cap\Omega)}$ is bounded. Indeed, we have

$$\begin{aligned} &\frac{(N-p_{i-})(p_{i-}-1)^2}{N+p_{i-}^2-2p_{i-}} + p_{i-} > p_{i+} \\ &\iff \frac{Np_{i-}^2 - Np_{i-} + N - p_{i-}}{N+p_{i-}^2-2p_{i-}} > p_{i+} \\ &\iff \quad (p_{i-}-1)(N-p_{i+})p_{i-} - (N-p_{i-})(p_{i+}-1) > 0 \\ &\iff \quad \bar{q}_{i-} = \frac{N(p_{i-}-1)}{N-1} > \frac{(N-p_{i-})}{(N-1)p_{i-}}\frac{N(p_{i+}-1)}{N-p_{i+}} \ge \sigma_1\sigma_2. \end{aligned}$$

We conclude that there exists some constants $r_i > 0$ such that $\int_{B_i\cap\Omega}|\nabla u_n|^{\bar{q}(x)}dx \le r_i$ for all $i = 1,\dots,m$, it follows that

$$\int_\Omega |\nabla u_n|^{\bar{q}(x)}dx = \sum_{i=1}^m \int_{B_i\cap\Omega}|\nabla u_n|^{\bar{q}(x)}dx \le C_6, \tag{5.52}$$

and by the Poincaré inequality, we obtain

$$\big\|u_n\big\|_{1,\bar{q}(x)} \le C_7,$$

with C_7 is a constant that does not depend on n, we deduce that

$$\|u\|_{1,\bar{q}(x)} \leq C_7,$$

where u is solution of (5.1). □

Theorem 5.4. *Let $p(.) \in C_+(\bar{\Omega})$. Assume (3.2)-(3.6) hold with $f \in W^{-1,p'(x)}(\Omega)$ and $\phi \in C^0(\mathbb{R}^N)$. Then (5.1) has at least one solution $u \in W_0^{1,p(x)}(\Omega)$.*

Proof. Let $u_n \in W_0^{1,p(x)}(\Omega)$ a weak solution of the approximate problem

$$\begin{gathered} Au_n + g_n(x, u_n, \nabla u_n) = f - \operatorname{div} \phi_n(u_n) \\ u_n \in W_0^{1,p(x)}(\Omega), \end{gathered} \tag{5.53}$$

where $\phi_n(s) = \phi(T_n(s))$ and $g_n(x, s, \xi) = \frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}$. By taking u_n as a test function in (5.53), we obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla u_n dx + \int_\Omega g_n(x, u_n, \nabla u_n)u_n dx = \int_\Omega f u_n dx + \int_\Omega \phi_n(u_n)\nabla u_n dx.$$

By the Divergence theorem, $\int_\Omega \phi_n(u_n)\nabla u_n dx = 0$, and since $g_n(x, u_n, \nabla u_n)u_n \geq 0$, we obtain

$$\begin{aligned} \alpha \int_\Omega |\nabla u_n|^{p(x)} dx &\leq \int_\Omega a(x, u_n, \nabla u_n)\nabla u_n dx \\ &\leq (\frac{1}{p_-} + \frac{1}{p'_-})\|f\|_{-1,p'(x)}\|u_n\|_{1,p(x)}, \end{aligned}$$

it follows that

$$\|\nabla u_n\|^\gamma_{p(x)} \leq C_1\|f\|_{-1,p'(x)}\|u_n\|_{1,p(x)} \quad \text{with } \gamma = \begin{cases} p_- & \text{if } \|\nabla u_n\|_{p(x)} > 1, \\ p_+ & \text{if } \|\nabla u_n\|_{p(x)} \leq 1, \end{cases}$$

by using the Poincaré inequality, we obtain

$$\|u_n\|^\gamma_{1,p(x)} \leq C_2\|u_n\|_{1,p(x)}.$$

Then $\|u_n\|_{1,p(x)} \leq C_3$, with C_3 independent of n, and

$$\|u\|_{1,p(x)} \leq C_3,$$

where u is solution of the problem (5.1). □

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