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FRACTIONAL DIFFERENTIAL EQUATIONS WITH FRACTIONAL NON-SEPARATED BOUNDARY CONDITIONS

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Abstract. We study boundary-value problems of nonlinear fractional differential equations with fractional non-separated (integral) boundary conditions. Existence and uniqueness results are obtained by using fixed point theorems and examples are given to illustrate the results.

1. Introduction

The study of fractional differential equations ranges from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. A strong motivation for studying fractional differential equations comes from the fact that they have been proved to be valuable tools in the modeling of many phenomena in engineering and sciences such as physics, mechanics, chemistry, economics and biology, etc. [10, 11, 12]. For some recent developments on the existence results of fractional differential equations, we can refer to, for instance, [2, 3, 4, 9, 13, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 28, 29, 30, 31, 33] and the references therein.

Ahmad and Nieto [5] investigated the existence and uniqueness of solutions for an anti-periodic fractional boundary-value problem

$$\begin{gathered} {}^cD^q x(t) = f(t,x(t)), \quad t\in[0,T],\ T>0,\ 1<q\leq 2, \\ x(0) = -x(T), \quad {}^cD^p x(0) = -{}^cD^p x(T), \quad 0<p<1, \end{gathered} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, f is a given continuous function.

Fractional differential equations with non-separated integral boundary conditions of the following form was considered in [6] by Ahmad, Nieto and Alsaedi.

$$\begin{gathered} {}^cD^q x(t) = f(t,x(t)), \quad t\in[0,T],\ T>0,\ 1<q\leq 2, \\ x(0)-\lambda_1 x(T) = \mu_1\int_0^T g(s,x(s))ds, \\ x'(0)-\lambda_2 x'(T) = \mu_2\int_0^T h(s,x(s))ds, \end{gathered}$$

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where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f, g, h : [0, T]\times\mathbb{R} \to \mathbb{R}$ are given continuous functions and $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ with $\lambda_1 \neq 1$, $\lambda_2 \neq 1$. The results obtained in [5, 6] are based on some standard fixed point theorems.

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multi-point and non-local BVP as special cases. Integral boundary conditions appear in the study of population dynamics [15] and cellular systems [1]. We can see the papers [3, 6, 7, 8, 14], etc., for fractional differential equations with integral boundary conditions.

Motivated by the above papers, in this article, we are concerned with the existence, uniqueness of solutions to fractional differential equations with a new class of non-separated (integral) boundary value conditions.

We first study fractional differential equations with fractional non-separated boundary values in the form

$$\begin{gathered} {}^cD^\alpha x(t) = f(t, x(t)), \quad t \in [0, T],\ 1 < \alpha \leq 2,\ T > 0, \\ a_1x(0) + b_1x(T) = c_1, a_2({}^cD^\gamma x(0)) + b_2({}^cD^\gamma x(T)) = c_2, 0 < \gamma < 1, \end{gathered} \tag{1.2}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, f is a continuous function on $[0, T] \times \mathbb{R}$ and a_i, b_i, c_i, $i = 1, 2$ are real constants such that $a_1 + b_1 \neq 0$ and $b_2 \neq 0$.

Then the results obtained for problem (1.2) in this paper are extended to fractional differential equations with fractional non-separated integral boundary conditions of the form

$$\begin{gathered} {}^cD^\alpha x(t) = f(t, x(t)), \quad t \in [0, T],\ T > 0,\ 1 < \alpha \leq 2, \\ a_1x(0) + b_1x(T) = c_1 \int_0^T g(s, x(s))ds, \\ a_2({}^cD^\gamma x(0)) + b_2({}^cD^\gamma x(T)) = c_2 \int_0^T h(s, x(s))ds, \quad 0 < \gamma < 1, \end{gathered} \tag{1.3}$$

where $g, h : [0, T] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

We remark that when $a_1 = 1$, $b_1 = 1$, $c_1 = 0$, $a_2 = 1$, $b_2 = 1$ and $c_2 = 0$, the problem (1.2) reduces to an anti-periodic fractional boundary value problem (1.1) (cf.[27]).

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel, Sections 3, 4 are dedicated to the existence results of the problem (1.2), respectively, the problem (1.3), in the final Section 5, two examples are given to illustrate the results.

2. Preliminaries

Definition 2.1 ([21]). *The Riemann-Liouville fractional integral of order q for a function f is defined as*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the integral exists.

Definition 2.2 ([21]). *For a continuous function f, the Caputo derivative of order q is defined as*

$$^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)ds, \quad n-1 < q < n,\ n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Lemma 2.3 ([32]). *Let $\alpha > 0$, then the differential equation*

$$^cD^\alpha h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1}t^{n-1}$ and

$$I^\alpha\, {}^cD^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1}t^{n-1},$$

here $c_i \in \mathbb{R}$, $i = 0, 1, 2, \cdots, n-1$, $n = [\alpha]+1$.

Lemma 2.4. *For any $y \in C([0,T], \mathbb{R})$, the unique solution of the fractional non-separated boundary-value problem*

$$\begin{gathered} ^cD^\alpha x(t) = y(t), \quad t \in [0,T],\ 1 < \alpha \le 2, \\ a_1 x(0) + b_1 x(T) = c_1, a_2({}^cD^\gamma x(0)) + b_2({}^cD^\gamma x(T)) = c_2, \quad 0 < \gamma < 1, \end{gathered} \tag{2.1}$$

is given by

$$\begin{aligned} x(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \frac{t\Gamma(2-\gamma)}{T^{1-\gamma}} \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds \\ &+ \frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2} - \frac{b_1}{a_1+b_1}\Big\{ \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \\ &- T^\gamma \Gamma(2-\gamma) \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds \Big\} \\ &- \frac{1}{a_1+b_1}\Big(\frac{b_1 c_2 T^\gamma \Gamma(2-\gamma)}{b_2} - c_1 \Big). \end{aligned} \tag{2.2}$$

Proof. For $1 < \alpha \le 2$, by Lemma 2.3, we know that the general solution of the equation $^cD^\alpha x(t) = y(t)$ can be written as

$$x(t) = I^\alpha y(t) - k_1 - k_2 t = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - k_1 - k_2 t, \tag{2.3}$$

where $k_1, k_2 \in \mathbb{R}$ are arbitrary constants. Since $^cD^\gamma k = 0$ (k is a constant), $^cD^\gamma t = \frac{t^{1-\gamma}}{\Gamma(2-\gamma)}$, $^cD^\gamma I^\alpha y(t) = I^{\alpha-\gamma} y(t)$ (see [21]), from (2.3) we have

$$^cD^\gamma x(t) = I^{\alpha-\gamma} y(t) - \frac{k_2 t^{1-\gamma}}{\Gamma(2-\gamma)} = \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds - \frac{k_2 t^{1-\gamma}}{\Gamma(2-\gamma)}.$$

Using the boundary conditions, we obtain

$$\begin{gathered} a_1(-k_1) + b_1\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - k_1 - k_2 T \Big) = c_1, \\ a_2 \times 0 + b_2\Big(\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds - \frac{k_2 T^{1-\gamma}}{\Gamma(2-\gamma)} \Big) = c_2. \end{gathered}$$

Therefore, we have

$$
\begin{aligned}
k_1 = &\frac{1}{a_1+b_1}\Big(\frac{b_1c_2T^\gamma\Gamma(2-\gamma)}{b_2}-c_1\Big)\\
&+\frac{b_1}{a_1+b_1}\Big(\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds-T^\gamma\Gamma(2-\gamma)\int_0^T\frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}y(s)ds\Big),\\
&k_2=\frac{\Gamma(2-\gamma)}{T^{1-\gamma}}\Big(\int_0^T\frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}y(s)ds-\frac{c_2}{b_2}\Big).
\end{aligned}
$$

Substituting the values of k_1,k_2 in (2.3), we obtain (2.2). This completes the proof. □

From the proof of the above Lemma, we notice that the solution (2.2) of problem (2.1) does not depend on the parameter a_2, that is to say, the parameter a_2 is of arbitrary nature for this problem.

Theorem 2.5 (Schauder fixed point theorem). *Let U be a closed, convex and non-empty subset of a Banach space X, let $P:U\to U$ be a continuous mapping such that $P(U)$ is a relatively compact subset of X. Then P has at least one fixed point in U.*

Theorem 2.6 (Nonlinear alternative of Leray-Schauder type). *Let X be a Banach space, C a closed, convex subset of X, U an open subset of C and $0\in U$. Suppose that $P:\overline{U}\to C$ is a continuous and compact map. Then either (a) P has a fixed point in $\overline{U}$, or (b) there exist a $x\in\partial U$ (the boundary of U) and $\lambda\in(0,1)$ with $x=\lambda P(x)$.*

3. Existence results for problem (1.2)

Let $\mathcal{C}=C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ equipped with the norm $\|x\|=\sup_{t\in[0,T]}|x(t)|$.

In view of Lemma 2.4, we define an operator $\mathcal{F}:\mathcal{C}\to\mathcal{C}$ as

$$
\begin{aligned}
&(\mathcal{F}x)(t)\\
&=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds-\frac{t\Gamma(2-\gamma)}{T^{1-\gamma}}\int_0^T\frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}f(s,x(s))ds\\
&\quad+\frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2}-\frac{b_1}{a_1+b_1}\Big\{\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds\\
&\quad-T^\gamma\Gamma(2-\gamma)\int_0^T\frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}f(s,x(s))ds\Big\}\\
&\quad-\frac{1}{a_1+b_1}\Big(\frac{b_1c_2T^\gamma\Gamma(2-\gamma)}{b_2}-c_1\Big).
\end{aligned}
\tag{3.1}
$$

Observe that problem (1.2) has solutions if and only if the operator equation $\mathcal{F}x$ has fixed points. We put $\mathcal{F}x=\mathcal{F}_1x+\mathcal{F}_2x$ where

$$
(\mathcal{F}_1x)(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds,\quad(\mathcal{F}_2x)(t)=-k_2^xt-k_1^x.
$$

Here k_1^x and k_2^x are constants given by

$$\begin{aligned}
k_1^x &= \frac{b_1c_2T^\gamma\Gamma(2-\gamma)}{(a_1+b_1)b_2} - \frac{c_1}{a_1+b_1} + \frac{b_1}{a_1+b_1}\Big\{\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds \\
&\quad - T^\gamma\Gamma(2-\gamma)\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}f(s,x(s))ds\Big\}, \\
k_2^x &= \frac{\Gamma(2-\gamma)}{T^{1-\gamma}}\Big(\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}f(s,x(s))ds - \frac{c_2}{b_2}\Big).
\end{aligned}$$

Now we are in a position to present our main results. The methods used to prove the existence results are standard, however, their exposition in the framework of problem (1.2) is new.

Theorem 3.1. *Suppose that $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies*

$$|f(t,x)-f(t,y)| \le m(t)|x-y|$$

for $t\in[0,T]$, $x,y\in\mathbb{R}$ with $m\in L^\infty([0,T],\mathbb{R}^+)$. If

$$\|m\|_{L^\infty}T^\alpha\Big(1+\frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)}+\frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big) < 1, \tag{3.2}$$

then problem (1.2) *has a unique solution.*

Proof. For $x,y\in\mathcal{C}$ and for each $t\in[0,T]$, we have

$$\begin{aligned}
|(\mathcal{F}_1x)(t)-(\mathcal{F}_1y)(t)| &\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big|f(s,x(s))-f(s,y(s))\big|ds \\
&\le \|m\|_{L^\infty}\|x-y\|\frac{T^\alpha}{\Gamma(\alpha+1)},
\end{aligned}$$

$$|(\mathcal{F}_2x)(t)-(\mathcal{F}_2y)(t)| \le T|k_2^x-k_2^y| + |k_1^x-k_1^y|,$$

$$\begin{aligned}
T|k_2^x-k_2^y| &\le T^\gamma\Gamma(2-\gamma)\Big|\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\big(f(s,x(s))-f(s,y(s))\big)ds\Big| \\
&\le \|m\|_{L^\infty}\|x-y\|\frac{\Gamma(2-\gamma)T^\alpha}{\Gamma(\alpha-\gamma+1)},
\end{aligned}$$

$$\begin{aligned}
|k_1^x-k_1^y| &\le \frac{|b_1|}{|a_1+b_1|}\Big|\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\big(f(s,x(s))-f(s,y(s))\big)ds\Big| \\
&\quad + \frac{|b_1|T^\gamma\Gamma(2-\gamma)}{|a_1+b_1|}\Big|\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\big(f(s,x(s))-f(s,y(s))\big)ds\Big| \\
&\le \|m\|_{L^\infty}\|x-y\|\frac{|b_1|}{|a_1+b_1|}\Big(\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{\Gamma(2-\gamma)T^\alpha}{\Gamma(\alpha-\gamma+1)}\Big).
\end{aligned}$$

Therefore, we have

$$\|\mathcal{F}x-\mathcal{F}y\| \le \|m\|_{L^\infty}T^\alpha\Big(1+\frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)}+\frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big)\|x-y\|.$$

This together with (3.2) implies that the map $\mathcal{F}$ is a contraction mapping. Hence the contraction mapping principle yields that $\mathcal{F}$ has a unique fixed point which is the unique solution of problem (1.2). The proof is complete. $\square$

Corollary 3.2. *Suppose that $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies*

$$|f(t,x) - f(t,y)| \le L|x-y|,$$

for $t \in [0,T]$, $x, y \in \mathbb{R}$ and $L > 0$. Then problem (1.2) has a unique solution provided

$$LT^\alpha\Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big) < 1.$$

Theorem 3.3. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

$$|f(t,x)| \le m(t) + d|x|^\rho$$

for $t \in [0,T]$, $x \in \mathbb{R}$ with $m \in L^\infty([0,T], \mathbb{R}^+)$, $d \ge 0$ and $0 \le \rho < 1$. Then problem (1.2) has at least one solution on $[0,T]$.

Proof. Define $B_r = \{x : x \in \mathcal{C} \text{ and } \|x\| \le r\}$, where

$$r \ge \max\{2A, (2Bd)^{\frac{1}{1-\rho}}\},$$

$$\begin{aligned}
A = {}& \frac{T^\gamma\Gamma(2-\gamma)|c_2|}{|b_2|} + \Big|\frac{b_1c_2T^\gamma\Gamma(2-\gamma)}{(a_1+b_1)b_2} - \frac{c_1}{a_1+b_1}\Big| \\
& + \|m\|_{L^\infty}T^\alpha\Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big), \\
B = {}& T^\alpha\Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big).
\end{aligned} \tag{3.3}$$

It is obvious that B_r is a closed, bounded and convex subset of the Banach space $\mathcal{C}$.

Firstly, we prove that $\mathcal{F} : B_r \to B_r$. For any $x \in B_r$, we have

$$|(\mathcal{F}_1x)(t)| \le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big(m(s) + d|x(s)|^\rho\big)ds \le \big(\|m\|_{L^\infty} + dr^\rho\big)\frac{T^\alpha}{\Gamma(\alpha+1)},$$

$$|(\mathcal{F}_2x)(t)| \le T|k_2^x| + |k_1^x|,$$

$$\begin{aligned}
T|k_2^x| &\le T^\gamma\Gamma(2-\gamma)\Big|\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}f(s,x(s))ds - \frac{c_2}{b_2}\Big| \\
&\le T^\gamma\Gamma(2-\gamma)\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}|f(s,x(s))|ds + \frac{T^\gamma\Gamma(2-\gamma)|c_2|}{|b_2|} \\
&\le \big(\|m\|_{L^\infty} + dr^\rho\big)\frac{\Gamma(2-\gamma)T^\alpha}{\Gamma(\alpha-\gamma+1)} + \frac{T^\gamma\Gamma(2-\gamma)|c_2|}{|b_2|},
\end{aligned}$$

$$\begin{aligned}
|k_1^x| \le {}& \Big|\frac{b_1c_2T^\gamma\Gamma(2-\gamma)}{(a_1+b_1)b_2} - \frac{c_1}{a_1+b_1}\Big| + \frac{|b_1|}{|a_1+b_1|}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s))|ds \\
& + \frac{|b_1|T^\gamma\Gamma(2-\gamma)}{|a_1+b_1|}\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}|f(s,x(s))|ds \\
\le {}& \big(\|m\|_{L^\infty} + dr^\rho\big)\frac{|b_1|}{|a_1+b_1|}\Big(\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)T^\alpha}{\Gamma(\alpha-\gamma+1)}\Big) \\
& + \Big|\frac{b_1c_2T^\gamma\Gamma(2-\gamma)}{(a_1+b_1)b_2} - \frac{c_1}{a_1+b_1}\Big|.
\end{aligned}$$

Hence we obtain

$$\|\mathcal{F}x\| \le \frac{T^\gamma\Gamma(2-\gamma)|c_2|}{|b_2|} + \Big|\frac{b_1c_2T^\gamma\Gamma(2-\gamma)}{(a_1+b_1)b_2} - \frac{c_1}{a_1+b_1}\Big|$$

$$
\begin{aligned}
&+ \|m\|_{L^\infty} T^\alpha\Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big)\\
&+ dr^\rho T^\alpha\Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big)\\
&\le A + dr^\rho B \le \frac{r}{2} + \frac{r}{2} = r.
\end{aligned}
$$

This implies that $\mathcal{F} : B_r \to B_r$.

Secondly, we show that $\mathcal{F}$ maps bounded sets into equicontinuous sets. Let $\overline{B}$ be any bounded subset of $\mathcal{C}$. Since f is continuous, we can assume without any loss of generality that there is positive constant N such that

$$|f(t, x(t))| \le N$$

for any $t \in [0, T]$ and $x \in \overline{B}$. Now let $0 \le t_1 < t_2 \le T$. For each $x \in \overline{B}$, we have the following facts:

$$
\begin{aligned}
&|(\mathcal{F}_1 x)(t_2) - (\mathcal{F}_1 x)(t_1)|\\
&\le \Big|\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))ds\Big| + \Big|\int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))ds\Big|\\
&\le \frac{N(t_2-t_1)^\alpha}{\Gamma(\alpha+1)} + \frac{N\big(t_2^\alpha - (t_2-t_1)^\alpha - t_1^\alpha\big)}{\Gamma(\alpha+1)}\\
&\le \frac{N(t_2^\alpha - t_1^\alpha)}{\Gamma(\alpha+1)},
\end{aligned}
$$

$$
\begin{aligned}
|(\mathcal{F}_2 x)(t_2) - (\mathcal{F}_2 x)(t_1)| &\le |k_2^x|(t_2-t_1)\\
&\le \frac{\Gamma(2-\gamma)}{T^{1-\gamma}}\Big(\frac{NT^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{|c_2|}{|b_2|}\Big)(t_2-t_1).
\end{aligned}
$$

Therefore, as $t_2 \to t_1$,

$$|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| \to 0$$

independently of $x \in \overline{B}$.

In view of the continuity of the function f, it is clear that the operator $\mathcal{F}$ is continuous. Now consider $\mathcal{F} : B_r \to B_r$. From the above analysis, Arzela-Ascoli theorem tells us that $\mathcal{F}(B_r)$ is a relatively compact subset of $\mathcal{C}$. Thus the conclusion of Theorem 2.5 implies that the problem (1.2) has at least one solution. This completes the proof. $\square$

Corollary 3.4. *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

$$|f(t, x)| \le \nu(t)$$

for $t \in [0, T]$, $x \in \mathbb{R}$ with $\nu \in C([0, T], \mathbb{R}^+)$. Then problem (1.2) *has at least one solution.*

In this situation, since $\nu \in L^\infty([0, T], \mathbb{R}^+)$, we let $d = 0$ in Theorem 3.3, we obtain the following result.

Corollary 3.5. *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exists a function $m \in L^\infty([0, T], \mathbb{R}^+)$ such that*

$$|f(t, x)| \le m(t) + d|x|, \quad d \ge 0.$$

If $dB < 1$ (B is defined by (3.3)*), then problem* (1.2) *has at least one solution.*

The proof of this Corollary is similar to that of Theorem 3.3.

Theorem 3.6. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:*
(1) there exist a function $m \in L^\infty([0,T], \mathbb{R}^+)$ and a nondecreasing function $\varphi : [0,\infty) \to [0,\infty)$ such that

$$|f(t,x)| \le m(t)\varphi(|x|), \quad \text{for } t \in [0,T],\ x \in \mathbb{R}.$$

(2) there exists a constant $M > 0$ such that

$$\frac{M}{O + \varphi(M)Q} > 1, \tag{3.4}$$

where

$$\begin{gathered} O = \frac{T^\gamma \Gamma(2-\gamma)|c_2|}{|b_2|} + \Big| \frac{b_1 c_2 T^\gamma \Gamma(2-\gamma)}{(a_1+b_1)b_2} - \frac{c_1}{a_1+b_1} \Big|, \\ Q = \|m\|_{L^\infty} T^\alpha \Big(1 + \frac{|b_1|}{|a_1+b_1|} \Big) \Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)} \Big). \end{gathered}$$

Then problem (1.2) *has at least one solution.*

Proof. We, firstly, prove that $\mathcal{F}$ maps bounded sets into bounded sets in $\mathcal{C}$. Let $\overline{B}$ be a bounded subset of $\mathcal{C}$ and assume that $\|x\| \le r$ for any $x \in \overline{B}$. As in the proof of the above theorems, we have the following estimates

$$\begin{gathered} |(\mathcal{F}_1 x)(t)| \le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,x(s))| ds \le \varphi(r)\|m\|_{L^\infty} \frac{T^\alpha}{\Gamma(\alpha+1)}, \\ |(\mathcal{F}_2 x)(t)| \le T|k_2^x| + |k_1^x|, \\ T|k_2^x| \le \varphi(r)\|m\|_{L^\infty} \frac{\Gamma(2-\gamma)T^\alpha}{\Gamma(\alpha-\gamma+1)} + \frac{T^\gamma\Gamma(2-\gamma)|c_2|}{|b_2|}, \\ \begin{aligned} |k_1^x| \le & \frac{|b_1|}{|a_1+b_1|} \Big(\varphi(r)\|m\|_{L^\infty} \frac{T^\alpha}{\Gamma(\alpha+1)} + \varphi(r)\|m\|_{L^\infty} \frac{\Gamma(2-\gamma)T^\alpha}{\Gamma(\alpha-\gamma+1)} \Big) \\ & + \Big| \frac{b_1 c_2 T^\gamma \Gamma(2-\gamma)}{(a_1+b_1)b_2} - \frac{c_1}{a_1+b_1} \Big|. \end{aligned} \end{gathered}$$

Hence we have

$$\begin{aligned} \|\mathcal{F}x\| \le & \frac{T^\gamma \Gamma(2-\gamma)|c_2|}{|b_2|} + \Big| \frac{b_1 c_2 T^\gamma \Gamma(2-\gamma)}{(a_1+b_1)b_2} - \frac{c_1}{a_1+b_1} \Big| \\ & + \varphi(r)\|m\|_{L^\infty} T^\alpha \Big(1 + \frac{|b_1|}{|a_1+b_1|} \Big) \Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)} \Big) \\ \le & O + \varphi(r)Q. \end{aligned}$$

This implies that $\mathcal{F}(\overline{B})$ is bounded in $\mathcal{C}$.

Secondly, we claim that $\mathcal{F}$ is equicontinuous on bounded subsets of $\mathcal{C}$. The proof of this claim is the same as the corresponding part in the proof of Theorem 3.3.

Finally, let $x = \lambda \mathcal{F} x$ for some $\lambda \in (0,1)$. Then for each $t \in [0,T]$, we have

$$|x(t)| = |\lambda(\mathcal{F}x)(t)| \le O + \varphi(\|x\|)Q.$$

That is to say, we have

$$\frac{\|x\|}{O + \varphi(\|x\|)Q} \le 1.$$

Due to (3.4), we know that there exists M such that $\|x\| \neq M$. Let

$$U = \{y \in \mathcal{C} : \|y\| < M\}.$$

The operator $\mathcal{F} : \overline{U} \to \mathcal{C}$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda \mathcal{F} x$ for some $\lambda \in (0,1)$. As a consequence of Theorem 2.6, we deduce that $\mathcal{F}$ has a fixed point $x \in \overline{U}$ which is a solution of (1.2). This completes the proof. □

4. Existence results for problem (1.3)

Lemma 4.1. *For any $y, \xi, \chi \in C([0,T],\mathbb{R})$, the unique solution of the fractional non-separated integral boundary-value problem*

$$\begin{gathered}
{}^cD^\alpha x(t) = y(t), \quad t \in [0,T],\ 1 < \alpha \le 2, \\
a_1 x(0) + b_1 x(T) = c_1 \int_0^T \xi(s)ds, \\
a_2({}^cD^\gamma x(0)) + b_2({}^cD^\gamma x(T)) = c_2 \int_0^T \chi(s)ds, 0 < \gamma < 1,
\end{gathered}$$

is given by

$$\begin{aligned}
x(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \frac{t\Gamma(2-\gamma)}{T^{1-\gamma}} \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds \\
&+ \frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2} \int_0^T \chi(s)ds - \frac{b_1}{a_1+b_1}\Big\{ \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \\
&- T^\gamma \Gamma(2-\gamma) \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds\Big\} \\
&- \frac{b_1 T^\gamma \Gamma(2-\gamma)c_2}{b_2(a_1+b_1)} \int_0^T \chi(s)ds + \frac{c_1}{a_1+b_1} \int_0^T \xi(s)ds.
\end{aligned}$$

To obtain the existence results of problem (1.3), in view of Lemma 4.1, we define an operator $\mathcal{S} : \mathcal{C} \to \mathcal{C}$ as

$$\begin{aligned}
&(\mathcal{S}x)(t) \\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds - \frac{t\Gamma(2-\gamma)}{T^{1-\gamma}} \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} f(s,x(s))ds \\
&+ \frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2} \int_0^T h(s,x(s))ds - \frac{b_1}{a_1+b_1}\Big\{ \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \\
&- T^\gamma \Gamma(2-\gamma) \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} f(s,x(s))ds\Big\} \\
&- \frac{b_1 T^\gamma \Gamma(2-\gamma)c_2}{b_2(a_1+b_1)} \int_0^T h(s,x(s))ds + \frac{c_1}{a_1+b_1} \int_0^T g(s,x(s))ds.
\end{aligned} \tag{4.1}$$

Observe that problem (1.3) has solutions if and only if the operator equation $\mathcal{S}x = x$ has solution.

From the definitions of the operators $\mathcal{F}$ and $\mathcal{S}$, we know that the difference between them is very apparent; i.e., c_1, c_2 in (3.1) were replaced by $c_1 \int_0^T g(s,x(s))ds$ and $c_2 \int_0^T h(s,x(s))ds$ in (4.1). It is easy to prove the following theorems, since they are similar to the ones obtained in section 3. Therefore, we omit the proofs of the following theorems.

Theorem 4.2. *Suppose that $f,g,h:[0,T]\times\mathbb{R}\to\mathbb{R}$ are continuous and satisfy*

$$\begin{gathered}
|f(t,x)-f(t,y)|\le m_1(t)|x-y|,\\
|g(t,x)-g(t,y)|\le m_2(t)|x-y|,\\
|h(t,x)-h(t,y)|\le m_3(t)|x-y|,
\end{gathered}$$

for each $t\in[0,T]$ and all $x,y\in\mathbb{R}$ with $m_1\in L^\infty([0,T],\mathbb{R}^+)$ and $m_2,m_3\in L^1([0,T],\mathbb{R}^+)$. If

$$\begin{aligned}
&\|m_1\|_{L^\infty}T^\alpha\Big(1+\frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)}+\frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big)\\
&+\frac{T^\gamma\Gamma(2-\gamma)|c_2|\|m_3\|_{L^1}}{|b_2|}\Big(1+\frac{|b_1|}{|a_1+b_1|}\Big)+\frac{|c_1|\|m_2\|_{L^1}}{|a_1+b_1|}<1,
\end{aligned}$$

then problem (1.3) has a unique solution.

Theorem 4.3. *Let $f,g,h:[0,T]\times\mathbb{R}\to\mathbb{R}$ be continuous functions. Assume that*

$$\begin{gathered}
|f(t,x)|\le m_1(t)+d_1|x|^{\rho_1},\\
|g(t,x)|\le m_2(t)+d_2|x|^{\rho_2},\\
|h(t,x)|\le m_3(t)+d_3|x|^{\rho_3},
\end{gathered}$$

for each $t\in[0,T]$ and $x\in\mathbb{R}$ with $m_1\in L^\infty([0,T],\mathbb{R}^+)$, $m_2,m_3\in L^1([0,T],\mathbb{R}^+)$ and $d_i\ge 0$, $0\le\rho_i<1$, $i=1,2,3$. Then problem (1.3) has at least one solution on $[0,T]$.

Theorem 4.4. *Let $f,g,h:[0,T]\times\mathbb{R}\to\mathbb{R}$ be continuous functions. Assume that:*
(1) there exist functions $m_1\in L^\infty([0,T],\mathbb{R}^+)$, $m_2,m_3\in L^1([0,T],\mathbb{R}^+)$ and three nondecreasing functions $\varphi_i:[0,\infty)\to[0,\infty)$, $i=1,2,3$, such that for $t\in[0,T]$, $x\in\mathbb{R}$

$$\begin{gathered}
|f(t,x)|\le m_1(t)\varphi_1(|x|),\\
|g(t,x)|\le m_2(t)\varphi_2(|x|),\\
|h(t,x)|\le m_3(t)\varphi_3(|x|).
\end{gathered}$$

(2) there exists a constant $M>0$ such that

$$\frac{M}{\varphi_1(M)Q+\varphi_3(M)\|m_3\|_{L^1}O+\frac{|c_1|}{|a_1+b_1|}\varphi_2(M)\|m_2\|_{L^1}}>1,$$

where

$$\begin{gathered}
O=\frac{T^\gamma\Gamma(2-\gamma)|c_2|}{|b_2|}\Big(1+\frac{|b_1|}{|a_1+b_1|}\Big),\\
Q=\|m_1\|_{L^\infty}T^\alpha\Big(1+\frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)}+\frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big).
\end{gathered}$$

Then problem (1.3) has at least one solution.

5. Examples

In this section, we give two simple examples to illustrate the main results. subsection*Example 1 Consider the fractional boundary-value problem

$$\begin{gathered} {}^cD^{\frac{3}{2}}x(t) = \frac{1}{(t+4)^2}(\sin x(t) + \frac{|x(t)|}{1+|x(t)|}), \quad t \in [0,1], \\ 3x(0) + \frac{1}{2}x(1) = 2.5, \\ 2({}^cD^{1/2}x(0)) + \frac{1}{3}({}^cD^{1/2}x(1)) = -\frac{1}{3}, \end{gathered} \tag{5.1}$$

Here $\alpha = \frac{3}{2}$, $\gamma = \frac{1}{2}$, $a_1 = 3$, $b_1 = \frac{1}{2}$, $c_1 = 2.5$, $a_2 = 2$, $b_2 = \frac{1}{3}$, $c_2 = -\frac{1}{3}$, $T = 1$ and $f(t,x) = \frac{1}{(t+4)^2}(\sin x + \frac{|x|}{1+|x|})$. Since

$$\begin{aligned} &|f(t,x) - f(t,y)| \le \frac{1}{(t+4)^2}|x-y| \le \frac{1}{16}|x-y|, \\ &LT^\alpha\Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big) \\ &\approx \frac{1}{16} \times \frac{8}{7} \times (0.7523 + 0.8862) = 0.1170 < 1. \end{aligned}$$

Thus, by Corollary 3.2, the boundary value problem (5.1) has a unique solution on $[0,1]$.

Example 2. Let $\alpha = \frac{5}{4}$, $\gamma = \frac{1}{3}$ and $T = \pi$. Consider the fractional integral boundary-value problem

$$\begin{gathered} {}^cD^{\frac{5}{4}}x(t) = 2t^3 - 3\ln(3+t) + (3t+1)^2\frac{|x(t)|^{1/2}}{2+\cos^2 x(t)}, \quad t \in [0,\pi], \\ \frac{1}{2}x(0) + x(\pi) = \int_0^\pi \frac{x^{1/3}(t)}{7(1+|x(t)|)}ds, \\ 2({}^cD^{1/3}x(0)) + 3({}^cD^{1/3}x(\pi)) = \int_0^\pi (3t^3 - 5 + e^{-t}|x(t)|^{2/5})ds, \end{gathered} \tag{5.2}$$

Since $f(t,x) = 2t^3 - 3\ln(3+t) + (3t+1)^2\frac{|x|^{1/2}}{2+\cos^2 x}$, $g(t,x) = \frac{x^{1/3}}{7(1+|x|)}$, $h(t,x) = (3t^3 - 5 + e^{-t}|x|^{2/5})$, $a_1 = \frac{1}{2}$, $b_1 = c_1 = c_2 = 1$, $a_2 = 2$ and $b_2 = 3$, we have

$$\begin{gathered} |f(t,x)| \le |2t^3 - 3\ln(3+t)| + (3\pi+1)^2|x|^{1/2}, \\ |g(t,x)| \le \frac{1}{7}|x|^{1/3}, \quad |h(t,x)| \le |3t^3 - 5| + |x|^{2/5}. \end{gathered}$$

Now it is easy to verify that all conditions of Theorem 4.3 are satisfied. Therefore, the fractional boundary value problem (5.2) has at least one solution on $[0,\pi]$.

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