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EXISTENCE OF SOLUTIONS FOR NONLINEAR IMPULSIVE NEUTRAL INTEGRO-DIFFERENTIAL EQUATIONS OF SOBOLEV TYPE WITH NONLOCAL CONDITIONS IN BANACH SPACES

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Abstract. In this article, we prove the existence of mild and strong solutions for nonlinear impulsive integro-differential equations of Sobolev type with nonlocal initial conditions. The results are obtained by using semigroup theory and the Schauder fixed point theorem. An example is provided to illustrate the theory.

1. Introduction

Differential equations arise in many areas of science and technology, specifically whenever a deterministic relation involving some continuously varying quantities (modeled by functions) and their rates of change in space and/or time are known or postulated. This is illustrated in classical mechanics where the motion of a body is described by its position and velocity as the time varies. It is well known that the systems described by partial differential equations can be expressed as abstract differential equations [20]. These equations occur in various fields of study and each system can be represented by different forms of differential or integro-differential equations in Banach spaces. Using the method of semigroups, various solutions of nonlinear and semilinear evolution equations have been discussed by Pazy [20]. The study of abstract nonlocal semilinear initial value problems was initiated by Byszewski [9, 10, 11]. Because it is demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problems. Such problems with nonlocal conditions have been extensively studied in the literature [1, 2, 4, 5, 6, 23]. Showalter [22] established the existence of solutions of semilinear evolution equations of Sobolev type in Banach spaces. This type of equations arise in various applications such as in the flow of fluid through fissured rocks, thermodynamics, and shear in second-order fluids. For more details, we refer the reader to [8, 16, 17].

Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention during the last few decades. There are also a number of applications in which the delayed argument

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occurs in the derivative of the state variable as well as in the independent variable, as in the so called neutral differential difference equations. A neutral functional differential equation is one in which the derivatives of the past history or derivatives of functionals of the past history are involved as well as the present state of the system. A good guide to the literature for neutral functional differential equations is the book by Hale and Verduyn Lunel [13] and the references therein. Hernandez [14] established the existence results for partial neutral functional differential equations with nonlocal conditions modeled as

$$\begin{gathered} \frac{d}{dt}\big[u(t) + F(t, u(t))\big] = Au(t) + G(t, u(t)) \\ u_\sigma = \varphi + q(x_{t_1}, x_{t_2}, \dots, x_{t_n}) \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where A is the infinitesimal generator of an analytic semigroup $T(t)$ on a Banach space. He made use of fixed point theorems and the results mentioned in Pazy [20]. For results on neutral partial differential equations with nonlocal and classical conditions, we refer to the papers of Hernandez and Henryquez [15], Fu and Ezzinbi [12], and references therein. Controllability of functional differential systems of Sobolev type in Banach spaces has been first studied by Balachandran and Dauer [3].

Differential equations arise in many real world problems such as physics, population dynamics, ecology,biological systems, biotechnology, optimal control and so forth. Much has been done the assumption that the state variables and systems parameters change continuously. However, one may easily visualize that abrupt changes such as shock, harvesting and disasters may occur in nature. These phenomena are short time perturbations whose duration is negligible in comparison with the duration of the whole evolution process. Consequently, it is natural to assume, in modeling these problems, that these perturbations act instantaneously, that is in the form of impulses. The theory of impulsive differential equation [18, 21] is much richer than the corresponding theory of differential equations without impulsive effects. The impulsive condition

$$\Delta u(t_i) = u(t_i^+) - u(t_i^-) = I_i(u(t_i^-)), \quad i = 1,\ 2, \dots, m,$$

is a combination of traditional initial value problems and short-term perturbations whose duration is negligible in comparison with the duration of the process. Lin and Liu [19] discussed the iterative methods for the solution of impulsive functional differential systems.

Motivated by the above approach, the goal of this paper is to use the fixed point theorem to obtain the mild solution of the nonlinear impulsive neutral integrodifferential equation of Sobolev type with nonlocal conditions.

2. Preliminaries

Consider the nonlinear impulsive neutral integrodifferential equation of Sobolev type with nonlocal conditions of the form

$$\begin{gathered} \frac{d}{dt}\big[Bu(t) + e(t, u(t))\big] + Au(t) = f(t, u(t)) + \int_0^t k(t, s, u(s))ds, \\ t \in (0, a],\ t \neq t_k, \end{gathered} \tag{2.1}$$

$$u(0) + \sum_{i=1}^n c_i u(t_i) = u_0 \tag{2.2}$$

$$\Delta u(t_k) = I_k(u_{t_k}), \quad k = 1, 2, \dots, m, \tag{2.3}$$

where $0 \le t_1 < t_2 < \dots < t_p \le a$, B and A are linear operators with domains contained in a Banach space X and ranges contained in a Banach space Y and the nonlinear operators $f : I \times X \to Y$, $k : I^2 \times X \to Y$, $e : I \times X \to Y$ and $I_k : X \to Y$ are appropriate functions and the symbol $\Delta u(t_k)$ represent the jump of the function u at t, which is defined by $\Delta u(t_k) = u(t^+) - u(t^-)$. Here $I = [0, a]$. In this paper, we establish the existence of a nonlinear impulsive neutral integro-differential equation of Sobolev type with nonlocal conditions using Schauder fixed point theorem.

To prove our main theorem we assume certain conditions on the operators A and B. Let X and Y be Banach spaces with norm $|\cdot|$ and $\|\cdot\|$ respectively. The operators $A : \mathcal{D}(A) \subset X \to Y$ and $B : \mathcal{D}(B) \subset X \to Y$ satisfy the following hypothesis:

- (M1) A and B are closed linear operators,
- (M2) $\mathcal{D}(B) \subset \mathcal{D}(A)$ and B is bijective,
- (M3) $B^{-1} : Y \to \mathcal{D}(B)$ is continuous.

The hypothesis (M1)–(M3) and the closed graph theorem imply the boundedness of the linear operator $AB^{-1} : Y \to Y$ and $-AB^{-1}$ generates a uniformly continuous semigroup $S(t), t \ge 0$, of bounded linear operators from Y into Y and so $\max_{t\in I} \|S(t)\|$ is finite. We denote $M = \max_{t\in I} \|S(t)\|$, $R = \|B^{-1}\|$. Let $\mathbb{B}_r = \{x \in X : |x| \le r\}$ and $c = \sum_{i=1}^{p} |c_i|$.

In this article, we assume that there exists an operator E on $\mathcal{D}(E) = X$ given by the formula

$$E = \Big[I + \sum_{i=1}^{n} c_i B^{-1} S(t_i) B\Big]^{-1} \quad \text{and} \quad Eu_0 \in \mathcal{D}(B),$$

with

$$\begin{aligned}
&E\Big\{B^{-1}e(t, u(t)) - B^{-1}S(t_i)e(0, u(0)) + \int_0^{t_i} AS(t_i - s)B^{-1}e(s, u(s))ds \\
&+ \int_0^{t_i} B^{-1}S(t_i - s)[f(s, u(s)) + \int_0^{s} k(s, \tau, u(\tau))d\tau]ds \\
&- \sum_{0<t_k<t_i} B^{-1}S(t_i - t_k)I_k u(t_k)\Big\} \in \mathcal{D}(B),
\end{aligned}$$

for $i = 1, 2, \dots, p$.

The existence of E can be observed from the following fact (see [9]). Suppose that $\{S(t)\}$ is a C_0 semigroup of operators on X such that $\|B^{-1}S(t_i)B\| \le Ce^{-\delta t_i}(i = 1, 2, \dots, n)$ where δ is a positive constant and $C \le 1$. If $\sum_{i=1}^{p} |c_i|e^{-\delta t_i} < 1/C$ then $\|\sum_{i=1}^{p} c_i B^{-1}S(t_i)B\| < 1$. So such an operator E exists on X.

Definition 2.1. A continuous solution u of the integral equation

$$\begin{aligned}
u(t) = {}& B^{-1}S(t)BEu_0 + \sum_{i=1}^{n} c_i B^{-1}S(t)BE \\
& \times \Big\{B^{-1}e(t,u(t)) - B^{-1}S(t_i)e(0,u(0)) - \sum_{0<t_i<t} B^{-1}S(t_i - t_k)I_k u(t_k)\Big\} \\
& + \sum_{i=1}^{n} c_i B^{-1}S(t)BE\Big\{\int_0^{t_i} B^{-1}S(t_i - s)\Big[Ae(s,u(s)) + f(s,u(s)) \\
& + \int_0^s k(s,\tau,u(\tau))d\tau\Big]ds\Big\} + B^{-1}S(t)e(0,u(0)) - B^{-1}e(t,u(t)) \\
& + \int_0^t S(t-s)B^{-1}\Big[Ae(s,u(s)) + f(s,u(s)) + \int_0^s k(s,\tau,u(\tau))d\tau\Big]ds \\
& + \sum_{0<t_i<t} B^{-1}S(t-t_k)I_k u(t_k)
\end{aligned} \tag{2.4}$$

is said to be a mild solution of problem (2.1)-(2.3) on I.

Definition 2.2. A function u is said to be a strong solution of (2.1)-(2.3) on I if u is differentiable almost everywhere on I, $u' \in L^1(I,X)$, $u(0) + \sum_{i=1}^{n} c_i u(t_i) = u_0$ and

$$\frac{d}{dt}\big[(Bu(t) + e(t,u(t))\big] + Au(t) = f(t,u(t)) + \int_0^t k(t,s,u(s))ds, \quad t \in (0,a],\ t \neq t_k$$
$$\Delta u(t_k) = I_k(u_{t_k}), \quad k = 1,2,,\dots,m$$

almost everywhere on I.

Remark 2.3. A mild solution of the neutral integro-differential (2.1)-(2.3) satisfies the condition (2.2), for (2.4)

$$\begin{aligned}
u(0) = {}& Eu_0 + \sum_{i=1}^{n} c_i E\Big\{B^{-1}e(t,u(t)) - B^{-1}S(t_i)e(0,u(0)) \\
& - \sum_{0<t_i<t} B^{-1}S(t_i - t_k)I_k u(t_k)\Big\} \\
& + \sum_{i=1}^{n} c_i E\Big\{\int_0^{t_i} S(t_i - s)B^{-1}\Big[Ae(s,u(s)) + f(s,u(s)) \\
& + \int_0^s k(s,\tau,u(\tau))d\tau\Big]ds\Big\}
\end{aligned}$$

and

$$\begin{aligned}
u(t_j) = {}& B^{-1}S(t_j)BEu_0 + \sum_{i=1}^{n} c_i B^{-1}S(t_j)BE\Big\{B^{-1}e(t,u(t)) - B^{-1}S(t_i)e(0,u(0)) \\
& - \sum_{0<t_i<t} B^{-1}S(t_i - t_k)I_k u(t_k)\Big\} \\
& + \sum_{i=1}^{n} c_i B^{-1}S(t_j)BE\Big\{\int_0^{t_i} S(t_i - s)B^{-1}[Ae(s,u(s)) + f(s,u(s))
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^s k(s,\tau,u(\tau))d\tau]ds\Big\} + B^{-1}S(t_j)e(0,u(0)) - B^{-1}e(t_j,u(t_j))\\
&+ \int_0^{t_j} S(t_j-s)B^{-1}\Big[Ae(s,u(s)) + f(s,u(s)) + \int_0^s k(s,\tau,u(\tau))d\tau\Big]ds\\
&+ \sum_{0<t_i<t} B^{-1}S(t_j-t_k)I_k u(t_k).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&u(0) + \sum_{j=1}^n c_j u(t_j)\\
&= \Big[I + \sum_{j=1}^n c_j B^{-1}S(t_j)B\Big]Eu_0 + \Big[I + \sum_{j=1}^n c_j B^{-1}S(t_j)B\Big]\sum_{i=1}^n c_i E\Big\{B^{-1}e(t,u(t))\\
&\quad - B^{-1}S(t_i)e(0,u(0)) - \sum_{0<t_i<t} B^{-1}S(t_i-t_k)I_k u(t_k)\Big\}\\
&\quad + \Big[I + \sum_{j=1}^n c_j B^{-1}S(t_j)B\Big]\sum_{i=1}^n c_i E\Big\{\int_0^{t_i} S(t_i-s)B^{-1}[Ae(s,u(s)) + f(s,u(s))\\
&\quad + \int_0^s k(s,\tau,u(\tau))d\tau]ds\Big\} + \sum_{j=1}^n c_j\Big[B^{-1}S(t_j)e(0,u(0)) - B^{-1}e(t_j,u(t_j))\\
&\quad + \int_0^{t_j} S(t_j-s)B^{-1}\Big[Ae(s,u(s)) + f(s,u(s)) + \int_0^s k(s,\tau,u(\tau))d\tau\Big]ds\\
&\quad + \sum_{0<t_i<t} B^{-1}S(t_j-t_k)I_k u(t_k)\Big]\\
&= u_0
\end{aligned}
$$

To prove the existence result, we use the following hypotheses:

(M4) The function $f : I \times X \to Y$ is continuous in t and there exists a constant $L_f > 0$ such that
$$\|f(t,u)\| \le L_f,\ \text{for } t \in I \text{ and } u \in X.$$

(M5) The function $k : I^2 \times X \to Y$ is continuous in t and there exists a constant $L_k > 0$ such that
$$\|k(t,s,u)\| \le L_k,\ \text{for } s,t \in I \text{ and } u \in X.$$

(M6) The function $e : I \times X \to Y$ is continuous in t and there exist constants $L_e > 0$, $L_0 > 0$ and $L_1 > 0$ such that
$$
\begin{gathered}
\|e(t,u(t))\| \le L_e, \quad \text{for } t \in I \text{ and } u \in X\\
\|e(0,u(0))\| \le L_0, \quad \text{for } t \in I \text{ and } u \in X\\
\|Ae(t,u(t))\| \le L_1, \quad \text{for } t \in I \text{ and } u \in X.
\end{gathered}
$$

(M7) The maps $I_k : X \to Y$ are continuous and there exists a constant $\mathcal{I} > 0$ such that
$$\|I_k(u)\| \le \mathcal{I}, \quad \text{for } k \in \mathbb{N} \text{ and } y \in X.$$

(M8)
$$\begin{aligned}&R\|BEu_o\|M+cR^2\|BE\|M[L_e+M\mathcal{I}+ML_0+aM(L_1+L_f+L_ka)]\\&+RM[L_0+L_1+L_f+L_ka+a(L_1+G_1)+\mathcal{I}+RK_1]\leq r.\end{aligned}$$

3. Main Results

Theorem 3.1. *If assumptions* (M1)-(M7) *hold, then Problem* (2.1)-(2.3) *has a mild solution on* I.

Proof. Let $E=\mathcal{C}(I,Y)$ and $\mathcal{Y}_0=\{u\in Y: u(t)\in\mathbb{B}_r,\ t\in I\}$. Clearly, $\mathcal{Y}_0$ is a bounded closed convex subset of Y. We define a mapping $F:\mathcal{Y}_0\to\mathcal{Y}_0$ by
$$\begin{aligned}(Fu)(t)&=B^{-1}S(t)BEu_0+\sum_{i=1}^n c_iB^{-1}S(t)BE\Big\{B^{-1}e(t,u(t))\\&\quad-B^{-1}S(t_i)e(0,u(0))-\sum_{0<t_i<t}B^{-1}S(t_i-t_k)I_ku(t_k)\Big\}\\&\quad+\sum_{i=1}^n c_iB^{-1}S(t)BE\Big\{\int_0^{t_i}S(t_i-s)B^{-1}[Ae(s,u(s))+f(s,u(s))\\&\quad+\int_0^s k(s,\tau,u(\tau))d\tau]ds\Big\}+B^{-1}S(t)e(0,u(0))-B^{-1}e(t,u(t))\\&\quad+\int_0^t S(t-s)B^{-1}\Big[Ae(s,u(s))+f(s,u(s))+\int_0^s k(s,\tau,u(\tau))d\tau\Big]ds\\&\quad+\sum_{0<t_i<t}B^{-1}S(t-t_k)I_ku(t_k)\end{aligned}$$
Now we shown that $F:\mathcal{Y}_0\to\mathcal{Y}_0$ is continuous. Let $\{u_n\}_0^\infty\subset\mathcal{Y}_0$ with $u_n\to u$ in $\mathcal{Y}_0$. Then there is an integer r such that $\|u_n(t)\|\leq r$, for all n and $t\in I$, so $u_n\in\mathbb{B}_r$ and $u\in\mathbb{B}_r$. From the assumptions $(M_1)-(M_7)$, we have

(a) I_k, $k=1,2,\dots,p$ is continuous.
(b) $e(t,u_n(t))\to e(t,u(t))$, for $t\in I$ and since
$$\|e(t,u_n(t))-e(t,u(t))\|<2[L_e+L_0].$$
(c) $Ae(t,u_n(t))\to Ae(t,u(t))$, for $t\in I$ and since
$$\|Ae(t,u_n(t))-Ae(t,u(t))\|<2[L_1+L_3].$$
(d) $f(t,u_n(t))\to f(t,u(t))$, for $t\in I$ and since
$$\|f(t,u_n(t))-f(t,u(t))\|<2[L_f+F_0].$$
(e) $k(t,s,u_n(s))\to k(t,s,u(s))$, for $t,\ s\in I$ and since
$$\|k(t,s,u_n(s))-k(t,s,u(s))\|<2[L_k+K_0].$$

By the dominated convergence theorem, we have
$$\begin{aligned}\|Fu_n-Fu\|&\leq R^2Mc\|BE\|\{\|e(t,u_n(t))-e(t,u(t))\|\}\\&\quad+R^2Mc\|BE\|\int_0^{t_i}S(t_i-s)\Big[\{\|Ae(s,u_n(s))-Ae(s,u(s))\|\}\\&\quad+\{\|f(s,u_n(s))-f(s,u(s))\|\}\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^s \{\|k(s,\tau,u_n(\tau)) - k(s,\tau,u(\tau))\|\} d\tau \Big] ds \\
&+ R^2 Mc\|BE\| \sum_{0<t_i<t} S(t_i - t_k)\{\|I_k(u_n(t_k)) - I_k(u(t_k))\|\} \\
&+ R\{\|e(t,u_n(t)) - e(t,u(t))\|\} \\
&+ RM \int_0^t \Big[\{\|Ae(s,u_n(s)) - Ae(s,u(s))\|\} \\
&+ \{\|f(s,u_n(s)) - f(s,u(s))\|\} \\
&+ \int_0^s \{\|k(s,\tau,u_n(\tau)) - k(s,\tau,u(\tau))\|\} d\tau \Big] ds \\
&+ RM \sum_{0<t_i<t} \{\|I_k(u_n(t_k)) - I_k(u(t_k))\|\} \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Thus F is continuous. Moreover, F maps $\mathcal{Y}_0$ into a precompact subset of $\mathcal{Y}_0$. We prove that the set $\mathcal{Y}_0(t) = \{(Fu)(t) : u \in \mathcal{Y}_0\}$ is precompact in X for every fixed $t \in I$. We shall show that $F(\mathcal{Y}_0) = \mathcal{Z} = \{Fu : u \in \mathcal{Y}_0\}$ is an equicontinuous family of functions.

For $0 < s < t$, we have

$$
\begin{aligned}
&\|(Fu)(t) - (Fu)(s)\| \\
&\le \|B^{-1}(S(t) - S(s))BEu_0\| \\
&\quad + \sum_{i=1}^n c_i \|B^{-1}(S(t) - S(s))BE\| \Big\{ \|B^{-1}e(t,u(t)) - B^{-1}S(t_i)e(0,u(0)) \\
&\quad - \sum_{0<t_i<t} B^{-1}S(t_i - t_k)I_k u(t_k)\| \Big\} + \sum_{i=1}^n c_i \|B^{-1}(S(t) - S(s))BE\| \\
&\quad \times \{ \int_0^{t_i} \|S(t_i - s)B^{-1}\Big[Ae(s,u(s)) + f(s,u(s)) + \int_0^s k(s,\tau,u(\tau))d\tau\Big]\| ds \} \\
&\quad + \|B^{-1}(S(t) - S(s))e(0,u(0))\| + \|B^{-1}(e(t,u(t)) - e(s,u(s)))\| \\
&\quad + \int_0^t \|(S(t-\theta) - S(s-\theta)B^{-1}\big\{Ae(\theta,u(\theta)) + f(\theta,u(\theta)) + \int_0^\theta k(s,\tau,u(\tau))d\tau\big\}\| d\theta \\
&\quad + \int_\theta^t \|S(t-\theta)B^{-1}\big[Ae(\theta,u(\theta)) + f(\theta,u(\theta)) + \int_0^\theta k(\theta,\tau,u(\tau))d\tau\big]\| d\theta \\
&\quad + \sum_{0<t_i<t} \|B^{-1}(S(t-s))I_k u(t_k)\| \\
&\le \Big\{ R\|BEu_0\| + R^2\|BE\|[L_e + M\mathcal{I} + ML_0]c \\
&\quad + R^2 Ma\|BE\|[L_1 + L_f + L_k a]c + RL_0 \Big\} \|s(t) - S(s)\| \\
&\quad + \{RL_0 + RM[L_1 + L_f + L_k a]\}|t-s| \\
&\quad + R(L_e + L_f + L_k a) \int_0^t \|S(t-\theta) - S(s-\theta)\| d\theta.
\end{aligned}
$$

The right hand side of the above inequality is independent of $u \in \mathcal{Y}_0$ and tends to zero as $s \to t$ as a consequence of the continuity of $S(t)$ in the uniform operator topology for $t > 0$ which follows from the compactness of $S(t)$, $t > 0$. It is also clear that Z is bounded in Y. Thus by Arzela-Ascoli's theorem, Z is precompact. Hence by the Schauder fixed point theorem, F has a fixed point in Y_0 and any fixed point of F is a mild solution of (2.1)-(2.3) on I such that $u(t) \in X$, for $t \in I$. □

Next we prove that the problem (2.1)-(2.3) has a strong solution.

Theorem 3.2. *Assume that*

(i) *Conditions* (M1)–(M8) *hold.*
(ii) Y *is a reflexive Banach space with norm* $\|\cdot\|$.
(iii) $f : I \times X \to Y$ *is continuous in* t *on* I *and there exists a constant* $G_1 > 0$ *such that*
$$\|f(t, u) - f(s, v)\| \le G_1[|t - s| + \|u - v\|],$$
for $t, s \in I$ *and* $u, v \in X$.
(iv) $k : I^2 \times X \to Y$ *is continuous in* t *and there exists a constant* $K_1 > 0$ *such that*
$$\|k(t, \tau, u) - k(s, \tau, u)\| \le K_1[|t - s|],$$
for $\tau, s, t \in I$, $u \in X$,
(v) $e : I \times X \to Y$ *is continuous and there exist constants* $K > 0$ *and* $K_1 > 0$ *such that*
$$\begin{gathered}\|Ae(t, u(t) - Ae(s, u(s))\| \le L_2[|t - s|], \quad \text{for } s, t \in I,\ u \in X,\\ \|e(t, u(t) - e(s, u(s))\| \le L[|t - s|], \quad \text{for } s, t \in I,\ u \in X.\end{gathered}$$
(vi) $Eu_0 \in \mathcal{D}(B)$,
$$\begin{aligned}&E\Big\{B^{-1}e(t, u(t)) - B^{-1}S(t_i)e(0, u(0))\\ &+ \int_0^{t_i} B^{-1}S(t_i - s)\Big[Ae(s, u(s)) + f(s, u(s)) + \int_0^s k(s, \tau, u(\tau))d\tau\Big]\,ds\\ &- \sum_{0<t_i<t} B^{-1}S(t_i - t_k)I_k u(t_k)\Big\} \in D(B),\end{aligned}$$
for $i = 1, 2, \dots, p$.

Then u *is a strong solution of problem* (2.1)–(2.3) *on* I.

Proof. Since all the assumptions of Theorem 3.1 are satisfied, then (2.1)-(2.3) has a mild solution belonging to $\mathcal{C}(I, X)$. Now we shall show that u is a strong solution of (2.1)-(2.3) on I. For any $t \in I$, we have
$$\begin{aligned}&\|u(t + h) - u(t)\|\\ &\le \|B^{-1}[T(t + h) - T(t)]BEu_0\\ &\quad + \sum_{i=1}^n c_i\|B^{-1}(S(t + h) - S(t))BE\|\Big\{\|B^{-1}e(t, u(t)) - B^{-1}S(t_i)e(0, u(0))\\ &\quad - \sum_{0<t_i<t} B^{-1}S(t_i - t_k)I_k u(t_k)\|\Big\} + \sum_{i=1}^n c_i\|B^{-1}(S(t + h) - S(t))BE\|\\ &\quad \times \Big\{\int_0^{t_i} \|S(t_i - s)B^{-1}[Ae(s, u(s)) + f(s, u(s)) + \int_0^s k(s, \tau, u(\tau))d\tau]\|ds\Big\}\\ &\quad + \|B^{-1}(S(t + h) - S(t))e(0, u(0))\| + \|B^{-1}(e(t + h, u) - e(t, u))\|\end{aligned}$$

$$
\begin{aligned}
&+\int_0^h \|S(t+h-s)B^{-1}\big[Ae(s,u(s))+f(s,u(s)+\int_0^s k(s,\tau,u(\tau))d\tau\big]\|ds\\
&+\int_h^{t+h} \|(S(t+h-s)B^{-1}\big[Ae(s,u(s))+f(s,u(s))+\int_0^s k(s,\tau,u(\tau))d\tau\big]\|ds\\
&+\int_0^t \|S(t-s)B^{-1}\big[Ae(s,u(s))+f(s,u(s))+\int_0^s k(s,\tau,u(\tau))d\tau\big]\|ds\\
&+\sum_{0<t_i<t}\|B^{-1}(S(t+h-t_k)-S(t-t_k))I_k u(t_k)\|\\
\le{}& \|B^{-1}S(t)[S(h)-I]BEu_0\|\\
&+\sum_{i=1}^n c_i\|B^{-1}S(t)(S(h)-I)BE\|\Big\{\|B^{-1}e(t,u(t))\|+\|B^{-1}S(t_i)e(0,u(0))\|\\
&+\sum_{0<t_i<t}\|B^{-1}S(t_i-t_k)I_k u(t_k)\|\Big\}+\sum_{i=1}^n c_i\|B^{-1}S(t)(S(h)-I)BE\|\\
&\times\Big\{\int_0^{t_i}\|S(t_i-s)B^{-1}\|\Big[\|Ae(s,u(s))\|+\|f(s,u(s))\|\\
&+\int_0^s\|k(s,\tau,u(\tau))\|d\tau\Big]ds\Big\}+\|B^{-1}S(t)(S(h)-I)e(0,u(0))\|\\
&+\|B^{-1}(e(t+h,u)-e(t,u))\|+\int_0^h\|(S(t+h-s)B^{-1}\|\Big[\|Ae(s,u(s))\|\\
&+\|f(s,u(s))\|+\int_0^s\|k(s,\tau,u(\tau))\|d\tau\Big]ds\\
&+\int_h^{t+h}\|(S(t+h-s)B^{-1}\|\Big[\|Ae(s,u(s))\|+\|f(s,u(s))\|\\
&+\int_0^s\|k(s,\tau,u(\tau))\|d\tau\Big]ds+\int_0^t\|S(t-s)B^{-1}\|\Big[\|Ae(s,u(s))\|+\|f(s,u(s))\|\\
&+\int_0^s\|k(s,\tau,u(\tau))\|d\tau\Big]ds+\sum_{0<t_i<t}\|B^{-1}S(t-t_k)(S(h)-I)I_k u(t_k)\|\\
\le{}& \|B^{-1}S(t)[S(h)-I]BEu_0\\
&+\sum_{i=1}^n c_i\|B^{-1}S(t)(S(h)-I)BE\|\{\|B^{-1}e(t,u(t))\|+\|B^{-1}S(t_i)e(0,u(0))\|\\
&+\sum_{0<t_i<t}\|B^{-1}S(t_i-t_k)I_k u(t_k)\|\}+\sum_{i=1}^n c_i\|B^{-1}S(t)(S(h)-I)BE\|\\
&\times\Big\{\int_0^{t_i}\|S(t_i-s)B^{-1}\|\Big[\|Ae(s,u(s))\|+\|f(s,u(s))\|\\
&+\int_0^s\|k(s,\tau,u(\tau))\|d\tau\Big]ds\Big\}+\|B^{-1}S(t)(S(h)-I)e(0,u(0))\|\\
&+\|B^{-1}(e(t+h,u)-e(t,u))\|\\
&+\int_0^h\|S(t+h-s)B^{-1}\|\Big[\|Ae(s,u(s))\|+\|f(s,u(s))\|+\int_0^s\|k(s,\tau,u(\tau))\|d\tau\Big]ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t \|S(t-s)B^{-1}\|\Big[\|Ae(s+h,u(s+h))-Ae(s,u(s))\| \\
&+\|f(s+h,u(s+h))-f(s,u(s))\|+\int_0^s \|k(s+h,\tau,u(\tau))-k(s,\tau,u(\tau))\|d\tau\Big]ds \\
&+\sum_{0<t_i<t}\|B^{-1}S(t-t_k)(S(h)-I)I_k u(t_k)\|
\end{aligned}
$$

using our assumptions we observe that

$$
\begin{aligned}
&\|u(t+h)-u(t)\| \\
&\le R\|BEu_0\|Mh\|AB^{-1}\|+R^2Mhc\|BE\|[L_e+M\mathcal{I}+ML_0]\|AB^{-1}\| \\
&\quad +R^2Mhc\|BE\|[Ma(L_1+L_e+L_ka)]\|AB^{-1}\| \\
&\quad +RMhL_0+RLh+RMh(L_1+L_e+K_1a)+RMh\mathcal{I} \\
&\quad +RM\int_0^t\{L_2[h+\|u(s+h)-u(s)]+G_1[h+\|u(s+h)-u(s)]\}ds \\
&\quad +RM\int_0^t\Big\{\int_0^s\|k(s+h,\tau,u(\tau))-k(s,\tau,u(\tau))\|d\tau \\
&\quad +\int_s^{s+h}\|k(s+h,\tau,u(\tau))\|d\tau\Big\}ds \\
&\le h\{R\|BEu_0\|M\|AB^{-1}\| \\
&\quad +R^2Mc\|BE\|[L_e+M\mathcal{I}+ML_0+Ma(L_1+L_e+L_ka)]\|AB^{-1}\| \\
&\quad +RM[L_0+L_1+L_e+L_ka+a(L_2+G_1+K_1+K_1a)+\mathcal{I}]+RL_k\} \\
&\quad +RM(L_2+G_1)\int_0^t\|u(s+h)-u(s)\|ds \\
&\le Ph+\mathcal{Q}\int_0^t\|u(s+h)-u(s)\|ds
\end{aligned}
$$

where

$$
\begin{gathered}
\mathcal{P}=R\|BEu_0\|M\|AB^{-1}+R^2Mc\|BE\|\Big[L_e+M\mathcal{I}+ML_0 \\
+Ma(L_1+L_e+L_ka)\Big]\|AB^{-1}\|+RM\big[L_0+L_1 \\
+L_e+L_ka+a(L_2+G_1+K_1+K_1a)+\mathcal{I}\big]+RL_k, \\
\mathcal{Q}=RM(L_2+G_1).
\end{gathered}
$$

Hence by Gronwall's inequality

$$
\|u(t+h)-u(t)\|\le \mathcal{P}he^{\mathcal{Q}},\quad \text{for } t\in I.
$$

Therefore, u is Lipschitz continuous on I. The Lipschitz continuity of u on I combined with (iii)–(v) implies that

$$
t\to f(t,u(t)),\quad t\to e(t,u(t)),\quad t\to\int_0^t k(t,s,u(s))ds.
$$

Hence, u is strong solution of the problem (2.1)-(2.3) on $(0,a]$. □

4. Example

Consider the partial integro-differential equation of neutral type

$$
\begin{gathered}
\frac{\partial}{\partial t}\Big[z(t,x) - z_{xx}(t,x) + \int_{-\infty}^{t} a_1(s-t) z_t(s,x)ds\Big] - z_{xx}(t,x) \\
= \rho(t, z(t,x)) + \int_0^t a(t,s) z(s,x) ds, \quad x \in [0,\pi],\ t \in I, \\
z(t,0) = z(t,\pi) = 0, \quad t \in I, \\
z(0,x) + \sum_{k=1}^{p} z(t_k, x) = z_0(x) \quad 0 < t_1 < t_2 < \dots < t_p < b;\ x \in [0,a] \\
\Delta z|_{t=t_i} = I_i(z(x)) = (\gamma_i(z(x)) + t_i)^{-1}, \quad z \in X,\ 1 \le i \le p,
\end{gathered}
\tag{4.1}
$$

where $a(t,s)$ is continuous such that $\|a(t,s)\| \le L_1$ and the constant γ_i is small.

Let us take $X = Y = \mathcal{L}^2[0,\pi]$ to be endowed with the usual norm $\|\cdot\|_{\mathcal{L}_2}$. and let

$$
\begin{gathered}
e(t,z) = \int_{-\infty}^{t} a_1(s-t) z_t(s,x) ds \\
f(t,z) = \rho(t, z(t,x)) \\
\int_0^t k(t,s,z) ds = \int_0^t a(t,s) z(s,x) ds \\
I_i(z(x)) = (\gamma_i(z(x)) + t_i)^{-1}.
\end{gathered}
$$

Define the operator $A : \mathcal{D}(A) \subset X \to Y$ and $B : \mathcal{D}(B) \subset X \to Y$ by

$$
Az = -z_{xx}, \quad Bz = z - z_{xx},
$$

where each domain $\mathcal{D}(A)$ and $\mathcal{D}(B)$ is given by

$$
\{z \in X : z, z_x \text{ are absolutely continuous, } z_{xx} \in X, z(0) = z(\pi) = 0\}.
$$

Then the above problem can be formulated abstractly as

$$
\begin{gathered}
\frac{d}{dt}\big[Bu(t) + e(t,u(t))\big] + Au(t) = f(t,u(t)) + \int_0^t k(t,s,u(s)) ds, \\
t \in (0,a],\ t \ne t_k, \\
u(0) + \sum_{i=1}^{n} c_i u(t_i) = u_0 \\
\Delta u(t_k) = I_k(u_{t_k}), \quad k = 1,2,\dots,m,
\end{gathered}
$$

Then A and B can be written, respectively, as

$$
\begin{gathered}
Az = \sum_{n=1}^{\infty} n^2 \langle z, z_n\rangle z_n,\ z \in \mathcal{D}(A) \\
Bz = \sum_{n=1}^{\infty} (1+n^2) \langle z, z_n\rangle z_n,\ z \in \mathcal{D}(B),
\end{gathered}
$$

where $z_n(x) = \sqrt{2/\pi}\sin(nx)$, $n = 1, 2, \dots$, is the orthogonal set of vectors of A. Furthermore for $z \in X$, we have

$$B^{-1}z = \sum_{n=1}^{\infty} \frac{1}{1+n^2}\langle z, z_n\rangle z_n,$$

$$-AB^{-1}z = \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2}\langle z, z_n\rangle z_n,$$

$$S(t)z = \sum_{n=1}^{\infty} \exp\Big(\frac{-n^2 t}{1+n^2}\Big)\langle z, z_n\rangle z_n.$$

It is easy to see that AB^{-1} generates a strongly continuous semigroup $S(t)$ on Y and $S(t)$ is compact such that $|S(t)| \le e^{-t}$ for each $t > 0$. For this $S(t), B, B^{-1}$ we assume that the operator E exists. So all the conditions of the Theorem 3.1 are satisfied. Hence the equation (4.1) has a mild solution.

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