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ASYMPTOTIC BEHAVIOR OF STOCHASTIC GILPIN-AYALA MUTUALISM MODEL WITH JUMPS

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Abstract. This article concerns the study of stochastic Gilpin-Ayala mutualism models with white noise and Poisson jumps. Firstly, an explicit solution for one-dimensional Gilpin-Ayala mutualism model with jumps is obtained and the asymptotic pathwise behavior is analyzed. Then, sufficient conditions for the existence of global positive solutions, stochastically ultimate boundedness and stochastic permanence are established for the n-dimensional model. Asymptotic pathwise behavior of n-dimensional Gilpin-Ayala mutualism model with jumps is also discussed. Finally numerical examples are introduced to illustrate the results developed.

1. Introduction

In nature, mutualism is a usual phenomena. Rhinos and tick birds are an example of a mutualism relationship. Tick birds eat the ticks on a rhino while the rhino loses annoying parasites, the relationship is positive for both the rhino and the tick birds because they both get what they want. Therefore it is important to study the mutualism models for multi-species. As is well known now, the most important model among several cooperative models is the following Lotka-Volterra mutualism system:

$$dx_i(t) = x_i(t)\Big[r_i - \sum_{j=1}^{d} a_{ij}x_j(t)\Big]dt, \quad 1 \le i \le n,$$

where $x_i(t)$ is the population size of species i, r_i is the intrinsic growth rate of species i, a_{ij} $(i \neq j)$ represents the effect of species j upon the growth rate of species i, a_{ii} stands for the intraspecies interaction, $a_{ii} > 0$, $a_{ij} < 0$, $i \neq j$. There is an extensive literature concerned this model, for example, [1, 9, 10, 13, 21, 24, 26, 28]. However, in the practical case, population systems are often subject to various stochastic small perturbation. The growth rates, interaction coefficients and so on may be influenced by environmental noise. In recent years, stochastic differential equations have received much attention, many results have been derived to reveal how environmental noise affects the population systems. In particular, Mao, Marion and Renshaw [22] revealed that the environmental noise can suppress a potential

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population explosion. Suppose the growth rates are perturbed by white noise $r_i \to r_i + \sum_{j=1}^m \sigma_{ij}\dot{B}_j(t)$, here $\dot{B}_j(t)$ is a white noise, i.e., $(B_1(t), \dots, B_m(t))$ is an m-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathbb{F}, \mathbb{P})$, σ_{ij}^2 stands for the intensity of the noise. Then the stochastic Lotka-Volterra mutualism system becomes

$$dx_i(t) = x_i(t)\Big[(r_i - \sum_{j=1}^d a_{ij}x_j(t))dt + \sum_{j=1}^m \sigma_{ij}dB_j(t)\Big], \quad 1 \le i \le n.$$

Various forms of cooperative Lotka-Volterra system have been extensively studied and we here mention Hung [11], Cheng[6], Liu and Wang [19, 20] and the references cited therein. The key method used in our paper is motivated by them.

Unfortunately, in the Lotka-Volterra model, the rate of change of the size of each species is linear function of sizes of the interacting species [16, 7]. However in complex ecosystem this is almost impossible. Therefore, to meet the practical situations, in 1973, Gilpin and Ayala [8] provided a modification for Lotka-Volterra model, called Gilpin-Ayala model. For various forms about the Gilpin-Ayala system readers can see [5, 17, 29, 30] and references therein for details. But here are few works about stochastic Gilpin-Ayala mutualism model.

On the other hand, the population systems may suffer sudden environmental perturbations, that is, some jump type stochastic perturbations; e.g., earthquakes, hurricanes, epidemics and so on [3, 4]. These phenomena can not be described by stochastic integrals driven only by Brownian motion. So it is feasible to introduce a jump process into the underlying population system. SDEs with jumps have received considerable attention in the past few years. We here mention Applebaum [2], Situ [27], Bao et al [3, 4]. Particularly, the books by Applebaum [2] and Situ [27] are good references in this area. To the best of the authors knowledge, to this day, n-dimensional Gilpin-Ayala mutualism model with jumps has not been studied. Motivated by these, the following n-dimensional Gilpin-Ayala mutualism model with jumps is considered in this article:

$$\begin{aligned} dx_i(t) = x_i(t^-)\Big\{\Big[r_i - a_{ii}x_i^{\theta_i}(t^-) - \sum_{j\neq i}^d a_{ij}x_j(t^-)\Big]dt + \sum_{j=1}^m \sigma_{ij}dB_j(t) \\ + \int_{\mathbb{Y}} \gamma_i(u)N(dt,du)\Big\}, \end{aligned} \tag{1.1}$$

for $1 \le i \le n$, where $x(t^-)$ is the left limit of $x(t)$, $\theta_i \ge 1$ is the parameter to modify the classical Lotka-Volterra model, $\gamma_i(u) > -1$ is a bounded function, $i = 1, \dots, n$, N is a Poisson counting measure with characteristic measure ν on a measurable subset $\mathbb{Y}$ of $(0, \infty)$ with $\nu(\mathbb{Y}) < \infty$, and $\widetilde{N}(dt, du) := N(dt, du) - \nu(du)dt$. We assume Brownian motion and N are independent.

Throughout this article $\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x_i > 0 \text{ for } i = 1, \dots, n\}$, $\bar{A} = (a_{ij})$, $\bar{A}^T$ denotes the transpose of $\bar{A}$. If $x \in \mathbb{R}^n$, its norm is denoted by $|x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$. If Q is a matrix, $|Q| = \sqrt{\text{trace}(Q^TQ)}$ represents its trace norm. If $Q = (q_{ij})_{n\times n}$ is a symmetric matrix, then $\lambda_{\max}^+(Q) = \sup_{x\in\mathbb{R}_+^n, |x|=1} x^TQx$. K is a positive constant and may be different at different places. We impose the following assumptions:

(A1) There is a positive constant κ such that

$$\int_{\mathbb{Y}}(|\ln(1+\gamma(y))|\vee|\ln(1+\gamma(y))|^2)\nu(dy)<\kappa.$$

(A2) There are positive constants $p_1,\dots,p_n$ such that

$$-2\tau:=\lambda_{\max}^{+}(-\bar{P}\bar{A}-\bar{A}^T\bar{P})<0,$$

where $\bar{P}=\operatorname{diag}(p_1,\dots,p_n)$.

From Assumption (A2), it is easy to see that $\tau\le a_{ii}p_i$ for $i=1,\dots,n$.

The aim of our work is to study the properties of n-dimensional stochastic Gilpin-Ayala mutualism model with jumps. The significance of this paper is mainly: (1) Gilpin-Ayala system is more suitable for the real situations than Lotka-Volterra system, but more complicated; (2) The white noise and Poisson jumps are taken into account. The remaining part of this paper is organized as follows. In section 2, we provide an explicit solution for one-dimensional Gilpin-Ayala model with jumps and study its asymptotic pathwise behavior. In section 3, we show that (1.1) will have a unique global positive solution under certain conditions. Section 4 and 5 deal with the asymptotic moment properties and asymptotic pathwise behavior of the solution, respectively. In section 6, we show that the system is stochastically permanent if the white noise and Poisson jumps satisfy our conditions. Finally we introduce some simulation figures to illustrate our main results.

2. One-dimensional Gilpin-Ayala model

As the single population is the basic unit of the whole ecological system, the establishment and theoretical analysis of the single population model can help us to understand the overall structure of the complex model. So we firstly analyze one-dimensional Gilpin-Ayala model with jumps.

Lemma 2.1 ([3]). *Consider the following system of equations with jumps:*

$$dY_i(t)=Y_i(t^-)\Big[(a_i-b_{ii}Y_i(t^-))dt+\sigma_i dB(t)+\int_{\mathbb{Y}}c_i(u)\widetilde{N}(dt,du)\Big],$$

where $a_i>0$, $b_{ii}>0$, $c_i(u)>-1$, $B(t)$ is a one-dimensional Brownian motion. Then for any initial value $Y_i(0)\in\mathbb{R}_+^n$, this equation admits a unique positive solution $Y_i(t)$, $t\ge 0$, which is global and admits the explicit formula

$$Y_i(t)=\frac{\varphi_i(t)}{\frac{1}{Y_i(0)}+\int_0^t b_{ii}\varphi_i(s)ds},$$

where

$$\varphi_i(t):=\exp\Big(\int_0^t\Big[a_i-\frac{1}{2}\sigma_i^2+\int_{\mathbb{Y}}(\ln(1+c_i(u))-c_i(u))\nu(du)\Big]ds+\int_0^t\sigma_i dB(s)+\int_0^t\int_{\mathbb{Y}}\ln(1+c_i(u))\widetilde{N}(dt,du)\Big).$$

Remark 2.2. In general, the intrinsic growth rate a_i is positive, but the above explicit solution holds for $a_i\le 0$.

The one-dimensional Gilpin-Ayala model with jumps is

$$\begin{aligned} dy_i(t) &= y_i(t^-)(r_i - a_{ii}y_i^{\theta_i}(t^-))dt + y_i \sum_{j=1}^m \sigma_{ij} dB_j(t) \\ &\quad + \int_{\mathbb{Y}} y_i(t^-)\gamma_i(u)N(dt,du), \quad \theta_i > 1, \\ y_i(0) &= x_i(0). \end{aligned} \tag{2.1}$$

Set $z_i = y_i^{\theta_i}$, by Itô's formula, we have

$$\begin{aligned} dz_i(t) &= z_i(t^-)\Big[\theta_i r_i + \frac{\theta_i(\theta_i-1)}{2}\sum_{j=1}^m \sigma_{ij}^2 + \int_{\mathbb{Y}}((1+\gamma_i(u))^{\theta_i} - 1)\nu(du) \\ &\quad - \theta_i a_{ii} z_i(t^-)\Big]dt + z_i(t^-)\theta_i \sum_{j=1}^m \sigma_{ij} dB_j(t) \\ &\quad + \int_{\mathbb{Y}} z_i(t^-)((1+\gamma_i(u))^{\theta_i} - 1)\widetilde{N}(dt,du), \quad \theta_i > 1, \\ z_i(0) &= x_i^{\theta_i}(0). \end{aligned} \tag{2.2}$$

From Lemma 2.1 and Remark 2.2, it follows that (2.2) has an explicit solution

$$z_i(t) = \frac{\Phi_i(t)}{\frac{1}{x_i^{\theta_i}(0)} + \int_0^t a_{ii}\theta_i \Phi_i(s)ds},$$

where

$$\begin{aligned} \Phi_i(t) &:= \exp\Big\{\int_0^t \theta_i\Big[r_i - \frac{1}{2}\sum_{j=1}^m \sigma_{ij}^2 + \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\nu(du)\Big]ds \\ &\quad + \int_0^t \theta_i \sum_{j=1}^m \sigma_{ij} dB_j(s) + \int_0^t \int_{\mathbb{Y}} \theta_i \ln(1+\gamma_i(u))\widetilde{N}(dt,du)\Big\} \end{aligned}$$

Combining [3, Lemma 4.4 and Theorem 4.4], we can deduce the following lemma.

Lemma 2.3. *Assume* (A1) *holds. If* $c_i := r_i - \frac{1}{2}\sum_{j=1}^m \sigma_{ij}^2 + \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\nu(du) \geq 0$*, then for* $i = 1, \dots, n$*, we have*

$$\lim_{t\to\infty} \frac{\ln z_i(t)}{t} = 0 \quad a.s. \tag{2.3}$$

Remark 2.4. Based on the above analysis, (2.1) has a unique positive solution $y_i(t)$ for any value $y_i(0) = x_i(0) > 0$ which is global and represented by

$$y_i(t) = \Big(\frac{\Phi_i(t)}{\frac{1}{x_i^{\theta_i}(0)} + \int_0^t a_{ii}\theta_i \Phi_i(s)ds}\Big)^{1/\theta_i},$$

where $\Phi_i(t)$ is defined as above. Under the conditions of Lemma 2.3, we obtain $\lim_{t\to\infty} \frac{\ln y_i(t)}{t} = 0$ a.s.

Theorem 2.5. *Suppose that* $y_i(t)$ *is a positive solution of* (2.1)*. If* $c_i \geq 0$*, then*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t y_i^{\theta_i}(s)ds = \frac{r_i - \frac{1}{2}\sum_{j=1}^m \sigma_{ij}^2 + \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\nu(du)}{a_{ii}} \quad a.s.$$

Proof. Applying Itô's formula to $\ln y_i(t)$ results in

$$\begin{aligned} d\ln y_i(t) = (r_i - \frac{1}{2}\sum_{j=1}^m \sigma_{ij}^2 + \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\nu(du) - a_{ii}y_i^{\theta_i})dt \\ + \sum_{j=1}^m \sigma_{ij} dB_j(t) + \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\widetilde{N}(dt,du). \end{aligned}$$

Integrating from 0 to t yields

$$\begin{aligned} \ln y_i(t) - \ln y_i(0) = \Big(r_i - \frac{1}{2}\sum_{j=1}^m \sigma_{ij}^2 + \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\nu(du)\Big)t \\ - a_{ii}\int_0^t y_i^{\theta_i}(s)ds + M_1(t) + M_2(t), \end{aligned}$$

where $M_1(t) = \int_0^t \sum_{j=1}^m \sigma_{ij} dB_j(s)$, $M_2(t) = \int_0^t \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\widetilde{N}(ds,du)$ are real valued local martingales vanishing at $t=0$. Hence

$$\begin{aligned} \frac{\ln y_i(t)}{t} - \frac{\ln y_i(0)}{t} = \Big(r_i - \frac{1}{2}\sum_{j=1}^m \sigma_{ij}^2 + \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\nu(du)\Big) \\ - \frac{a_{ii}}{t}\int_0^t y_i^{\theta_i}(s)ds + \frac{M_1(t)}{t} + \frac{M_2(t)}{t}. \end{aligned} \tag{2.4}$$

Then by [14, Proposition 2.4],

$$\langle M_1\rangle(t) = \int_0^t \sum_{j=1}^m \sigma_{ij}^2 ds = \sum_{j=1}^m \sigma_{ij}^2 t,$$

$$\langle M_2\rangle(t) = \int_0^t \int_{\mathbb{Y}} [\ln(1+\gamma_i(u))]^2\nu(du)ds = t\int_{\mathbb{Y}} [\ln(1+\gamma_i(u))]^2\nu(du),$$

where $\langle M\rangle(t) := \langle M, M\rangle$ is Meyer's angle bracket process. We have

$$\int_0^t \frac{1}{(1+s)^2}ds = \frac{t}{t+1} < \infty,$$

by the strong law of large numbers for local martingales [18], we then obtain

$$\lim_{t\to\infty} \frac{M_1(t)}{t} = 0 \text{ a.s.}, \quad \lim_{t\to\infty} \frac{M_2(t)}{t} = 0 \text{ a.s.}$$

Taking limits on both sides of (2.4) and combining Remark 2.4 lead to

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t y_i^{\theta_i}(s)ds = \frac{r_i - \frac{1}{2}\sum_{j=1}^m \sigma_{ij}^2 + \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\nu(du)}{a_{ii}} \quad \text{a.s.}$$

This completes the proof. $\square$

3. Global positive solutions of (1.1)

As $x_i(t)$ in (1.1) denotes the size of species i, it should be nonnegative. To guarantee that the stochastic differential equations (SDEs) have a unique global solution for any given initial value, the coefficients of the equation are generally required to satisfy both the linear growth condition and the local Lipschitz condition (see e.g.[23, 12]). But we can find that the coefficients of (1.1) are locally Lipschitz continuous, and they do not satisfy the linear growth condition. So the solution of

(1.1) may explode in a finite time. The following theorem gives sufficient condition for global positive solutions.

Theorem 3.1. *Let* (A1), (A2) *hold and* $\theta_i \geq 1$, $i = 1, \dots, n$. *Then for any initial value* $x_0 \in \mathbb{R}_+^n$, *Equation* (1.1) *has a unique global solution* $x(t) \in \mathbb{R}_+^n$ *for all* $t \geq 0$ *almost surely.*

Proof. Our proof is motivated by Mao, Marion and Renshaw [22]. Clearly, the coefficients of (1.1) are locally Lipschitz continuous, so for any initial value $x_0 \in \mathbb{R}_+^n$ Equation (1.1) has a unique maximal local solution $x(t)$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. If we show that $\tau_e = \infty$ a.s., then the solution is global. Now let k_0 be big enough for every component of x_0 lying within the interval $[1/k_0, k_0]$. For any integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in [0, \tau_e) : x_i(t) \notin (1/k, k) \text{ for some } i = 1, \dots, n\},$$

where we set $\inf \emptyset = \infty$. Obviously, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \leq \tau_e$ a.s. Now all we need to show is $\tau_\infty = \infty$ a.s. If this assertion is false, then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \epsilon$. Therefore, there is an integer $k_1 \geq k_0$ such that

$$\mathbb{P}\{\tau_k \leq T\} \geq \epsilon \quad \text{for all } k \geq k_1. \tag{3.1}$$

Define a C^2-function $V : \mathbb{R}_+^n \to \mathbb{R}_+$ by

$$V(x) = \sum_{i=1}^n p_i(x_i - 1 - \ln x_i).$$

If $x(t) \in \mathbb{R}_+^n$, Itô's formula shows that

$$\begin{aligned}
&dV(x(t)) \\
&= \sum_{i=1}^n \Big\{ p_i \Big[(x_i - 1)(r_i - a_{ii}x_i^{\theta_i} - \sum_{j\neq i}^n a_{ij}x_j) + 0.5 \sum_{j=1}^m \sigma_{ij}^2 \Big] \Big\} dt \\
&\quad + \sum_{i=1}^n p_i(x_i - 1) \sum_{j=1}^m \sigma_{ij} dB_j(t) + \sum_{i=1}^n \int_{\mathbb{Y}} p_i[x_i\gamma_i(u) - \ln(1 + \gamma_i(u))] N(dt, du) \\
&= \sum_{i=1}^n \Big\{ p_i \Big[(x_i - 1)(r_i - a_{ii}x_i^{\theta_i} - \sum_{j\neq i}^n a_{ij}x_j) + 0.5 \sum_{j=1}^m \sigma_{ij}^2 \\
&\quad + \int_{\mathbb{Y}} (x_i\gamma_i(u) - \ln(1 + \gamma_i(u)))\nu(du) \Big] \Big\} dt + \sum_{i=1}^n p_i(x_i - 1) \sum_{j=1}^m \sigma_{ij} dB_j(t) \\
&\quad + \sum_{i=1}^n \int_{\mathbb{Y}} p_i[x_i\gamma_i(u) - \ln(1 + \gamma_i(u))] \widetilde{N}(dt, du),
\end{aligned}$$

where we drop t^- from $x(t^-)$. Using (A1) and (A2), we get that there exists a positive constant K such that

$$\begin{aligned}
&\sum_{i=1}^n p_i \Big[(x_i - 1)(r_i - a_{ii}x_i^{\theta_i} - \sum_{j\neq i}^n a_{ij}x_j) + 0.5 \sum_{j=1}^m \sigma_{ij}^2 \\
&+ \int_{\mathbb{Y}} (x_i\gamma_i(u) - \ln(1 + \gamma_i(u)))\nu(du) \Big]
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^n p_i\Big[(r_i+\int_{\mathbb{Y}}\gamma_i(u)\nu(du))x_i + a_{ii}x_i^2 - a_{ii}x_i^{\theta+1} + a_{ii}x_i^{\theta} - r_i + 0.5\sum_{j=1}^m \sigma_{ij}^2 \\
&\quad - \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\nu(du)\Big] + 0.5x^T(-\bar{P}\bar{A}-\bar{A}^T\bar{P})x \\
&\le \sum_{i=1}^n \Big\{ -a_{ii}p_i x_i^{\theta_i+1} - (\tau - a_{ii}p_i)x_i^2 + p_i a_{ii}x_i^{\theta_i} + p_i\Big(r_i + \int_{\mathbb{Y}}\gamma_i(u)\nu(du)\Big)x_i \\
&\quad + p_i\Big(-r_i + 0.5\sum_{j=1}^m \sigma_{ij}^2 - \int_{\mathbb{Y}}\ln(1+\gamma_i(u))\nu(du)\Big)\Big\} \le K.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_0^{\tau_k\wedge T} dV(x(t)) &\le \int_0^{\tau_k\wedge T} K dt + \int_0^{\tau_k\wedge T}\sum_{i=1}^n p_i(x_i(t)-1)\sum_{j=1}^m \sigma_{ij}dB_j(t) \\
&\quad + \int_0^{\tau_k\wedge T}\int_{\mathbb{Y}}\sum_{i=1}^n\int_{\mathbb{Y}} p_i[x_i\gamma_i(u) - \ln(1+\gamma_i(u))]\widetilde{N}(dt,du).
\end{aligned}
$$

Taking expectations on both sides results in

$$
\mathbb{E}V(x(\tau_k\wedge T)) \le V(x_0) + K\mathbb{E}(\tau_k\wedge T) \le V(x_0) + KT. \tag{3.2}
$$

Set $\Omega_k = \{\tau_k \le T\}$ for $k \ge k_1$, from (3.1) we have $\mathbb{P}(\Omega_k) \ge \epsilon$. Note that for every $\omega \in \Omega_k$, there is some i such that $x_i(\tau_k,\omega)$ equals either k or $1/k$, hence

$$
V(x(\tau_k,\omega)) \ge p_i[k-1-\ln(k)] \wedge p_i[1/k - 1 - \ln(1/k)].
$$

Using (3.2), yields

$$
V(x_0) + KT \ge \mathbb{E}\left(I_{\Omega_k}V(x(\tau_k,\omega))\right) \ge \epsilon\left(p_i[k-1-\ln(k)] \wedge p_i[1/k-1-\ln(1/k)]\right),
$$

where I_{Ω_k} is the indicator function of Ω_k. When $k\to\infty$ we obtain

$$
\infty > V(x_0) + KT = \infty,
$$

it results in $\tau_\infty = \infty$ a.s. The proof is complete. $\square$

4. Ultimate boundedness

In the previous section, we saw that (1.1) has a unique global solution $x(t) \in \mathbb{R}_+^n$ for any $t \ge 0$ almost surely. Based on this fundamental theorem, we discuss the ultimate boundedness and asymptotic boundedness in any pth moment of the solutions.

Theorem 4.1. *Under the assumptions of Theorem 3.1, for any initial value $x_0 \in \mathbb{R}_+^n$, the solution of* (1.1) *satisfies*

$$
\limsup_{t\to\infty} \mathbb{E}|x(t)| \le K.
$$

Proof. Define the Lyapunov function

$$
V(x) := \sum_{i=1}^n p_i x_i, \quad x \in \mathbb{R}_+^n.
$$

Applying Itô's formula, we obtain

$$\begin{aligned} dV(x(t)) &= \sum_{i=1}^{n} p_i x_i (r_i - a_{ii} x_i^{\theta_i} - \sum_{j\neq i}^{n} a_{ij} x_j) dt + \sum_{i=1}^{n} p_i x_i \sum_{j=1}^{m} \sigma_{ij} dB_j(t) \\ &\quad + \int_{\mathbb{Y}} \sum_{i=1}^{n} p_i x_i \gamma_i(u) N(dt, du) \\ &= LV(x)dt + \sum_{i=1}^{n} p_i x_i \sum_{j=1}^{m} \sigma_{ij} dB_j(t) + \int_{\mathbb{Y}} \sum_{i=1}^{n} p_i x_i \gamma_i(u) \widetilde{N}(dt, du), \end{aligned} \tag{4.1}$$

where we write $x(t^-) = x$, and

$$\begin{aligned} LV(x) &= \sum_{i=1}^{n} p_i \Big(r_i - a_{ii} x_i^{\theta_i} - \sum_{j\neq i}^{n} a_{ij} x_j + \int_{\mathbb{Y}} \gamma_i(u)\nu(du) \Big) x_i \\ &= \sum_{i=1}^{n} p_i \Big(r_i - a_{ii} x_i^{\theta_i} + a_{ii} x_i + \int_{\mathbb{Y}} \gamma_i(u)\nu(du) \Big) x_i + x^T(-\bar{P}\bar{A})x \\ &= \sum_{i=1}^{n} p_i \Big(r_i - a_{ii} x_i^{\theta_i} + a_{ii} x_i + \int_{\mathbb{Y}} \gamma_i(u)\nu(du) \Big) x_i + 0.5 x^T(-\bar{P}\bar{A} - \bar{A}^T\bar{P})x \\ &\le \sum_{i=1}^{n} \Big[- a_{ii} p_i x_i^{\theta_i + 1} - (\tau - a_{ii} p_i) x_i^2 + p_i \Big(r_i + \int_{\mathbb{Y}} \gamma_i(u)\nu(du) \Big) x_i \Big]. \end{aligned}$$

For arbitrary $\alpha > 0$, making use of the conditions of this Theorem, applying Itô's formula once again yields

$$\begin{aligned} & d(e^{\alpha t} V(x(t))) \\ &= \alpha e^{\alpha t} V(x(t)) dt + e^{\alpha t} dV(x(t)) \\ &\le e^{\alpha t} \sum_{i=1}^{n} [-a_{ii} p_i x_i^{\theta_i+1} - (\tau - a_{ii} p_i) x_i^2 + p_i(\alpha + r_i + \int_{\mathbb{Y}} \gamma_i(u)\nu(du)) x_i] dt \\ &\quad + e^{\alpha t} \sum_{i=1}^{n} p_i x_i \sum_{j=1}^{m} \sigma_{ij} dB_j(t) + e^{\alpha t} \int_{\mathbb{Y}} \sum_{i=1}^{n} p_i x_i \gamma_i(u) \widetilde{N}(dt, du) \\ &\le K_0 e^{\alpha t} dt + e^{\alpha t} \sum_{i=1}^{n} p_i x_i \sum_{j=1}^{m} \sigma_{ij} dB_j(t) + e^{\alpha t} \int_{\mathbb{Y}} \sum_{i=1}^{n} p_i x_i \gamma_i(u) \widetilde{N}(dt, du), \end{aligned}$$

where K_0 is a positive constant. Therefore,

$$\mathbb{E}(e^{\alpha t} V(x(t))) \le V(x_0) + \frac{K_0}{\alpha}(e^{\alpha t} - 1);$$

that is to say

$$\limsup_{t\to\infty} \mathbb{E} V(x(t)) \le \frac{K_0}{\alpha}. \tag{4.2}$$

Noting that $|x(t)| \le \sum_{i=1}^{n} x_i(t) \le \frac{V(x(t))}{\min_{1\le i\le n} p_i}$, we obtain

$$\limsup_{t\to\infty} \mathbb{E}|x(t)| \le \frac{K_0}{\alpha \min_{1\le i\le n} p_i} =: K.$$

This completes the proof. $\square$

Definition 4.2 ([15])**.** The solution of(1.1) is said to be stochastically ultimately bounded if for any $\epsilon \in (0,1)$, there is a constant $H = H(\epsilon)$ such that for any $x_0 \in \mathbb{R}^n_+$,

$$\limsup_{t\to\infty} \mathbb{P}\left\{|x(t)| > H\right\} < \epsilon.$$

As an application of Theorem 4.1, together with the Chebyshev inequality, we have the following corollary.

Corollary 4.3. *Under the conditions of Theorem 4.1, the solution of* (1.1) *is stochastically ultimately bounded.*

Furthermore, we can get the following property.

Theorem 4.4. *Assume* (A1), (A2) *hold and $\theta_i > 1$, $i = 1, \dots, n$. Then for $p > 0$, there exists a positive constant $K = K(p)$, for any initial value $x_0 \in \mathbb{R}^n_+$, the solution of* (1.1) *has the property*

$$\limsup_{t\to\infty} \mathbb{E}x_i^p(t) \le K(p), \quad t \ge 0,\ i = 1, \dots, n.$$

Proof. Define the Lyapunov function

$$V(x,t) := \sum_{i=1}^n e^t x_i^p, \quad x \in \mathbb{R}^n_+.$$

Applying Itô's formula, we obtain

$$\begin{aligned} dV(x(t),t) &= LV(x(t))dt + e^t p \sum_{i=1}^n x_i^p \sum_{j=1}^m \sigma_{ij} dB_j(t) \\ &\quad + e^t \sum_{i=1}^n x_i^p \int_{\mathbb{Y}} [(1+\gamma_i(u))^p - 1] \widetilde{N}(dt, du), \end{aligned}$$

where we write $x(t^-) = x$, and

$$\begin{aligned} LV(x) &= e^t \sum_{i=1}^n p\Big\{\Big[\frac{1}{p} + r_i + \frac{p-1}{2}\sum_{j=1}^m \sigma_{ij}^2 + \frac{1}{p}\int_{\mathbb{Y}}[(1+\gamma_i(u))^p - 1]\nu(du)\Big]x_i^p \\ &\quad - a_{ii}x_i^{p+\theta_i} - \sum_{j\neq i}^n a_{ij} x_i^p x_j\Big\} \\ &\le e^t \sum_{i=1}^n p\Big\{\Big[\frac{1}{p} + r_i + \frac{p-1}{2}\sum_{j=1}^m \sigma_{ij}^2 + \frac{1}{p}\int_{\mathbb{Y}}[(1+\gamma_i(u))^p - 1]\nu(du)\Big]x_i^p \\ &\quad - a_{ii}x_i^{p+\theta_i} - \sum_{j\neq i}^n a_{ij}\Big[\frac{p x_i^{p+1}}{p+1} + \frac{x_j^{p+1}}{p+1}\Big]\Big\} \\ &\le e^t \sum_{i=1}^n p\Big\{\Big[\frac{1}{p} + r_i + \frac{p-1}{2}\sum_{j=1}^m \sigma_{ij}^2 + \frac{1}{p}\int_{\mathbb{Y}}[(1+\gamma_i(u))^p - 1]\nu(du)\Big]x_i^p \\ &\quad - a_{ii}x_i^{p+\theta_i} - \Big[\sum_{j\neq i}^n \Big(\frac{p}{p+1}(a_{ij}) + \frac{1}{p+1}(a_{ji})\Big)\Big]x_i^{p+1}\Big\} \\ &\le e^t \sum_{i=1}^n K_i(p), \end{aligned}$$

where $K_i(p)$ is a positive constant. Hence

$$e^t\mathbb{E}[\sum_{i=1}^n x_i^p(t)] \le \sum_{i=1}^n x_i^p(0) + \mathbb{E}\int_0^t e^s \sum_{i=1}^n K_i(p)ds = \sum_{i=1}^n x_i^p(0) + \sum_{i=1}^n K_i(p)(e^t-1).$$

It is not difficult to derive that

$$\limsup_{t\to\infty} \mathbb{E}[\sum_{i=1}^n x_i^p(t)] \le \sum_{i=1}^n K_i(p) =: K(p).$$

The required assertion follows immediately. □

5. Pathwise estimation

In this section we consider the asymptotic pathwise estimation of the solution to (1.1).

Theorem 5.1. *For $\theta_i > 1$, $i = 1,\dots,n$, under Assumptions* (A1), (A2), *for any initial value $x_0 \in \mathbb{R}_+^n$, the solution of* (1.1) *has the property*

$$\limsup_{t\to\infty} \frac{\ln x_i(t)}{\ln t} \le 1 \quad a.s.,\ i = 1,\dots,n.$$

Proof. Here we adopt the same notation as in the proof of Theorem 4.1. From (4.1), by simple manipulation, one has

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\le u\le t+1} V(x(u))\Big)\\
&\le \mathbb{E}(V(x(t))) + \mathbb{E}\Big(\sup_{t\le u\le t+1}\int_t^u \sum_{i=1}^n \Big[-a_{ii}p_i x_i^{\theta_i+1}(s) - (\tau - a_{ii}p_i)x_i^2(s)\\
&\quad + p_i\Big(r_i + \int_{\mathbb{Y}} \gamma_i(u)\nu(du)\Big)x_i(s)\Big]ds\Big) + \mathbb{E}\Big(\sup_{t\le u\le t+1}\int_t^u \sum_{i=1}^n p_i x_i(s)\sum_{j=1}^m \sigma_{ij}dB_j(s)\Big)\\
&\quad + \mathbb{E}\Big(\sup_{t\le u\le t+1}\int_t^u\int_{\mathbb{Y}}\sum_{i=1}^n p_i x_i(s)\gamma_i(u)\widetilde{N}(ds,du)\Big)\\
&\le \mathbb{E}(V(x(t))) + \mathbb{E}\Big(\sup_{t\le u\le t+1}\int_t^u \sum_{i=1}^n \Big[a_{ii}p_i x_i^2(s)\\
&\quad + p_i\Big(r_i + \int_{\mathbb{Y}} |\gamma_i(u)|\nu(du)\Big)x_i(s)\Big]ds\Big)\\
&\quad + \mathbb{E}\Big(\sup_{t\le u\le t+1}\int_t^u \sum_{i=1}^n p_i x_i(s)\sum_{j=1}^m \sigma_{ij}dB_j(s)\Big)\\
&\quad + \mathbb{E}\Big(\sup_{t\le u\le t+1}\Big|\int_t^u\int_{\mathbb{Y}}\sum_{i=1}^n p_i x_i(s)\gamma_i(u)\widetilde{N}(ds,du)\Big|\Big)\\
&\le \mathbb{E}(V(x(t))) + q_1\int_t^{t+1}\mathbb{E}|x(s)|ds + q_2\int_t^{t+1}\mathbb{E}|x(s)|^2ds\\
&\quad + \mathbb{E}\Big(\sup_{t\le u\le t+1}\int_t^u \sum_{i=1}^n p_i x_i(s)\sum_{j=1}^m \sigma_{ij}dB_j(s)\Big)
\end{aligned}$$

$$+\mathbb{E}\Big(\sup_{t\le u\le t+1}|\int_t^u\int_{\mathbb{Y}}\sum_{i=1}^n p_ix_i(s)\gamma_i(u)\widetilde{N}(ds,du)|\Big), \tag{5.1}$$

where $q_1=\sqrt{n}\max_{1\le i\le n}\{p_i(r_i+\int_{\mathbb{Y}}|\gamma_i(u)|\nu(du))\}$, $q_2=\max_{1\le i\le n}\{a_{ii}p_i\}$. By the Burkholder-Davis-Gundy inequality for local martingale (see, e.g., [23, 2]) and the Hölder inequality, we obtain that

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\le u\le t+1}\int_t^u\sum_{i=1}^n p_ix_i(s)\sum_{j=1}^m\sigma_{ij}dB_j(s)\Big)&\le 3\mathbb{E}\Big(\int_t^{t+1}|x^T\bar{P}\sigma|^2ds\Big)^{1/2}\\ &\le 3|\bar{P}\sigma|\Big(\mathbb{E}\int_t^{t+1}|x(s)|^2ds\Big)^{1/2},\end{aligned}$$

where $\sigma=(\sigma_{ij})$, and

$$\begin{aligned}&\mathbb{E}\Big(\sup_{t\le u\le t+1}|\int_t^u\int_{\mathbb{Y}}\sum_{i=1}^n p_ix_i(s)\gamma_i(u)\widetilde{N}(ds,du)|\Big)\\ &\le\sum_{i=1}^n p_iJ\mathbb{E}\Big(\int_t^{t+1}\int_{\mathbb{Y}}x_i^2(s)\gamma_i^2(u)N(ds,du)\Big)^{1/2}\\ &\le\sum_{i=1}^n p_iJ\Big(\mathbb{E}\int_t^{t+1}\int_{\mathbb{Y}}x_i^2(s)\gamma_i^2(u)N(ds,du)\Big)^{1/2}\\ &=\sum_{i=1}^n p_iJ\Big(\int_{\mathbb{Y}}\gamma_i^2(u)\nu(du)\mathbb{E}\int_t^{t+1}x_i^2(s)ds\Big)^{1/2}\\ &\le J\sum_{i=1}^n p_i\Big(\int_{\mathbb{Y}}\gamma_i^2(u)\nu(du)\Big)^{1/2}\Big(\mathbb{E}\int_t^{t+1}|x(s)|^2ds\Big)^{1/2},\end{aligned}$$

where J is a positive constant. Moreover, we can derive from Theorem 4.4 that there exists positive constants K_1, K_2 such that $\limsup_{t\to\infty}\mathbb{E}\int_t^{t+1}|x(s)|ds\le K_1$ and $\limsup_{t\to\infty}\mathbb{E}\int_t^{t+1}|x(s)|^2ds\le K_2$. Substituting the above inequalities into (5.1) and combining (4.2), we can see that

$$\begin{aligned}&\limsup_{t\to\infty}\mathbb{E}\Big(\sup_{t\le u\le t+1}V(x(u))\Big)\\ &\le\frac{K_0}{\alpha}+q_1K_1+q_2K_2+\Big(3|\bar{p}\sigma|+J\sum_{i=1}^n p_i\Big(\int_{\mathbb{Y}}\gamma_i^2(u)\nu(du)\Big)^{1/2}\Big)K_2^{1/2}.\end{aligned}$$

Hence there is a positive constant M such that

$$\mathbb{E}\Big(\sup_{n\le t\le n+1}|x(t)|\Big)\le M,\quad n=1,2,\dots.$$

Let $\varepsilon>0$ be arbitrary, by the Chebyshev inequality, we have

$$\mathbb{P}\big\{\sup_{n\le t\le n+1}|x(t)|>n^{1+\varepsilon}\big\}\le\frac{M}{n^{1+\varepsilon}}\quad n=1,2,\dots.$$

Since the series $\sum_{n=1}^\infty\frac{M}{n^{1+\varepsilon}}$ converges, then from the Borel-Cantella lemma [23] that there exists a $n_0:=n_0(\omega)$ such that for almost all $\omega\in\Omega$, whenever $n\ge n_0$ and $n\le t\le n+1$, we have

$$\sup_{n\le t\le n+1}|x(t)|\le n^{1+\varepsilon}.$$

So

$$\limsup_{t\to\infty} \frac{\ln|x(t)|}{\ln t} \le 1+\varepsilon.$$

Letting $\varepsilon \to 0$ leads to the desired assertion. $\square$

Remark 5.2. Noting that the limit $\lim_{t\to\infty}\frac{\ln t}{t}=0$, under the conditions of Theorem 5.1, we obtain $\limsup_{t\to\infty}\frac{\ln x_i(t)}{t} \le 0$, a.s., $i=1,\dots,n$.

On the other hand, by the positivity of solution of (1.1) and the comparison theorem [25, Theorem 3.1], we obtain that

$$x_i(t) \ge y_i(t), \quad i=1,\dots,n,$$

where $y_i(t)$ is the solution of (2.1). According to the analysis for (2.1) in section 2, we obtain the following results.

Theorem 5.3. *Let* (A1), (A2) *hold and* $\theta_i > 1$, $i=1,\dots,n$. *If* $r_i - \frac{1}{2}\sum_{j=1}^m \sigma_{ij}^2 + \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\nu(du) \ge 0$, *then for any initial value* $x_0 \in \mathbb{R}_+^n$, *the solution of* (1.1) *satisfies*

$$\begin{aligned}
\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i^{\theta_i}(s)ds &\ge \lim_{t\to\infty}\frac{1}{t}\int_0^t y_i^{\theta_i}(s)ds \\
&= \frac{r_i - \frac{1}{2}\sum_{j=1}^m \sigma_{ij}^2 + \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\nu(du)}{a_{ii}},
\end{aligned}$$

$$\liminf_{t\to\infty}\frac{\ln x_i(t)}{t} \ge 0 \quad a.s.,\ i=1,\dots,n.$$

Now combining Remark 5.2 and Theorem 5.3 leads to the following theorem.

Theorem 5.4. *Under the conditions of Theorem 5.3, for each* $i=1,\dots,n$,

$$\lim_{t\to\infty}\frac{\ln x_i(t)}{t} = 0 \quad a.s.$$

6. Stochastic permanence

Stochastic permanence is one of the most interesting and important topics. In this section, stochastic permanence is studied based on the results in Section 4. We firstly introduce the definition of stochastic permanence.

Definition 6.1 ([20])**.** If for arbitrary $\varepsilon \in (0,1)$, there are two positive constants β_1 and β_2 such that for any initial data $x_0 \in \mathbb{R}_+^n$, the solution $x(t)$ of Eq.(1.1) has the property that

$$\liminf_{t\to\infty}\mathbb{P}\{x_i(t)\ge\beta_1\} \ge 1-\varepsilon, \quad \liminf_{t\to\infty}\mathbb{P}\{x_i(t)\le\beta_2\}\ge 1-\varepsilon, \quad 1\le i\le n,$$

then (1.1) is said to be stochastically permanent.

Theorem 6.2. *Under the conditions of Theorem 3.1, if there exists a positive constant* α, *such that*

$$r_i - \frac{3+\alpha}{2}\sum_{j=1}^m \sigma_{ij}^2 - \int_{\mathbb{Y}}\Big[\frac{1}{(2+\alpha)(1+\gamma_i(u))^{2+\alpha}} - \frac{1}{2+\alpha}\Big]\nu(du) > 0,$$

then, for the case $1 \le \theta_i \le 2+\alpha$, *Equation* (1.1) *is stochastically permanent.*

Proof. Define $y_i = 1/x_i$ for $x_i > 0$, applying Itô's formula, we obtain

$$\begin{aligned}
dy_i^{2+\alpha} &= d(\frac{1}{x_i})^{2+\alpha}\\
&= -(2+\alpha)(\frac{1}{x_i})^{\alpha+3}x_i(r_i - a_{ii}x_i^{\theta_i} - \sum_{j\neq i}^{n} a_{ij}x_j)dt\\
&\quad + \frac{2+\alpha}{2}(\alpha+3)(\frac{1}{x_i})^{\alpha+4}x_i^2\sum_{j=1}^{m}\sigma_{ij}^2 dt - (2+\alpha)(\frac{1}{x_i})^{\alpha+3}x_i\sum_{j=1}^{m}\sigma_{ij}dB_j\\
&\quad + \int_{\mathbb{Y}}[\frac{1}{(x_i + x_i\gamma_i(u))^{2+\alpha}} - \frac{1}{x_i^{2+\alpha}}]N(dt,du)\\
&\le (2+\alpha)(\frac{1}{x_i})^{\alpha}\Big\{-\frac{1}{x_i^2}\Big[r_i - \frac{\alpha+3}{2}\sum_{j=1}^{m}\sigma_{ij}^2 - \int_{\mathbb{Y}}(\frac{1}{(2+\alpha)(1+\gamma_i(u))^{2+\alpha}}\\
&\quad - \frac{1}{2+\alpha})\nu(du)\Big] + \frac{a_{ii}}{x_i^{2-\theta_i}}\Big\}dt - (2+\alpha)(\frac{1}{x_i})^{\alpha+2}\sum_{j=1}^{m}\sigma_{ij}dB_j\\
&\quad + \int_{\mathbb{Y}}[\frac{1}{(x_i + x_i\gamma_i(u))^{2+\alpha}} - \frac{1}{x_i^{2+\alpha}}]\widetilde{N}(dt,du)\\
&= (2+\alpha)y_i^{\alpha}\Big\{-y_i^2\Big[r_i - \frac{\alpha+3}{2}\sum_{j=1}^{m}\sigma_{ij}^2 - \int_{\mathbb{Y}}(\frac{1}{(2+\alpha)(1+\gamma_i(u))^{2+\alpha}}\\
&\quad - \frac{1}{2+\alpha})\nu(du)\Big] + a_{ii}y_i^{2-\theta_i}\Big\}dt - (2+\alpha)y_i^{\alpha+2}\sum_{j=1}^{m}\sigma_{ij}dB_j\\
&\quad + \int_{\mathbb{Y}} y_i^{2+\alpha}[\frac{1}{(1+\gamma_i(u))^{2+\alpha}} - 1]\widetilde{N}(dt,du).
\end{aligned}$$

Choose a sufficiently small positive ζ such that

$$r_i - \frac{\alpha+3}{2}\sum_{j=1}^{m}\sigma_{ij}^2 - \int_{\mathbb{Y}}\Big(\frac{1}{(2+\alpha)(1+\gamma_i(u))^{2+\alpha}} - \frac{1}{2+\alpha}\Big)\nu(du) > \frac{\zeta}{2+\alpha}.$$

Define $V = e^{\zeta t}y_i^{2+\alpha}$, using Itô's formula results in

$$\begin{aligned}
dV &\le (2+\alpha)e^{\zeta t}y_i^{\alpha}\Big\{-y_i^2\Big[r_i - \frac{\alpha+3}{2}\sum_{j=1}^{m}\sigma_{ij}^2 - \int_{\mathbb{Y}}(\frac{1}{(2+\alpha)(1+\gamma_i(u))^{2+\alpha}}\\
&\quad - \frac{1}{2+\alpha})\nu(du)\Big] + a_{ii}y_i^{2-\theta_i}\Big\}dt + \zeta e^{\zeta t}y_i^{2+\alpha}dt\\
&\quad - (2+\alpha)e^{\zeta t}y_i^{\alpha+2}\sum_{j=1}^{m}\sigma_{ij}dB_j + e^{\zeta t}\int_{\mathbb{Y}} y_i^{2+\alpha}[\frac{1}{(1+\gamma_i(u))^{2+\alpha}} - 1]\widetilde{N}(dt,du)\\
&=: e^{\zeta t}F(y_i)dt - (2+\alpha)e^{\zeta t}y_i^{2+\alpha}\sum_{j=1}^{m}\sigma_{ij}dB_j\\
&\quad + e^{\zeta t}\int_{\mathbb{Y}} y_i^{2+\alpha}[\frac{1}{(1+\gamma_i(u))^{2+\alpha}} - 1]\widetilde{N}(dt,du),
\end{aligned}$$

where

$$F(y_i) = (2+\alpha)y_i^\alpha\Big\{ - y_i^2\Big[r_i - \frac{\alpha+3}{2}\sum_{j=1}^m \sigma_{ij}^2 - \int_{\mathbb{Y}}\big(\frac{1}{(2+\alpha)(1+\gamma_i(u))^{2+\alpha}} - \frac{1}{2+\alpha}\big)\nu(du) - \frac{\zeta}{2+\alpha}\Big] + a_{ii}y_i^{2-\theta_i}\Big\}$$

has an upper positive bound, say K. Integrating from 0 to t and taking expectations, we obtain

$$\mathbb{E}[e^{\zeta t}y_i^{2+\alpha}] \le y_i^{2+\alpha}(0) + \frac{K}{\zeta}(e^{\zeta t}-1).$$

Therefore

$$\limsup_{t\to\infty}\mathbb{E}[x_i^{-(\alpha+2)}(t)] \le \frac{K}{\zeta} =: K_0.$$

For any given $\varepsilon > 0$, set $\beta_1 = \varepsilon^{1/(2+\alpha)}/K_0^{1/(2+\alpha)}$, by the Chebyshev inequality, we obtain

$$\begin{aligned}\limsup_{t\to\infty}\mathbb{P}\big\{x_i(t) < \beta_1\big\} &= \limsup_{t\to\infty}\mathbb{P}\big\{x_i^{-(2+\alpha)}(t) > \beta_1^{-(2+\alpha)}\big\}\\ &\le \limsup_{t\to\infty}\beta_i^{2+\alpha}\mathbb{E}[x_i^{-(2+\alpha)}(t)] = \varepsilon.\end{aligned}$$

Hence $\liminf_{t\to\infty}\mathbb{P}\{x_i(t)\ge\beta_1\}\ge 1-\varepsilon$, $i=1,\dots,n$.

On the other hand, as an application of Theorem 4.1, and the Chebyshev inequality, we can easily show that for arbitrary $\varepsilon\in(0,1)$, there is a positive constant β_2 such that for any initial data $x_0\in\mathbb{R}_+^n$, $\liminf_{t\to\infty}\mathbb{P}\{x_i(t)\le\beta_2\}\ge 1-\varepsilon$, $1\le i\le n$. Therefore, (1.1) is stochastically permanent. □

7. Example and numerical simulations

Consider the two-species stochastic Gilpin-Ayala mutualism system with jumps

$$\begin{aligned}dx_1 &= x_1(r_1 - a_{11}x_1^{\theta_1} - a_{12}x_2)dt + \sigma_{11}dB_1(t) + \sigma_{12}dB_2(t) + \int_{\mathbb{Y}}\gamma_1(u)x_1N(dt,du),\\ dx_2 &= x_2(r_2 - a_{22}x_2^{\theta_2} - a_{21}x_1)dt + \sigma_{21}dB_1(t) + \sigma_{22}dB_2(t) + \int_{\mathbb{Y}}\gamma_2(u)x_2N(dt,du).\end{aligned} \tag{7.1}$$

In Figure 1, we choose $r_1 = 0.06$, $r_2 = 0.05$, $a_{11} = 0.08$, $a_{22} = 0.04$, $a_{12} = a_{21} = -0.005$, $\theta_1 = \theta_2 = 1.01$, $\sigma_{11} = 0.2$, $\sigma_{22} = 0.05$, $\sigma_{12} = \sigma_{21} = 0$, $\gamma_1(u) = 0.2$, $\gamma_2(u) = 0.24$, $x_1(0) = 1.1$, $x_2(0) = 1.5$, $\mathbb{Y} = (0,\infty)$, $\lambda(\mathbb{Y}) = 1$. Since $a_{11}a_{22} - a_{12}a_{21} > 0$, then (A1) and (A2) hold, so (7.1) has a unique global positive solution for any positive initial value by Theorem 3.1 and pth moment of the solution of (7.1) is asymptotic bounded, see Figure 1. Moreover,

$$\begin{aligned}r_1 - \frac{\sigma_{11}^2+\sigma_{12}^2}{2} + \int_{\mathbb{Y}}\ln(1+\gamma_1(u))\nu(du) = 0.22 > 0,\\ r_2 - \frac{\sigma_{21}^2+\sigma_{22}^2}{2} + \int_{\mathbb{Y}}\ln(1+\gamma_2(u))\nu(du) = 0.26 > 0.\end{aligned}$$

Then in view of Theorem 5.3, we obtain $\frac{\ln x_i(t)}{t}\to 0$, $i = 1,2$, Figure 1 confirms these.

In Figure 2, we choose $r_1 = 0.5$, $r_2 = 0.2$, $a_{11} = a_{22} = 0.9$, $a_{12} = a_{21} = -0.05$, $\theta_1 = \theta_2 = 1.01$, $\sigma_{11} = 0.05$, $\sigma_{22} = 0.1$, $\sigma_{12} = \sigma_{21} = 0$, $\gamma_1(u) = 0.2$, $\gamma_2(u) = 0.12$,

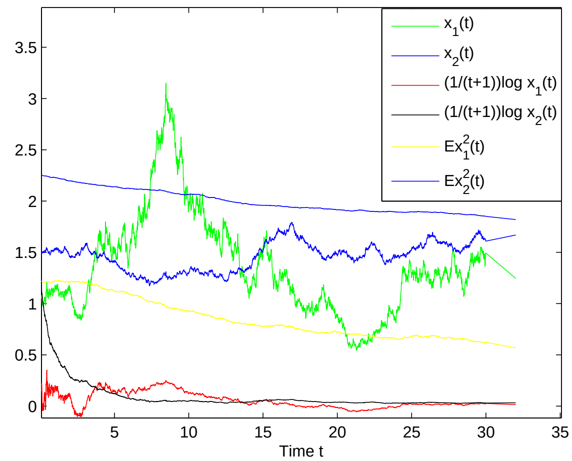

FIGURE 1. Solutions of (7.1) for $r_1 = 0.06$, $r_2 = 0.05$, $a_{11} = 0.08$, $a_{22} = 0.04$, $a_{12} = a_{21} = -0.005$, $\theta_1 = \theta_2 = 1.01$, $\sigma_{11} = 0.2$, $\sigma_{22} = 0.05$, $\sigma_{12} = \sigma_{21} = 0$, $\gamma_1(u) = 0.2$, $\gamma_2(u) = 0.24$, $x_1(0) = 1.1$, $x_2(0) = 1.5$, $\mathbb{Y} = (0, \infty)$, $\lambda(\mathbb{Y}) = 1$

$x_1(0) = 0.1$, $x_2(0) = 0.8$, $\mathbb{Y} = (0, \infty)$, $\lambda(\mathbb{Y}) = 1$. Since $a_{11}a_{22} - a_{12}a_{21} > 0$, then (A1) and (A2) hold. In Theorem 6.2, choose $\alpha = 1$, by simple calculation, we have

$$r_1 - \frac{3+\alpha}{2}\sum_{j=1}^{m}\sigma_{1j}^2 - \int_{\mathbb{Y}}\Big[\frac{1}{(2+\alpha)(1+\gamma_1(u))^{2+\alpha}} - \frac{1}{2+\alpha}\Big]\nu(du) = 0.64 > 0,$$
$$r_2 - \frac{3+\alpha}{2}\sum_{j=1}^{m}\sigma_{2j}^2 - \int_{\mathbb{Y}}\Big[\frac{1}{(2+\alpha)(1+\gamma_2(u))^{2+\alpha}} - \frac{1}{2+\alpha}\Big]\nu(du) = 0.28 > 0\,.$$

Theorem 6.2 tells us that (7.1) is stochastically permanent, and Figure 2 confirms this.

Conclusions. An stochastic Gilpin-Ayala mutualism model with jumps has been studied in this article. The high nonlinearity of Gilpin-Ayala model and Poisson jumps make the problem difficult. Sufficient criteria for the existence of global positive solution, stochastically ultimate boundedness and stochastic permanence are derived for the n-dimensional model by analysis of Lyapunov functions which has been used by many authors. We also investigate asymptotic pathwise estimation. The simulation results verify the effectiveness of the proposed results.

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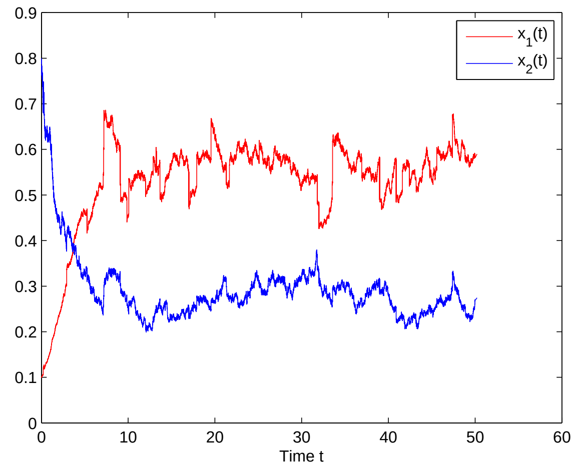

FIGURE 2. Solutions of (7.1) for $r_1 = 0.5$, $r_2 = 0.2$, $a_{11} = a_{22} = 0.9$, $a_{12} = a_{21} = -0.05$, $\theta_1 = \theta_2 = 1.01$, $\sigma_{11} = 0.05$, $\sigma_{22} = 0.1$, $\sigma_{12} = \sigma_{21} = 0$, $\gamma_1(u) = 0.2$, $\gamma_2(u) = 0.12$, $x_1(0) = 0.1$, $x_2(0) = 0.8$, $\mathbb{Y} = (0, \infty)$, $\lambda(\mathbb{Y}) = 1$

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