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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR ANISOTROPIC ELLIPTIC SYSTEMS WITH NON-STANDARD GROWTH CONDITIONS

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Abstract. In this article, we study the existence and multiplicity of solutions for a class of anisotropic elliptic systems with non-standard growth conditions. Our results extend the results in El Hamidi [9] to the anisotropic case.

1. Introduction

In this article, we are interested in the existence and multiplicity of solutions for the anisotropic elliptic system

$$
\begin{gathered}
-\sum_{i=1}^{N} \partial_{x_i}\Big(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u\Big) = F_u(x,u,v) \quad \text{in } \Omega,\\
-\sum_{i=1}^{N} \partial_{x_i}\Big(|\partial_{x_i} v|^{q_i(x)-2}\partial_{x_i} v\Big) = F_v(x,u,v) \quad \text{in } \Omega,\\
u = v = 0 \quad \text{on } \partial\Omega,
\end{gathered}
\tag{1.1}
$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial\Omega$, and p_i, q_i, $i = 1,2,\dots,N$ are continuous functions on $\overline{\Omega}$ such that $2 \leq p_i(x) < N$, $2 \leq q_i(x) < N$ for all $x \in \overline{\Omega}$, $\nabla F = (F_u, F_v)$ stands for the gradient of a C^1-function $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ in the variable $(u,v) \in \mathbb{R}^2$.

When $p_i(x) = p(x)$ for all $i = 1,2,\dots,N$, the operator involved in (1.1) has similar properties to the $p(x)$-Laplace operator; i.e., $\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. This differential operator is a natural generalization of the isotropic p-Laplace operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, where $p > 1$ is a real constant. However, the $p(x)$-Laplace operator possesses more complicated nonlinearities than the p-Laplace operator, due to the fact that $\Delta_{p(x)}$ is not homogeneous. The study of nonlinear elliptic problems (equations and systems) involving quasilinear homogeneous type operators like the p-Laplace operator is based on the theory of standard Sobolev spaces $W^{k,p}(\Omega)$ in order to find weak solutions. These spaces consist of functions that have weak derivatives and satisfy certain integrability conditions. In the case of

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nonhomogeneous $p(x)$-Laplace operators the natural setting for this approach is the use of the variable exponent Sobolev spaces. Differential and partial differential equations with nonstandard growth conditions have received specific attention in recent decades. The interest played by such growth conditions in elastic mechanics and electrorheological fluid dynamics has been highlighted in many physical and mathematical works. We refer to [6, 7, 9, 12, 13, 14, 15].

In a recent article [8], Fragalà et al. studied the anisotropic quasilinear elliptic problem

$$\begin{gathered} -\sum_{i=1}^{N} \partial_{x_i}\Big(|\partial_{x_i}u|^{p_i-2}\partial_{x_i}u\Big) = \lambda u^{p-1} \quad \text{in } \Omega, \\ u \geq 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $p_i > 1$ for all $i = 1, 2, \dots, N$ and $p > 1$. Note that if $p_i = 2$ for all $i = 1, 2, \dots, N$ then problem (1.2) reduces to the well-known semilinear equation $-\Delta u = \lambda u^{p-1}$. By proving an embedding theorem involving the critical exponent of anisotropic type, the authors obtained some existence and nonexistence results in the case when $p > p_+ = \max\{p_1, p_2, \dots, p_N\}$ or $p < p_- = \min\{p_1, p_2, \dots, p_N\}$. The results in [8] have been extended by A.D. Castro et al. [4], in which the authors study problem (1.2) in the case when $p_- < p < p_+$. In order to study the existence of solutions for (1.2) the above authors have found the solutions in the space $W_0^{1,\vec{p}}(\Omega)$ which is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\vec{p}} = \sum_{i=1}^{N} |\partial_{x_i}u|_{p_i},$$

where $\vec{p} = (p_1, p_2, \dots, p_N)$ and $|\cdot|_{p_i}$ denotes the norm in $L^{p_i}(\Omega)$ for all $i = 1, 2, \dots, N$.

In [2, 3, 10, 11], V. Rădulescu et al studied (1.2) when $p_i(x)$ are continuous functions in $\overline{\Omega}$, $i = 1, 2, \dots, N$. The goal of this paper is to extend the original results of El Hamidi [9] on elliptic systems with nonstandard growth conditions to the anisotropic case. To our best knowledge, the present paper is the first contribution in this direction. Regarding the elliptic systems with standard growth conditions, the readers may consult the excellent survey article of D.G.by de Figueiredo [5].

Our paper is organized as follows: In section 2, we introduce the theory of generalized Lebesgue-Sobolev spaces and the generalized anisotropic Sobolev spaces, in which we can seek the solutions of (1.1). In section 3, we will state and prove the main results.

2. Preliminaries

First, we recall some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. In that context, we refer to the book by Musielak [14], the papers by Kováčik and Rákosník [13] and Fan et al [6, 7]. Set

$$C_+(\overline{\Omega}) := \{h; h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define $h^+ = \sup_{x\in\overline{\Omega}} h(x)$, $h^- = \inf_{x\in\overline{\Omega}} h(x)$. For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \big\{u : u \text{ is measurable real-valued function}, \int_\Omega |u(x)|^{p(x)} dx < \infty\big\}.$$

We recall the so-called *Luxemburg norm* on this space, defined by

$$|u|_{p(x)} = \inf\big\{\mu > 0; \int_\Omega |\frac{u(x)}{\mu}|^{p(x)}\, dx \le 1\big\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \le p^+ < \infty$ and continuous functions are dense if $p^+ < \infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \le p_2(x)$ a.e. $x \in \Omega$ then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder inequality

$$\Big|\int_\Omega uv\, dx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)|u|_{p(x)}|v|_{p'(x)}$$

holds.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\, dx.$$

If $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold

$$|u|^{p^-}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^+}_{p(x)} \tag{2.1}$$

provided $|u|_{p(x)} > 1$ while

$$|u|^{p^+}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^-}_{p(x)} \tag{2.2}$$

provided $|u|_{p(x)} < 1$ and

$$|u_n - u|_{p(x)} \to 0 \iff \rho_{p(x)}(u_n - u) \to 0. \tag{2.3}$$

Next, we define the space $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}.$$

We point out that the above norm is equivalent with the following norm

$$\|u\|_{p(x)} = \sum_{i=1}^N |\partial_{x_i} u|_{p(x)},$$

provided $p(x) \ge 2$ for all $x \in \overline{\Omega}$. The space $\big(W_0^{1,p(x)}(\Omega), \|\cdot\|_{p(x)}\big)$ is a separable and Banach space. We note that if $s \in C_+(\overline{\Omega})$ and $s(x) < p^*(x)$ for all $\overline{\Omega}$ then the embedding

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$$

is compact and continuous, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^*(x) = \infty$ if $p(x) > N$.

We introduce a natural generalization of the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ that will enable us to study problem (1.1) with sufficient accuracy. Define $\vec{p}:\overline{\Omega}\to\mathbb{R}^N$ the vectorial function $\vec{p}=(p_1,p_2,\dots,p_N)$. We introduce the anisotropic variable exponent Sobolev space, $W_0^{1,\vec{p}(x)}(\Omega)$, as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\vec{p}(x)}=\sum_{i=1}^N|\partial_{x_i}u|_{p_i(x)}.$$

Then $W_0^{1,\vec{p}(x)}(\Omega)$ is a reflexive and separable Banach space. In the case when p_i are all constants functions the resulting anisotropic space is denoted by $W_0^{1,\vec{p}}(\Omega)$, where $\vec{p}$ is the constant vector $(p_1,p_2,\dots,p_N)$. The theory of such spaces has been developed in [4, 8]. Finally, we introduce $\vec{P}_+,\vec{P}_-\in\mathbb{R}^N$ and $P_+^+,P_-^+,P_+^-,P_-^-\in\mathbb{R}^+$ as

$$\begin{gathered}
\vec{P}_+=(p_1^+,p_2^+,\dots,p_N^+),\quad \vec{P}_-=(p_1^-,p_2^-,\dots,p_N^-),\\
P_+^+=\max\{p_1^+,p_2^+,\dots,p_N^+\},\quad P_-^+=\max\{p_1^-,p_2^-,\dots,p_N^-\},\\
P_+^-=\min\{p_1^+,p_2^+,\dots,p_N^+\},\quad P_-^-=\min\{p_1^-,p_2^-,\dots,p_N^-\}.
\end{gathered}$$

Throughout this paper we assume that

$$\sum_{i=1}^N\frac{1}{p_i^-}>1 \tag{2.4}$$

and define $P_-^*\in\mathbb{R}$ and $P_{-,\infty}\in\mathbb{R}^+$ by

$$P_-^*=\frac{N}{\sum_{i=1}^N\frac{1}{p_i^-}-1},\quad P_{-,\infty}=\max\{P_-^+,P_-^*\}.$$

We recall that if $s\in C_+(\overline{\Omega})$ satisfies $1<s(x)<P_{-,\infty}$ for all $x\in\overline{\Omega}$ then the embedding $W_0^{1,\vec{p}(x)}(\Omega)\hookrightarrow L^{s(x)}(\Omega)$ is compact, see for example [11, Theorem 1].

For (u,v) and (φ,ψ) in $W=W_0^{1,\vec{p}(x)}(\Omega)\times W_0^{1,\vec{q}(x)}(\Omega)$, let

$$\mathcal{F}(u,v):=\int_\Omega F(x,u,v)dx.$$

Then

$$\mathcal{F}'(u,v)(\varphi,\psi)=D_1\mathcal{F}(u,v)(\varphi)+D_2\mathcal{F}(u,v)(\psi),$$

where

$$D_1\mathcal{F}(u,v)(\varphi)=\int_\Omega F_u(x,u,v)\varphi dx,\quad D_2\mathcal{F}(u,v)(\psi)=\int_\Omega F_v(x,u,v)\psi dx.$$

The Euler-Lagrange functional associated to system (1.1) is

$$J(u,v):=\sum_{i=1}^N\int_\Omega\frac{1}{p_i(x)}|\partial_{x_i}u|^{p_i(x)}dx+\sum_{i=1}^N\int_\Omega\frac{1}{q_i(x)}|\partial_{x_i}v|^{q_i(x)}dx-\int_\Omega F(x,u,v)dx.$$

It is easy to verify that $J\in C^1(W,\mathbb{R})$ and $(u,v)\in W$ is a weak solution of system (1.1) if and only if (u,v) is a critical point of J. Moreover, we have

$$J'(u,v)(\varphi,\psi)=D_1J(u,v)(\varphi)+D_2J(u,v)(\psi), \tag{2.5}$$

where

$$D_1J(u,v)(\varphi) = \sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\partial_{x_i}\varphi dx - D_1\mathcal{F}(u,v)(\varphi),$$

$$D_2J(u,v)(\psi) = \sum_{i=1}^{N}\int_\Omega |\partial_{x_i}v|^{q_i(x)-2}\partial_{x_i}v\partial_{x_i}\psi dx - D_2\mathcal{F}(u,v)(\psi).$$

Let us choose on W the norm $\|\cdot\|$ defined by

$$\|(u,v)\| := \max\big\{\|u\|_{\vec{p}(x)}, \|u\|_{\vec{q}(x)}\big\}.$$

The dual space of W will be denoted by W^* and $\|\cdot\|_*$ will stand for its norm. Therefore,

$$\|J'(u,v)\|_* = \|D_1J(u,v)\|_{*,\vec{p}(x)} + \|D_2J(u,v)\|_{*,\vec{q}(x)},$$

where $\|\cdot\|_{*,\vec{p}(x)}$ (respectively $\|\cdot\|_{*,\vec{q}(x)}$) is the norm of $(W_0^{1,\vec{p}(x)}(\Omega))^*$ (respectively $(W_0^{1,\vec{q}(x)}(\Omega))^*$).

3. Main results

Before stating our results, we introduce some natural growth hypotheses on the right-hand side of system (1.1):

(F1) There exists $C>0$ such that

$$F(x,s,t) \le C\big(1+|s|^{\alpha_1(x)}+|t|^{\beta_1(x)}+|s|^{\alpha_2(x)}|t|^{\beta_2(x)}\big)$$

for all $(x,s,t)\in\Omega\times\mathbb{R}^2$, where $\alpha_i,\beta_i\in C_+(\overline{\Omega})$ and $P_+^+ < \alpha_i^- \le \alpha_i^+ < P_-^*$, $Q_+^+ < \beta_i^- \le \beta_i^+ < Q_-^*$, $i=1,2$, and $\frac{\alpha_2(x)}{P_-^*}+\frac{\beta_2(x)}{Q_-^*} < 1$ for all $x\in\overline{\Omega}$;

(F2) There exist constants $R>0$, $\theta_1 > P_+^+$ and $\theta_2 > Q_+^+$ such that

$$0 < F(x,s,t) \le \frac{s}{\theta_1}F_s(x,s,t) + \frac{t}{\theta_2}F_t(x,s,t)$$

for all $(x,s,t)\in\Omega\times\mathbb{R}^2$ with $|s|^{\theta_1}+|t|^{\theta_2}\ge 2R$;

(F3) $F(x,s,t) = o\Big(|s|^{P_+^+}+|t|^{Q_+^+}\Big)$ as $(s,t)\to(0,0)$ uniformly with respect to $x\in\Omega$.

It should be noticed that from the condition (F1), we have $P_{-,\infty} = \max\{P_-^+, P_-^*\} = P_-^*$. Thus, if $s\in C_+(\overline{\Omega})$ satisfies $1<s(x)<P_-^*$ for all $x\in\overline{\Omega}$ then the embedding $W_0^{1,\vec{p}(x)}(\Omega)\hookrightarrow L^{s(x)}(\Omega)$ is compact. Similarly, if $s\in C_+(\overline{\Omega})$ satisfies $1<s(x)<Q_-^*$ for all $x\in\overline{\Omega}$ then the embedding $W_0^{1,\vec{q}(x)}(\Omega)\hookrightarrow L^{s(x)}(\Omega)$ is compact.

Theorem 3.1. *If the function F satisfies the condition*

$$|F(x,s,t)| \le c_1\Big(1+|s|^{\alpha_3(x)}+|t|^{\beta_3(x)}\Big)$$

for all $(x,s,t)\in\Omega\times\mathbb{R}^2$, where $\alpha_3,\beta_3\in C(\overline{\Omega})$ are two functions satisfying $1<\alpha_3(x)<\min\{P_-^-,Q_-^-\}$ and $1<\beta_3(x)<\min\{P_-^-,Q_-^-\}$ for all $x\in\overline{\Omega}$, then system (1.1) has a weak solution.

Proof. From the condition on F, using the Hölder inequality and the Sobolev type embeddings, we can show that the functional J is weakly lower semi-continuous in W. We will show that J is coercive. Indeed, let $\{(u_m,v_m)\}\in W$ be such that

$\|(u_m, v_m)\| \to \infty$ as $m \to \infty$. Without loss of generality, we may assume that $\|u_m\|_{\vec{p}(x)} \geq \|v_m\|_{\vec{q}(x)}$. Hence, using the Sobolev type embeddings, we have

$$\begin{aligned}
J(u_m, v_m) &= \sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u_m|^{p_i(x)} dx + \sum_{i=1}^N \int_\Omega \frac{1}{q_i(x)} |\partial_{x_i} v_m|^{q_i(x)} dx \\
&\quad - \int_\Omega F(x, u_m, v_m) dx \\
&\geq \frac{1}{P_+^+} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u_m|^{p_i(x)} dx + \frac{1}{Q_+^+} \sum_{i=1}^N \int_\Omega |\partial_{x_i} v_m|^{q_i(x)} dx \\
&\quad - c_1 \int_\Omega |u_m|^{\alpha_3(x)} dx - c_1 \int_\Omega |v_m|^{\beta_3(x)} dx - c_1 |\Omega| \\
&\geq \frac{1}{P_+^+} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u_m|^{p_i(x)} dx - c_1 \int_\Omega \Big(|u_m|^{\alpha_3^+} + |u_m|^{\alpha_3^-} \Big) dx \\
&\quad - c_1 \int_\Omega \Big(|v_m|^{\beta_3^+} + |v_m|^{\beta_3^-} \Big) dx - c_1 |\Omega| \\
&\geq \frac{1}{P_+^+} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u_m|^{p_i(x)} dx - c_2 \Big(\|u_m\|_{\vec{p}(x)}^{\alpha_3^+} + \|u_m\|_{\vec{p}(x)}^{\alpha_3^-} \Big) \\
&\quad - c_3 \Big(\|v_m\|_{\vec{q}(x)}^{\beta_3^+} + \|v_m\|_{\vec{q}(x)}^{\beta_3^-} \Big) - c_1 |\Omega|,
\end{aligned}$$

where $|\Omega|$ denotes the measure of Ω.

For each $i \in \{1, 2, \dots, N\}$ and $m \in \mathbb{N}$, we define

$$\alpha_{i,m} = \begin{cases} P_+^+ & \text{if } |\partial_{x_i} u_m|_{p_i(x)} < 1, \\ P_-^- & \text{if } |\partial_{x_i} u_m|_{p_i(x)} > 1. \end{cases}$$

Using (2.1), (2.2) and some simple computations, we infer that for any m, we have

$$\begin{aligned}
\int_\Omega |\partial_{x_i} u_m|^{p_i(x)} dx &\geq \sum_{i=1}^N |\partial_{x_i} u_m|_{p_i(x)}^{\alpha_{i,m}} \\
&\geq \sum_{i=1}^N |\partial_{x_i} u_m|_{p_i(x)}^{P_-^-} - \sum_{\{i : \alpha_{i,m} = P_+^+\}} \Big(|\partial_{x_i} u_m|_{p_i(x)}^{P_-^-} - |\partial_{x_i} u_m|_{p_i(x)}^{P_+^+} \Big) \\
&\geq N \Big(\frac{\sum_{i=1}^N |\partial_{x_i} u_m|_{p_i(x)}}{N} \Big)^{P_-^-} - N \\
&= \frac{\|u_m\|_{\vec{p}(x)}^{P_-^-}}{N^{P_-^- - 1}} - N.
\end{aligned} \tag{3.1}$$

Thus, we obtain

$$\begin{aligned}
J(u_m, v_m) &\geq \frac{1}{P_+^+} \Big(\frac{\|u_m\|_{\vec{p}(x)}^{P_-^-}}{N^{P_-^- - 1}} - N \Big) - c_2 \Big(\|u_m\|_{\vec{p}(x)}^{\alpha_3^+} + \|u_m\|_{\vec{p}(x)}^{\alpha_3^-} \Big) \\
&\quad - c_3 \Big(\|v_m\|_{\vec{q}(x)}^{\beta_3^+} + \|v_m\|_{\vec{q}(x)}^{\beta_3^-} \Big) - c_1 |\Omega|
\end{aligned}$$

$$\begin{aligned}&\geq \frac{\|u_m\|_{\vec{p}(x)}^{P_-^-}}{P_+^+ N^{P_-^- -1}} - c_2\Big(\|u_m\|_{\vec{p}(x)}^{\alpha_3^+} + \|u_m\|_{\vec{p}(x)}^{\alpha_3^-}\Big)\\ &\quad - c_3\Big(\|v_m\|_{\vec{q}(x)}^{\beta_3^+} + \|v_m\|_{\vec{q}(x)}^{\beta_3^-}\Big) - c_4.\end{aligned}$$

By the definition of the norm on W, we have $\|(u_m, v_m)\| = \|u_m\|_{\vec{p}(x)} \to \infty$ as $m \to \infty$. The above inequality and the assumptions on α_3, β_3 imply that J is coercive and thus, J has a minimum point $(u, v) \in W$ and (u, v) is a weak solution which may be trivial of problem (1.1). □

Theorem 3.2. *Assume that the conditions* (F1)-(F3) *are satisfied. Then syste* (1.1) *has at least one nontrivial weak solution.*

To prove Theorem 3.2, we will use the mountain pass theorem. We need to verify the following lemmas.

Lemma 3.3. *Let $\{(u_m, v_m)\}$ be a Palais-Smale sequence for the Euler-Lagrange functional J. If the condition* (F2) *is satisfied then $\{(u_m, v_m)\}$ is bounded.*

Proof. Let $\{(u_m, v_m)\}$ be a Palais-Smale sequence for the functional J. This means that $J(u_m, v_m)$ is bounded and $\|J'(u_m, v_m)\|_* \to 0$ as $m \to \infty$. By the condition (F2), there exists $\overline{c} > 0$ such that for all m,

$$\begin{aligned}\overline{c} &\geq J(u_m, v_m)\\ &= \sum_{i=1}^N \int_\Omega \frac{|\partial_{x_i} u_m|^{p_i(x)}}{p_i(x)} dx + \sum_{i=1}^N \int_\Omega \frac{|\partial_{x_i} v_m|^{q_i(x)}}{q_i(x)} dx - \int_\Omega F(x, u_m, v_m) dx\\ &\geq \int_\Omega \Big(\sum_{i=1}^N \frac{|\partial_{x_i} u_m|^{p_i(x)}}{p_i(x)} - \frac{u_m}{\theta_1} F_u(x, u_m, v_m)\Big) dx\\ &\quad + \int_\Omega \Big(\sum_{i=1}^N \frac{|\partial_{x_i} v_m|^{q_i(x)}}{q_i(x)} - \frac{v_m}{\theta_2} F_v(x, u_m, v_m)\Big) dx - c_5,\end{aligned}$$

where c_5 is a positive constant. Then,

$$\begin{aligned}\overline{c} &\geq \Big(\frac{1}{P_+^+} - \frac{1}{\theta_1}\Big) \sum_{i=1}^N \int_\Omega |\partial_{x_i} u_m|^{p_i(x)} dx + \Big(\frac{1}{Q_+^+} - \frac{1}{\theta_2}\Big) \sum_{i=1}^N \int_\Omega |\partial_{x_i} v_m|^{q_i(x)} dx\\ &\quad + \frac{1}{\theta_1} \int_\Omega \Big(\sum_{i=1}^N |\partial_{x_i} u_m|^{p_i(x)} - F_u(x, u_m, v_m) u_m\Big) dx\\ &\quad + \frac{1}{\theta_2} \int_\Omega \Big(\sum_{i=1}^N |\partial_{x_i} v_m|^{q_i(x)} - F_v(x, u_m, v_m) v_m\Big) dx - c_5\\ &\geq \Big(\frac{1}{P_+^+} - \frac{1}{\theta_1}\Big) \sum_{i=1}^N \int_\Omega |\partial_{x_i} u_m|^{p_i(x)} dx + \Big(\frac{1}{Q_+^+} - \frac{1}{\theta_2}\Big) \sum_{i=1}^N \int_\Omega |\partial_{x_i} v_m|^{q_i(x)} dx\\ &\quad - \frac{1}{\theta_1} \|D_1 J(u_m, v_m)\|_{*,\vec{p}(x)} . \|u_m\|_{\vec{p}(x)}\\ &\quad - \frac{1}{\theta_2} \|D_2 J(u_m, v_m)\|_{*,\vec{q}(x)} . \|v_m\|_{\vec{q}(x)} - c_5.\end{aligned} \tag{3.2}$$

Now, suppose that the sequence $\{(u_m, v_m)\}$ is not bounded. With loss of generality, we may assume that $\|u_m\|_{\vec{p}(x)} \geq \|v_m\|_{\vec{q}(x)}$ and $\|u_m\|_{\vec{p}(x)} > 1$ for all m. Then, by (3.2),

$$\begin{aligned}
\overline{c} &\geq \Big(\frac{1}{P_+^+} - \frac{1}{\theta_1}\Big) \sum_{i=1}^N \int_\Omega |\partial_{x_i} u_m|^{p_i(x)} dx + \Big(\frac{1}{Q_+^+} - \frac{1}{\theta_2}\Big) \sum_{i=1}^N \int_\Omega |\partial_{x_i} v_m|^{q_i(x)} dx \\
&\quad - \frac{1}{\theta_1} \|D_1 J(u_m, v_m)\|_{*,\vec{p}(x)} \|u_m\|_{\vec{p}(x)} - \frac{1}{\theta_2} \|D_2 J(u_m, v_m)\|_{*,\vec{q}(x)} \|v_m\|_{\vec{q}(x)} - c_5 \\
&\geq \Big(\frac{1}{P_+^+} - \frac{1}{\theta_1}\Big) \sum_{i=1}^N |\partial_{x_i} u_m|_{p_i(x)}^{P_-^-} \\
&\quad - \Big(\frac{1}{\theta_1} \|D_1 J(u_m, v_m)\|_{*,\vec{p}(x)} + \frac{1}{\theta_2} \|D_2 J(u_m, v_m)\|_{*,\vec{q}(x)}\Big) \|u_m\|_{\vec{p}(x)} - c_5.
\end{aligned}$$

Using (3.1), we have

$$\begin{aligned}
\overline{c} &\geq \Big(\frac{1}{P_+^+} - \frac{1}{\theta_1}\Big) \frac{\|u_m\|_{\vec{p}(x)}^{P_-^-}}{N^{P_-^- - 1}} \\
&\quad - \Big(\frac{1}{\theta_1} \|D_1 J(u_m, v_m)\|_{*,\vec{p}(x)} + \frac{1}{\theta_2} \|D_2 J(u_m, v_m)\|_{*,\vec{q}(x)}\Big) \|u_m\|_{\vec{p}(x)} - c_6.
\end{aligned}$$

But this cannot hold true since $P_-^- > 1$, $\theta_1 > P_+^+$ and $\|(u_m, v_m)\| = \|u_m\|_{\vec{p}(x)} \to \infty$ as $m \to \infty$. Hence, the sequence $\{(u_m, v_m)\}$ is bounded in W. $\square$

In the following lemma, we show every bounded Palais-Smale sequence for the functional J contains a convergent subsequence.

Lemma 3.4. *Let $\{(u_m, v_m)\}$ be a bounded Palais-Smale sequence for the functional J. If the condition* (F1) *is satisfied then $\{(u_m, v_m)\}$ contains a convergent subsequence.*

Proof. Let $\{(u_m, v_m)\}$ be a bounded Palais-Smale sequence for the functional J. Then there is a subsequence, still denoted by $\{(u_m, v_m)\}$ which converges weakly in W to a function $(u, v) \in W$. Then $J'(u_m, v_m) \to 0$ in W^* as $m \to \infty$. Thus, we have

$$\begin{aligned}
J'(u_m, v_m)(u_m - u, 0) &= \sum_{i=1}^N \int_\Omega |\partial_{x_i} u_m|^{p_i(x)-2} \partial_{x_i} u_m (\partial_{x_i} u_m - \partial_{x_i} u) dx \\
&\quad - \int_\Omega F_u(x, u_m, v_m)(u_m - u) dx
\end{aligned}$$

and

$$\begin{aligned}
J'(u_m, v_m)(0, v_m - v) &= \sum_{i=1}^N \int_\Omega |\partial_{x_i} v_m|^{q_i(x)-2} \partial_{x_i} v_m (\partial_{x_i} v_m - \partial_{x_i} v) dx \\
&\quad - \int_\Omega F_v(x, u_m, v_m)(v_m - v) dx,
\end{aligned}$$

which tends to 0 as $m \to \infty$.

On the other hand, let $\widetilde{\alpha_2}$, $\widetilde{\beta_2}$ be two continuous and positive functions on $\overline{\Omega}$ such that

$$\frac{\alpha_2(x) + \widetilde{\alpha_2}(x)}{P_-^*} + \frac{\beta_2(x) + \widetilde{\beta_2}(x)}{Q_-^*} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

Using the Young inequality, we obtain

$$|s|^{\alpha_2(x)}|t|^{\beta_2(x)} \le |s|^{\frac{\alpha_2(x)P_-^*}{\alpha_2(x)+\widetilde{\alpha_2}(x)}} + |t|^{\frac{\beta_2(x)Q_-^*}{\beta_2(x)+\widetilde{\beta_2}(x)}} = |s|^{\alpha_4(x)} + |t|^{\beta_4(x)},$$

where

$$\alpha_4(x) = \frac{\alpha_2(x)P_-^*}{\alpha_2(x)+\widetilde{\alpha_2}(x)} < P_-^*, \quad \beta_4(x) = \frac{\beta_2(x)Q_-^*}{\beta_2(x)+\widetilde{\beta_2}(x)} < Q_-^*$$

for all $x \in \overline{\Omega}$. From (F1), we can obtain that there exists $c_7 > 0$ such that

$$|F(x,s,t)| \le c_7\Big(1 + |s|^{\alpha_1(x)} + |t|^{\beta_1(x)} + |s|^{\alpha_4(x)} + |t|^{\beta_4(x)}\Big)$$

for all $(x,s,t) \in \Omega \times \mathbb{R}^2$. From this inequality, using the reflexivity of the spaces, the boundedness of the sequences and the Hölder inequality, we can show that

$$\lim_{m\to\infty} \int_\Omega F_u(x,u_m,v_m)(u_m-u)dx = 0,$$
$$\lim_{m\to\infty} \int_\Omega F_v(x,u_m,v_m)(v_m-v)dx = 0.$$

Therefore,

$$\lim_{m\to\infty} \sum_{i=1}^N \int_\Omega |\partial_{x_i}u_m|^{p_i(x)-2}\partial_{x_i}u_m(\partial_{x_i}u_m - \partial_{x_i}u)dx = 0, \tag{3.3}$$

$$\lim_{m\to\infty} \sum_{i=1}^N \int_\Omega |\partial_{x_i}v_m|^{q_i(x)-2}\partial_{x_i}v_m(\partial_{x_i}v_m - \partial_{x_i}v)dx = 0. \tag{3.4}$$

Since $\{u_m\}$ converges weakly to u in $W_0^{1,\vec{p}(x)}(\Omega)$, by (3.3), we find

$$\lim_{m\to\infty} \sum_{i=1}^N \int_\Omega \Big(|\partial_{x_i}u_m|^{p_i(x)-2}\partial_{x_i}u_m - |\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\Big)(\partial_{x_i}u_m - \partial_{x_i}u)dx = 0.$$

Next, we apply the inequality (see [16])

$$(|\xi|^{r-2}\xi - |\eta|^{r-2}\eta)\cdot(\xi-\eta) \ge 2^{-r}|\xi-\eta|^r, \quad \xi,\eta \in \mathbb{R}^N, \tag{3.5}$$

valid for all $r \ge 2$. Relations (3.3) and (3.5) show actually $\{u_m\}$ converges strongly to u in $W_0^{1,\vec{p}(x)}(\Omega)$. Similarly, from (3.4) we conclude that $\{v_m\}$ converges strongly to v in $W_0^{1,\vec{q}(x)}(\Omega)$. Thus, $\{(u_m,v_m)\}$ converges strongly to (u,v) in W. □

Proof Theorem 3.2. Let us show that the functional J satisfies the conditions of the mountain pass theorem in [1]. By Lemmas 3.3 and 3.4, J satisfied the Palais-Smale condition in W.

First, let $s \in C_+(\overline{\Omega})$ and function vector $\vec{r} = (r_1, r_2, \dots, r_N)$ be such that $s(x) < R_{-,\infty}$ in $\overline{\Omega}$. Using the continuous embedding $W_0^{1,\vec{r}(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$, we deduce that there exist $\delta_1, \delta_2 \in (0,1)$ such that for all $w \in W_0^{1,\vec{r}(x)}(\Omega)$ with $\|w\|_{\vec{r}(x)} < \delta_1$ it follows that $\|w\|_{s(x)} < \delta_2$.

For $(u,v) \in W$ with $\|(u,v)\| \ll 1$, using the Young and Hölder inequalities, the fact that $\frac{\alpha_2(x)+\widetilde{\alpha_2}(x)}{P_-^*} + \frac{\beta_2(x)+\widetilde{\beta_2}(x)}{Q_-^*} = 1$ for all $x \in \overline{\Omega}$, it follows that

$$\int_\Omega |u|^{\alpha_2(x)}|v|^{\beta_2(x)}dx$$

$$
\begin{aligned}
&\le \int_\Omega \Big(\frac{(|u|^{\alpha_2(x)})^{\frac{P_-^*}{\alpha_2(x)+\widetilde{\alpha_2}(x)}}}{P_-^*}(\alpha_2(x)+\widetilde{\alpha}_2(x)) + \frac{(|v|^{\beta_2(x)})^{\frac{Q_-^*}{\beta_2(x)+\widetilde{\beta_2}(x)}}}{Q_-^*}(\beta_2(x)+\widetilde{\beta_2}(x))\Big)dx \\
&\le \int_\Omega (|u|^{\alpha_2(x)})^{\frac{P_-^*}{\alpha_2(x)+\widetilde{\alpha_2}(x)}}dx + \int_\Omega (|v|^{\beta_2(x)})^{\frac{Q_-^*}{\beta_2(x)+\widetilde{\beta_2}(x)}}dx \\
&\le \Big(\frac{1}{\big(\frac{P_-^*}{\alpha_2(x)+\widetilde{\alpha_2}(x)}\big)^+} + \frac{1}{\big(\frac{Q_-^*}{\beta_2(x)+\widetilde{\beta_2}(x)}\big)^+}\Big)\Big|\big|(|u|^{\alpha_2(x)})^{\frac{P_-^*}{\alpha_2(x)+\widetilde{\alpha_2}(x)}}\big|\Big|_{\frac{\alpha_2(x)+\widetilde{\alpha_2}(x)}{P_-^*}}|1|_{\frac{\beta_2(x)+\widetilde{\beta_2}(x)}{Q_-^*}} \\
&\quad + \Big(\frac{1}{\big(\frac{Q_-^*}{\beta_2(x)+\widetilde{\beta_2}(x)}\big)^+} + \frac{1}{\big(\frac{P_-^*}{\alpha_2(x)+\widetilde{\alpha_2}(x)}\big)^+}\Big)\Big|\big|(|v|^{\beta_2(x)})^{\frac{Q_-^*}{\beta_2(x)+\widetilde{\beta_2}(x)}}\big|\Big|_{\frac{\beta_2(x)+\widetilde{\beta_2}(x)}{Q_-^*}}|1|_{\frac{\alpha_2(x)+\widetilde{\alpha_2}(x)}{P_-^*}} \\
&\le c_8 \int_\Omega \Big(|u|^{\alpha_2(x)} + |v|^{\beta_2(x)}\Big)dx \\
&\le c_9\Big(\|u\|_{\vec{p}(x)}^{\alpha_2^-} + \|v\|_{\vec{q}(x)}^{\beta_2^-}\Big).
\end{aligned}
$$

On the other hand, assuming (F1), $W_0^{1,\vec{p}(x)}(\Omega) \hookrightarrow L^{P_+^+}(\Omega)$ and $W_0^{1,\vec{q}(x)}(\Omega) \hookrightarrow L^{Q_+^+}(\Omega)$. Then there exist $c_{10} > 0$ and $c_{11} > 0$ such that

$$
\begin{gathered}
|u|_{P_+^+} \le c_{10}\|u\|_{\vec{p}(x)}, \text{ for all } u \in W_0^{1,\vec{p}(x)}(\Omega), \\
|v|_{Q_+^+} \le c_{11}\|v\|_{\vec{q}(x)}, \text{ for all } v \in W_0^{1,\vec{q}(x)}(\Omega).
\end{gathered}
$$

Let $\epsilon > 0$ be small enough such that

$$
\epsilon c_{10}^{P_+^+} < \frac{1}{2P_+^+ N^{P_+^+-1}}, \quad \epsilon c_{11}^{Q_+^+} < \frac{1}{2Q_+^+ N^{Q_+^+-1}}.
$$

By (F1) and (F3), there exists a constant $c(\epsilon) > 0$ such that

$$
|F(x,s,t)| \le \epsilon(|s|^{P_+^+} + |t|^{Q_+^+}) + c(\epsilon)\Big(|s|^{\alpha_1(x)} + |t|^{\beta_1(x)} + |s|^{\alpha_2(x)}|t|^{\beta_2(x)}\Big) \tag{3.6}
$$

for all $(x,s,t) \in \overline{\Omega}\times\mathbb{R}^2$. For $\|(u,v)\| < 1$ sufficiently small, we have $\|u\|_{\vec{p}(x)} < 1$ and $\|v\|_{\vec{q}(x)} < 1$. For such an element u we obtain $|\partial_{x_i} u|_{p_i(x)} < 1$ for all $i = 1,2,\dots,N$. Using (2.1) and some simple computations, we obtain

$$
\begin{aligned}
\sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)}dx &\ge \sum_{i=1}^N |\partial_{x_i} u|_{p_i(x)}^{p_i^+} \ge \sum_{i=1}^N |\partial_{x_i} u|_{p_i(x)}^{P_+^+} \\
&\ge N\Big(\frac{\sum_{i=1}^N |\partial_{x_i} u|_{p_i(x)}}{N}\Big)^{P_+^+} \\
&= \frac{\|u\|_{\vec{p}(x)}^{P_+^+}}{N^{P_+^+-1}}.
\end{aligned} \tag{3.7}
$$

Similarly, we deduce that

$$
\sum_{i=1}^N \int_\Omega |\partial_{x_i} v|^{q_i(x)}dx \ge \frac{\|v\|_{\vec{q}(x)}^{Q_+^+}}{N^{Q_+^+-1}}. \tag{3.8}
$$

Combining relations (3.6), (3.7) and (3.8), we obtain

$$
\begin{aligned}
J(u,v) &= \sum_{i=1}^{N}\int_\Omega \frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)}dx + \sum_{i=1}^{N}\int_\Omega \frac{|\partial_{x_i}v|^{q_i(x)}}{q_i(x)}dx - \int_\Omega F(x,u,v)dx \\
&\geq \frac{\|u\|_{\vec{p}(x)}^{P_+^+}}{P_+^+N^{P_+^+-1}} + \frac{\|v\|_{\vec{q}(x)}^{Q_+^+}}{Q_+^+N^{Q_+^+-1}} - \epsilon\int_\Omega |u|^{P_+^+}dx - \epsilon\int_\Omega |v|^{Q_+^+}dx \\
&\quad - c(\epsilon)\int_\Omega \Big(|u|^{\alpha_1(x)} + |v|^{\beta_1(x)} + |u|^{\alpha_2(x)}|v|^{\beta_2(x)}\Big)dx \\
&\geq \frac{\|u\|_{\vec{p}(x)}^{P_+^+}}{P_+^+N^{P_+^+-1}} + \frac{\|v\|_{\vec{q}(x)}^{Q_+^+}}{Q_+^+N^{Q_+^+-1}} - \epsilon c_{10}^{P_+^+}\|u\|_{\vec{p}(x)}^{P_+^+} - \epsilon c_{11}^{Q_+^+}\|v\|_{\vec{q}(x)}^{Q_+^+} \\
&\quad - \overline{c}(\epsilon)\Big(\|u\|_{\vec{p}(x)}^{\alpha_1^-} + \|v\|_{\vec{q}(x)}^{\beta_1^-} + \|u\|_{\vec{p}(x)}^{\alpha_2^-} + \|v\|_{\vec{q}(x)}^{\beta_2^-}\Big) \\
&\geq \frac{\|u\|_{\vec{p}(x)}^{P_+^+}}{2P_+^+N^{P_+^+-1}} + \frac{\|v\|_{\vec{q}(x)}^{Q_+^+}}{2Q_+^+N^{Q_+^+-1}} \\
&\quad - \overline{c}(\epsilon)\Big(\|u\|_{\vec{p}(x)}^{\alpha_1^-} + \|v\|_{\vec{q}(x)}^{\beta_1^-} + \|u\|_{\vec{p}(x)}^{\alpha_2^-} + \|v\|_{\vec{q}(x)}^{\beta_2^-}\Big).
\end{aligned}
$$

Since $\alpha_i^- > P_+^+$ and $\beta_i^- > Q_+^+$, $i = 1,2$, there exist $r \in (0,1)$ and $\delta > 0$ such that $J(u,v) \geq \delta > 0$ for any $\|(u,v)\| = r$.

On the other hand, we have known that the assumption (F2) implies the following assertion: for every $x \in \overline{\Omega}$, $s,t \in \mathbb{R}$, the inequality

$$F(x,s,t) \geq c_{12}\Big(|s|^{\theta_1} + |t|^{\theta_2} - 1\Big) \tag{3.9}$$

holds, see [9, page 38].

For $(u_0,v_0) \in W\backslash\{(0,0)\}$ and $t > 1$, we have

$$
\begin{aligned}
J(tu_0,tv_0) &= \sum_{i=1}^{N}\int_\Omega \frac{|\partial_{x_i}tu_0|^{p_i(x)}}{p_i(x)}dx + \sum_{i=1}^{N}\int_\Omega \frac{|\partial_{x_i}tv_0|^{q_i(x)}}{q_i(x)}dx - \int_\Omega F(x,tu_0,tv_0)dx \\
&\leq \frac{t^{P_+^+}}{P_-^-}\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u_0|^{p_i(x)}dx + \frac{t^{Q_+^+}}{Q_-^-}\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}v_0|^{q_i(x)}dx \\
&\quad - c_{12}t^{\theta_1}\int_\Omega |u_0|^{\theta_1}dx - c_{12}t^{\theta_2}\int_\Omega |v_0|^{\theta_2}dx - c_{12}.
\end{aligned}
$$

Since $\theta_1 > P_+^+$ and $\theta_2 > Q_+^+$, $J(tu_0,tv_0) \to -\infty$ as $t \to +\infty$. Moreover, $J(0,0) = 0$, considering Lemmas 3.3 and 3.4, we obtain that the functional J satisfies the conditions of the mountain pass theorem. So J admits at least one nontrivial critical point and thus system (1.1) has at least one nontrivial weak solution. $\square$

Next, we prove under some symmetry conditions on the function F that system (1.1) possesses infinitely many nontrivial weak solutions.

Theorem 3.5. *Assume that the conditions* (F1), (F2) *are satisfied, and* $F(x,s,t)$ *is even in* s,t*. Then system* (1.1) *possesses infinitely many (pairs) of solutions with unbounded energy.*

Because $W_0^{1,\vec{p}(x)}(\Omega)$ and $W_0^{1,\vec{q}(x)}(\Omega)$ are reflexive and separable Banach spaces, then W and W^* are too. There exist $\{e_j\} \subset W$ and $\{e_j^*\} \subset W^*$ such that

$$W = \overline{\mathrm{span}\{\mathrm{e_j} : \mathrm{j} = 1, 2, \dots\}}, \quad W^* = \overline{\mathrm{span}\{\mathrm{e_j^*} : \mathrm{j} = 1, 2, \dots\}},$$

$$\langle e_i, e_j^* \rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denoted the duality product between W and W^*. For convenience, we write $X_j = \mathrm{span}\{\mathrm{e_j}\}$, $Y_k = \oplus_{j=1}^{k} X_j$, $Z_k = \overline{\oplus_{j=k}^{\infty} X_j}$. In our arguments, we will use the following Fountain theorem.

Lemma 3.6 ([17, Fountain theorem]). *Assume that X is a separable Banach space, $I \in C^1(X, \mathbb{R})$ is an even functional satisfying the Palais-Smale condition. Moreover, for each $k = 1, 2, \dots$, there exist $\rho_k > r_k > 0$ such that*

(A1) $\max_{\{u \in Y_k:\ \|u\| = \rho_k\}} I(u) \le 0$;

(A2) $\inf_{\{u \in Z_k:\ \|u\| = r_k\}} I(u) \to +\infty$ *as* $k \to \infty$.

Then I has a sequence of critical values which tends to $+\infty$.

For every $a > 1$, $u, v \in L^a(\Omega)$, we define $|(u,v)|_a := \max\{|u|_a, |v|_a\}$. In the assumption (F1), let $\widetilde{\alpha_2}(x)$ and $\widetilde{\beta_2}(x)$ be two continuous and positive functions on $\overline{\Omega}$ such that

$$\frac{\alpha_2(x) + \widetilde{\alpha_2}(x)}{P_-^*} + \frac{\beta_2(x) + \widetilde{\beta_2}(x)}{Q_-^*} = 1 \text{ for all } x \in \overline{\Omega}.$$

Set

$$a := \max_{x \in \overline{\Omega}} \Big\{ \frac{\alpha_2(x) + \widetilde{\alpha_2}(x)}{P_-^*}, \frac{\beta_2(x) + \widetilde{\beta_2}(x)}{Q_-^*}, \alpha_1(x), \beta_1(x) \Big\}, \tag{3.10}$$

$$b := \min_{x \in \overline{\Omega}} \Big\{ \frac{\alpha_2(x) + \widetilde{\alpha_2}(x)}{P_-^*}, \frac{\beta_2(x) + \widetilde{\beta_2}(x)}{Q_-^*}, \alpha_1(x), \beta_1(x) \Big\}. \tag{3.11}$$

Then we obtain the following result whose proof can be found in [9].

Lemma 3.7 ([9]). *Denote*

$$C_k = \sup \{ |(u,v)|_a :\ \|(u,v)\| = 1,\ (u,v) \in Z_k \}.$$

Then $\lim_{k \to \infty} C_k = 0$.

Now, we are in the position to prove Theorem 3.5.

Proof Theorem 3.5. It suffices to show that J has an unbounded sequence of critical points. The proof is based on the Fountain theorem. According to the assumptions on F, Lemmas 3.3 and 3.4, J is an even functional and satisfies the Palais-Smale condition. We will show that if k is large enough, then there exist $\rho_k > r_k > 0$ such that (A1) and (A2) hold.

It is clear that for every $\gamma \in C_+(\overline{\Omega})$, $w \in L^{\gamma(x)}(\Omega)$, there exists $\xi \in \Omega$ such that

$$\int_\Omega |w|^{\gamma(x)} dx = |w|_{\gamma(x)}^{\gamma(\xi)}.$$

For any $(u_k, v_k) \in Z_k$, $\|u_k\|_{\vec{p}(x)} \ge 1$ and $\|v_k\|_{\vec{q}(x)} \ge 1$ and $\|(u_k, v_k)\| = r_k$ (r_k will be specified below), we have

$$J(u_k, v_k) = \sum_{i=1}^{N} \int_\Omega \frac{|\partial_{x_i} u_k|^{p_i(x)}}{p_i(x)} dx + \sum_{i=1}^{N} \int_\Omega \frac{|\partial_{x_i} v_k|^{q_i(x)}}{q_i(x)} dx - \int_\Omega F(x, u_k, v_k) dx$$

$$
\begin{aligned}
&\ge \frac{1}{P_+^+}\sum_{i=1}^N \int_\Omega |\partial_{x_i} u_k|^{p_i(x)}dx + \frac{1}{Q_+^+}\sum_{i=1}^N \int_\Omega |\partial_{x_i} v_k|^{q_i(x)}dx \\
&\quad - c\int_\Omega \Big(1+|u_k|^{\alpha_1(x)}+|v_k|^{\beta_1(x)}+|u_k|^{\alpha_2(x)}|v_k|^{\beta_2(x)}\Big)dx \\
&\ge \frac{1}{P_+^+}\Big(\frac{\|u_k\|_{\vec{p}(x)}^{P_-^-}}{N^{P_-^- -1}} - N\Big) + \frac{1}{Q_+^+}\Big(\frac{\|v_k\|_{\vec{q}(x)}^{Q_-^-}}{N^{Q_-^- -1}} - N\Big) - c|u_k|_{\alpha_1(x)}^{\alpha_1(\xi_1^k)} - c|v_k|_{\beta_1(x)}^{\beta_1(\xi_2^k)} \\
&\quad - c|u_k|_{\alpha_4(x)}^{\alpha_4(\eta_1^k)} - c|v_k|_{\beta_4(x)}^{\beta_4(\eta_2^k)} - c|\Omega|,
\end{aligned}
$$

where $\xi_1^k,\xi_2^k,\eta_1^k,\eta_2^k \in \Omega$ and $|\Omega|$ denotes the measure of Ω. Therefore,

$$
\begin{aligned}
J(u_k,v_k) &\ge \frac{\|u_k\|_{\vec{p}(x)}^{P_-^-}}{P_+^+N^{P_-^- -1}} + \frac{\|v_k\|_{\vec{q}(x)}^{Q_-^-}}{Q_+^+N^{Q_-^- -1}} - c_{13}|u_k|_a^{\alpha_1(\xi_1^k)} - c_{13}|v_k|_a^{\beta_1(\xi_2^k)} \\
&\quad - c_{13}|u_k|_a^{\alpha_4(\eta_1^k)} - c_{13}|v_k|_a^{\beta_4(\eta_2^k)} - c_{13} \\
&\ge \frac{1}{\max\{P_+^+,Q_+^+\}}\|(u_k,v_k)\|^{\min\{P_-^-,Q_-^-\}} - c_{13}(C_k\|(u_k,v_k)\|)^{\alpha_1(\xi_1^k)} \\
&\quad - c_{13}(C_k\|(u_k,v_k)\|)^{\beta_1(\xi_1^k)} - c_{13}(C_k\|(u_k,v_k)\|)^{\alpha_4(\eta_1^k)} \\
&\quad - c_{13}(C_k\|(u_k,v_k)\|)^{\beta_4(\eta_2^k)} - c_{13} \\
&\ge \frac{1}{\max\{P_+^+,Q_+^+\}}\|(u_k,v_k)\|^{\min\{P_-^-,Q_-^-\}} - c_{14}C_k^b\|(u_k,v_k)\|^a - c_{14},
\end{aligned}
$$

where a,b are defined by (3.10) and (3.11). At this stage, we fix r_k as follows:

$$
r_k := \Big(\frac{1}{2c_{14}C_k^b\max\{P_+^+,Q_+^+\}}\Big)^{\frac{1}{(a-\min\{P_-^-,Q_-^-\})}} \to +\infty \quad \text{as } k\to+\infty.
$$

Consequently, if $\|(u_k,v_k)\| = r_k$ then

$$
J(u_k,v_k) \ge \frac{1}{2\max\{P_+^+,Q_+^+\}}\|(u_k,v_k)\|^{\min\{P_-^-,Q_-^-\}} - c_{14} \to +\infty \quad \text{as } k\to+\infty.
$$

From (F2), we have $F(x,s,t) \ge c_{12}(|s|^{\theta_1}+|t|^{\theta_2}-1)$ for every $x\in\Omega$ and $s,t\in\mathbb{R}$. Therefore, for any $(u,v)\in Y_k\backslash\{(0,0)\}$ with $\|(u,v)\| = 1$ and $1<\rho_k = t_k$ with $t_k\to+\infty$, we have

$$
\begin{aligned}
J(t_ku,t_kv) &= \sum_{i=1}^N \int_\Omega |\partial_{x_i} t_ku|^{p_i(x)}dx + \sum_{i=1}^N \int_\Omega |\partial_{x_i} t_kv|^{q_i(x)}dx - \int_\Omega F(x,t_ku,t_kv)dx \\
&\le \frac{t_k^{P_+^+}}{P_-^-}\sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)}dx + \frac{t_k^{Q_+^+}}{Q_-^-}\sum_{i=1}^N \int_\Omega |\partial_{x_i} v|^{q_i(x)}dx \\
&\quad - c_{12}t_k^{\theta_1}\int_\Omega |u|^{\theta_1}dx - c_{12}t_k^{\theta_2}\int_\Omega |v|^{\theta_2}dx + c_{12}.
\end{aligned}
$$

Since $\theta_1 > P_+^+$ and $\theta_2 > Q_+^+$ and $\dim(Y_k) = k$, it is easy to see that $J(t_ku,t_kv)\to -\infty$ as $k\to+\infty$ for $(u,v)\in Y_k$. This implies

$$
\max\{J(u,v) : \|(u,v)\| = \rho_k,\ (u,v)\in Y_k\} \le 0
$$

for every ρ_k large enough. Then the proof of Theorem 3.5 is completed by the Fountain theorem. □

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