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ALMOST AUTOMORPHIC MILD SOLUTIONS OF HYPERBOLIC EVOLUTION EQUATIONS WITH STEPANOV-LIKE ALMOST AUTOMORPHIC FORCING TERM

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Abstract. This article concerns the existence and uniqueness of almost automorphic solutions to the semilinear parabolic boundary differential equations

$$\begin{aligned} x'(t) &= A_m x(t) + f(t, x(t)), \quad t \in \mathbb{R}, \\ Lx(t) &= \phi(t, x(t)), \quad t \in \mathbb{R}, \end{aligned}$$

where $A := A_m|_{\ker L}$ generates a hyperbolic analytic semigroup on a Banach space X, with Stepanov-like almost automorphic nonlinear term, defined on some extrapolated space $X_{\alpha-1}$, for $0 < \alpha < 1$ and ϕ takes values in the boundary space ∂X.

1. Introduction

In this article, we prove existence and uniqueness results of almost automorphic solutions to the following semilinear parabolic boundary differential equations, with Stepanov-like almost automorphic nonlinear term using the techniques initiated by Diagana and N'Guèrèkata in [4].

$$\begin{aligned} x'(t) &= A_m x(t) + h(t, x(t)), \quad t \in \mathbb{R}, \\ Lx(t) &= \phi(t, x(t)), \quad t \in \mathbb{R}, \end{aligned} \tag{1.1}$$

where the first equation stands in the complex Banach space X, called the state space and the second equation lies in a boundary space $\partial X; (A_m, D(A_m))$ is a densely defined linear operator on X and $L : D(A_m) \to \partial X$ is a bounded linear operator.

Motivation for this paper come basically from the following three sources.

The first one is a nice paper by Boulite et al [1]. They have established the existence and uniqueness of almost automorphic solutions to the semilinear boundary differential equation (1.1) using extrapolation methods.

The second source of motivation is a recent paper by Baroun et al [2], where the authors have considered the same equation as (1.1) and proved the existence of almost periodic (almost automorphic) solutions, when the nonlinear term h is almost periodic (almost automorphic), whereas we prove the assertion by taking

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h to be Stepanov-like almost automorphic function. The functions h and ϕ are defined on some continuous interpolation space X_β, $0 \le \beta < 1$, with respect to the sectorial operator $A := A_m|_{\ker L}$.

To prove our results, we make use of the techniques initiated by Diagana and N'Guèrèkata [4], which is also our third source of motivation.

Likewise [1, 2] we solve the (1.1) by transforming the semilinear boundary differential equation (1.1) into an equivalent semilinear evolution equation,

$$x'(t) = A_{\alpha-1}x(t) + h(t, x(t)) + (\lambda - A_{\alpha-1})L_\lambda\phi(t, x(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

where $A_{\alpha-1}$ $0 \le \beta < \alpha < 1$, is the continuous extension of $A := A_m|_{kerL}$ to the extrapolated Banach space $X_{\alpha-1}$ of X_α with respect to A and the semilinear term $h(t, x) + (\lambda - A_{\alpha-1})L_\lambda\phi(t, x) := f(t, x)$ is an $X_{\alpha-1}$ valued function. As in [1, 2] we also assume Greiner's assumption introduced by Greiner [8], which is stated in Section 4. Under Greiner's assumption on L, the operator $L_\lambda := (L|_{\ker(\lambda - A_m)})^{-1}$, called the Drichilet map of A_m, is a bounded linear map from ∂X to X, where $X_{\alpha-1}$ is a larger Banach space than X. The extrapolation theory was introduced by Da Prato, Grisvard [3] and Nagel [7] and is used for various purposes. One can see Section 2 for the mentioned notion (cf. [7, 11] for more details).

These days people have increasing interest in showing almost automorphy of the solutions of the functional differential equations see for e.g. [1, 2, 4, 6, 9, 10, 13]. We refer [9], for the more details on the topic.

Our results generalize the existing ones in [1], in the sense that the function h is assumed to be Stepanov-like almost automorphic functions.

2. Preliminaries

In this section, we begin with fixing some notation and recalling the definitions and basic results on generators of interpolation and extrapolation spaces. Let X be a complex Banach space and $(A, D(A))$ be a sectorial operator on X; that is, there exist the constants $\omega \in \mathbb{R}$, $\phi \in (\frac{\pi}{2}, \pi)$ and $M > 0$ such that

$$\|R(\lambda, A - \omega)\|_{\mathcal{L}(X)} \le \frac{M}{|\lambda - \omega|}, \quad \forall \lambda \in \Sigma_{\omega,\phi},$$

$$\text{where } \Sigma_{\omega,\phi} := \{\lambda \in \mathbb{C} : \lambda \ne \omega, \, |\arg(\lambda - \omega)| \le \phi\} \subset \rho(A).$$

The real *interpolation space* X_α for $\alpha \in (0, 1)$, is a Banach space endowedwith the norm,

$$\|x\|_\alpha := \sup_{\lambda > 0} \|\lambda^\alpha (A - \omega)R(\lambda, A - \omega)x\|. \tag{2.1}$$

Here we denote by, $X_0 := X$, $X_1 := D(A)$, $\|x\|_0 = \|x\|$, and $\|x\|_1 = \|(A - \omega)x\|$. The *extrapolation space* X_{-1} associated with A, is defined to be the completion of $(\widehat{X}, \|\cdot\|_{-1})$, where $\widehat{X} := \overline{D(A)}$, endowed with the norm $\|\cdot\|_{-1}$ given by

$$\|x\|_{-1} := \|(\omega - A)^{-1}x\|, \quad x \in X.$$

In a similar fashion, we can define the space $X_{\alpha-1} := (X_{-1})_\alpha = \overline{\widehat{X}}^{\|\cdot\|_{\alpha-1}}$, with $\|x\|_{\alpha-1} = \sup_{\lambda>0} \|\lambda^\alpha R(\lambda, A_{-1} - \omega)x\|$. The restriction $A_{\alpha-1} : X_\alpha \to X_{\alpha-1}$ of A_{-1} generates the analytic semigroup $(T_{\alpha-1}(t))_{t\ge0}$ on $X_{\alpha-1}$ which is the extension of $T(t)$ to $X_{\alpha-1}$. Observe that $\omega - A_{\alpha-1} : X_\alpha \to X_{\alpha-1}$ is an isometric isomorphism.

We have the following continuous embedding of the spaces, which will be frequently used here.

$$D(A) \hookrightarrow X_\beta \hookrightarrow D((\omega - A)^\alpha) \hookrightarrow X_\alpha \hookrightarrow X,$$
$$X \hookrightarrow X_{\beta-1} \hookrightarrow D((\omega - A_{-1})^\alpha) \hookrightarrow X_{\alpha-1} \hookrightarrow X_{-1},$$

for all $0 < \alpha < \beta < 1$.

Now we state certain propositions for the proofs of which one can see [2].

Proposition 2.1. *Assume that $0 < \alpha \leq 1$ and $0 \leq \beta \leq 1$. Then the following assertions hold for $0 < t \leq t_0$, $t_0 > 0$ and $\tilde{\epsilon} > 0$ such that $0 < \alpha - \tilde{\epsilon} < 1$ with constants possibly depending on t_0.*

(i) *The operator $T(t)$ has continuous extensions $T_{\alpha-1}(t) : X_{\alpha-1} \to X$ satisfying*
$$\|T_{\alpha-1}(t)\|_{\mathcal{L}(X_{\alpha-1},X)} \leq ct^{\alpha-1-\tilde{\epsilon}}, \tag{2.2}$$

(ii) *For $x \in X_{\alpha-1}$ we have*
$$\|T_{\alpha-1}(t)\|_\beta \leq ct^{\alpha-\beta-1-\tilde{\epsilon}}\|x\|_{\alpha-1}. \tag{2.3}$$

Remark 2.2. We can remove $\tilde{\epsilon}$ in Proposition 2.1 by extending $T(t)$ to operators from $D(\omega - A_{-1})^{\alpha\pm\tilde{\epsilon}}$ to X, with norms bounded by $t^{\alpha-1\pm\tilde{\epsilon}}$, where $0 < \alpha \pm \tilde{\epsilon} < 1$, and therefore by employing the reiteration theorem and the interpolation property, the inequality in the assertion (i) can be obtained without $\tilde{\epsilon}$. For a more general situation see [12].

Definition 2.3. An analytic semigroup $(T(t))_{t\geq 0}$ is said to be hyperbolic if it satisfies the following three conditions.

(i) there exist two subspaces X_s (the stable space) and X_u (the unstable space) of X such that $X = X_s \oplus X_u$;
(ii) $T(t)$ is defined on X_u, $T(t)X_u \subset X_u$, and $T(t)X_s \subset X_s$ for all $t \geq 0$;
(iii) there exist constants $M, \delta > 0$ such that
$$\|T(t)P_s\| \leq Me^{-\delta t},\ t \geq 0, \quad \|T(t)P_u\| \leq Me^{\delta t},\ t \leq 0, \tag{2.4}$$
where P_s and P_u are the projections onto X_s and X_u, respectively.

Recall that an analytic semigroup $(T(t))_{t\geq 0}$ is hyperbolic if and only if $\sigma(A) \cap i\mathbb{R} = \phi$, (cf. [7, Prop. 1.15]). In the next proposition, we show the hyperbolicity of the extrapolated semigroup $(T_{\alpha-1}(t))_{t\geq 0}$. Before stating the proposition, we assume that the part of A, $A|_{P_u} : P_u(X) \to P_u(X)$ is bounded, which implies

$$\|AP_u\| \leq C,$$

where C is some constant.

Proposition 2.4. *Let $T(\cdot)$ be hyperbolic and $0 < \alpha \leq 1$. Then the operators P_s and P_u admit continuous extensions $P_{u,\alpha-1} : X_{\alpha-1} \to X$ and $P_{s,\alpha-1} : X_{\alpha-1} \to X_{\alpha-1}$ respectively. Moreover we have the following assertions.*

(i) *$P_{u,\alpha-1}X_{\alpha-1} = P_uX$;*
(ii) *$T_{\alpha-1}(t)P_{s,\alpha-1} = P_{s,\alpha-1}T_{\alpha-1}(t)$;*
(iii) *$T_{\alpha-1}(t) : P_{u,\alpha-1}(X_{\alpha-1}) \to P_{u,\alpha-1}(X_{\alpha-1})$ is an invertible function with inverse $T_{\alpha-1}(-t)$;*

(iv) *for* $0 < \alpha - \tilde{\epsilon} < 1$, *we have*

$$\|T_{\alpha-1}(t)P_{s,\alpha-1}x\| \leq mt^{\alpha-1-\tilde{\epsilon}}e^{-\gamma t}\|x\|_{\alpha-1} \quad \text{for } x \in X_{\alpha-1} \text{ and } t \geq 0, \tag{2.5}$$

$$\|T_{\alpha-1}(t)P_{u,\alpha-1}x\| \leq Ce^{\delta t}\|x\|_{\alpha-1} \quad \text{for } x \in X_{\alpha-1} \text{ and } t \leq 0, \tag{2.6}$$

Proposition 2.5. *For* $x \in X_{\alpha-1}$ *and* $0 \leq \beta \leq 1$, $0 < \alpha < 1$, *we have the following assertions.*

(i) *there is a constant* $c(\alpha, \beta)$, *such that*

$$\|T_{\alpha-1}(t)P_{u,\alpha-1}x\|_\beta \leq c(\alpha, \beta)e^{\delta t}\|x\|_{\alpha-1} \quad \text{for } t \leq 0, \tag{2.7}$$

(ii) *there is a constant* $m(\alpha, \beta)$, *such that for* $t \geq 0$ *and* $0 < \alpha - \tilde{\epsilon} < 1$.

$$\|T_{\alpha-1}(t)P_{s,\alpha-1}x\|_\beta \leq m(\alpha, \beta)e^{-\gamma t}t^{\alpha-\beta-\tilde{\epsilon}-1}\|x\|_{\alpha-1}. \tag{2.8}$$

Definition 2.6. A continuous function $f : \mathbb{R} \to X$, is called almost automorphic, if for every sequence $(\sigma_n)_{n\in\mathbb{N}}$ of real numbers, there is a subsequence $(s_n)_{n\in\mathbb{N}} \subset (\sigma_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n,m\to\infty} f(t + s_n - s_m) = f(t), \quad \text{for each } t \in \mathbb{R}.$$

This is equivalent to

$$g(t) = \lim_{n\to\infty} f(t + s_n), \quad \text{and} \quad f(t) = \lim_{n\to\infty} g(t - s_n),$$

are well defined for each $t \in \mathbb{R}$. The function g in the above definition measurable but not necessarily continuous.

Remark 2.7. An almost automorphic function is continuous but may not be uniformly continuous, for e.g. let $p(t) = 2 + \cos(t) + \cos(\sqrt{2}t)$ and $f : \mathbb{R} \to \mathbb{R}$ defined as $f := \sin(1/p)$, then $f \in AA(X)$, but f is not uniformly continuous on $\mathbb{R}$, so $f \notin AP(X)$.

Lemma 2.8. *We have the following properties of almost automorphic functions:*

(a) *For* $f \in AA(X)$, *the range* $\mathcal{R}_f := \{f(t) : t \in \mathbb{R}\}$ *is precompact in* X, *so that* f *is bounded.*
(b) *For* $f, g \in AA(X)$ *then* $f + g \in AA(X)$.
(c) *Assume that* $f_n \in AA(X)$ *and* $f_n \to g$ *uniformly on* $\mathbb{R}$, *then* $g \in AA(X)$.
(d) $AA(X)$, *equipped with the sup norm given by*

$$\|f\| = \sup_{t\in\mathbb{R}} \|f(t)\|, \tag{2.9}$$

turns out to be a Banach space.

2.1. S^p-Almost automorphy.

Definition 2.9. [14] The Bochner transform $f^b(t, s)$, $t \in \mathbb{R}$, $s \in [0, 1]$ of a function $f : \mathbb{R} \to X$ is defined by $f^b(t, s) := f(t + s)$.

Definition 2.10. The Bochner transform $f^b(t, s, u)$, $t \in \mathbb{R}$, $s \in [0, 1]$, $u \in X$ of a function $f(t, u)$ on $\mathbb{R} \times X$, with values in X, is defined by

$$f^b(t, s, u) := f(t + s, u)$$

for each $x \in X$.

Definition 2.11. For $p \in (1,\infty)$, the space $BS^p(X)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to X$ such that f^b belongs to $L^\infty(\mathbb{R}; L^p((0,1), X))$. This is a Banach space with the norm

$$\|f\|_{S^p} := \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(\tau)\|^p d\tau \Big)^{1/p}. \tag{2.10}$$

Definition 2.12. [13] The space $AS^p(X)$ of Stepanov almost automorphic functions (or S^p-almost automorphic) consists of all $f \in BS^p(X)$ such that $f^b \in AA(L^p(0,1;X))$. That is, a function $f \in L^p_{\rm loc}(\mathbb{R}, X)$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p(0,1;X)$ is almost automorphic in the sense that, for every sequence $(s'_n)_{n\in\mathbb{N}}$ of real numbers, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $g \in L^p_{\rm loc}(\mathbb{R}, X)$ such that

$$\Big[\int_t^{t+1} \|f(s_n + s) - g(s)\|^p ds \Big]^{1/p} \to 0,$$
$$\Big[\int_t^{t+1} \|g(s - s_n) - f(s)\|^p ds \Big]^{1/p} \to 0,$$

as $n \to \infty$ pointwise on $\mathbb{R}$.

Remark 2.13. $AS^p(X_{\alpha-1})$ is the extrapolated space of $AS^p(X_\alpha)$ equipped with norm $\|\cdot\|_{S^p_{\alpha-1}}$, given by

$$\|f\|_{S^p_{\alpha-1}} := \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(\tau)\|^p_{\alpha-1} d\tau \Big)^{1/p}.$$

Remark 2.14. It is clear that if $1 \le p < q < \infty$ and $f \in L^q_{\rm loc}(\mathbb{R}; X)$ is S^q-almost automorphic, then f is S^p-almost automorphic. Also if $f \in AA(X)$, then f is S^p-almost automorphic for any $1 \le p < \infty$.

Let $(Y, \|\cdot\|_Y)$ be an abstract Banach space.

Definition 2.15. A function $F : \mathbb{R} \times Y \to X$, $(t,u) \mapsto F(t,u)$ with $F(\cdot,u) \in L^p_{\rm loc}(\mathbb{R}; X)$ for each $u \in Y$, is said to be S^p-almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in Y$ if $t \mapsto F(t,u)$ is S^p-almost automorphic for each $u \in Y$, that is for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $G(\cdot,u) \in L^p_{\rm loc}(\mathbb{R}, X)$ such that following statements hold

$$\Big[\int_t^{t+1} \|F(s_n + s) - G(s)\|^p ds \Big]^{1/p} \to 0,$$
$$\Big[\int_t^{t+1} \|G(s - s_n) - F(s)\|^p ds \Big]^{1/p} \to 0,$$

as $n \to \infty$ pointwise on $\mathbb{R}$ for each $u \in Y$.

The collection of all S^p-almost automorphic functions from $f : \mathbb{R} \times Y \mapsto X$ will be denoted by $AS^p(\mathbb{R} \times Y)$. Now we have the following composition theorem due to Diagana [6].

Theorem 2.16. [6] *Assume that $\phi \in AS^p(Y)$ such that $K := \overline{\{\phi(t) : t \in \mathbb{R}\}} \subset Y$ is a relatively compact subset of X. Let $F \in AS^p(\mathbb{R} \times Y)$ and let the function $(t,u) \mapsto F(t,u)$ be Lipschitz continuous that is there exists a constant $L > 0$ such that*

$$\|F(t,u) - F(t,v)\| \le L\|u - v\|_Y,$$

for all $t \in \mathbb{R}, (u,v) \in Y \times Y$. *Then the function* $\Gamma : \mathbb{R} \to X$ *defined by* $\Gamma(\cdot) := F(\cdot, \phi(\cdot))$ *belongs to* $AS^p(X)$.

3. Main results

In this section we discuss the existence and uniqueness of almost automorphic solutions of the following semilinear evolution equation,

$$x'(t) = A_{\alpha-1}x(t) + f(t, x(t)), \quad t \in \mathbb{R}, \tag{3.1}$$

with the following assumptions;

- (A1) A is the sectorial operator and the generator of a hyperbolic analytic semigroup $(T(t))_{t\geq 0}$.
- (A2) $f : \mathbb{R} \times X_\beta \to X_{\alpha-1}$, is Stepanov-like almost automorphic in t, for each $x \in X_\beta$.
- (A3) f is uniformly Lipschitz with respect to the second argument, that is
$$\|f(t,x) - f(t,y)\|_{\alpha-1} \leq k\|x-y\|_\beta, \tag{3.2}$$
for all $t \in \mathbb{R}, \ x, y \in X_\beta$, and some constant $k > 0$.

Definition 3.1. A continuous function $x : \mathbb{R} \to X_\beta$, is said to be a mild solution of (3.1), if it satisfies following variation of constants formula

$$x(t) = T(t-s)x(s) + \int_s^t T_{\alpha-1}(t-\sigma)f(\sigma, x(\sigma))d\sigma \tag{3.3}$$

for all $t \geq s, \ t, s \in \mathbb{R}$.

Definition 3.2. A function $u : \mathbb{R} \to X_\beta$, is said to be a bounded solution of (3.1) provided that

$$u(t) = \int_{-\infty}^t T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}f(\sigma, u(\sigma))d\sigma - \int_t^\infty T_{\alpha-1}(t-\sigma)P_{u,\alpha-1}f(\sigma, u(\sigma))d\sigma, \tag{3.4}$$

$t \in \mathbb{R}$.

Throughout the rest of this paper, we assume $\mathcal{H}u(t) := H_1u(t) + H_2u(t)$, where

$$H_1u(t) := \int_{-\infty}^t T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}f(\sigma, u(\sigma))d\sigma,$$
$$H_2u(t) := \int_t^\infty T_{\alpha-1}(t-\sigma)P_{u,\alpha-1}f(\sigma, u(\sigma))d\sigma,$$

for all $t \in \mathbb{R}$.

Lemma 3.3. *Assume that assumptions* (A1)–(A3) *are satisfied. If*

$$M(\alpha, \beta, q, \gamma) := \sum_{n=1}^\infty \Big[\int_{n-1}^n e^{-\gamma q \sigma}\sigma^{-q(\beta+1+\tilde{\epsilon}-\alpha)}d\sigma\Big]^{1/q} < \infty, \tag{3.5}$$

then the operator $\mathcal{H}$ *maps* $AA(X_\beta) \mapsto AA(X_\beta)$.

Proof. Let u be in $AA(X_\beta)$. Then $u \in AS^p(X_\beta)$ and by Lemma 2.8 the set $\overline{\{u(t) : t \in \mathbb{R}\}}$ is compact in X_β. Since f is Lipschitz, then it follows from Theorem 2.16 (also see [5, Theorem 2.21]) that the function $\phi(t) := f(t, u(t))$ belongs to $AS^p(X_\beta)$. Now we show that $\mathcal{H}u \in AA(X_\beta)$.

For that we first define a sequence of integral operators $\{\phi_n\}$ as follows

$$\phi_n(t) := \int_{n-1}^{n} T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}g(\sigma)d\sigma, \quad t\in\mathbb{R} \text{ and } n=1,2,3\dots \tag{3.6}$$

Putting $r=t-\sigma$,

$$\phi_n(t) := \int_{t-n}^{t-n+1} T_{\alpha-1}(r)P_{s,\alpha-1}g(t-r)dr. \tag{3.7}$$

Let $0<\tilde\epsilon+\beta<\alpha$, $0<\alpha-\tilde\epsilon<1$ and using Proposition 2.5 we have

$$\begin{aligned}
\|\phi_n(t)\|_\beta &\le \int_{t-n}^{t-n+1} m(\alpha,\beta)r^{\alpha-1-\beta-\tilde\epsilon}e^{-\gamma r}\|g(t-r)\|_{S^p_{\alpha-1}}dr\\
&\text{now, } r\to(t-r),\\
&\le \int_{n-1}^{n} m(\alpha,\beta)(t-r)^{\alpha-1-\beta-\tilde\epsilon}e^{-\gamma(t-r)}\|g(r)\|_{S^p_{\alpha-1}}dr,\\
&\le \int_{n-1}^{n} m(\alpha,\beta)\sigma^{\alpha-\beta-1-\tilde\epsilon}e^{-\gamma\sigma}\|g\|_{S^p_{\alpha-1}}d\sigma,\\
&\le q(\alpha,\beta)\Big[\int_{n-1}^{n} e^{-\gamma q\sigma}\sigma^{q(\alpha-\beta-1-\tilde\epsilon)}d\sigma\Big]^{1/q}\|g\|_{S^p_{\alpha-1}}.
\end{aligned}$$

By Weierstrass theorem and (3.5), it follows that the series

$$\Phi(t) := \sum_{n=1}^{\infty}\phi_n(t)$$

is uniformly convergent on $\mathbb{R}$. Moreover $\Phi\in C(\mathbb{R},X_\beta)$;

$$\|\Phi(t)\|_\beta \le \sum_{n=1}^{\infty}\|\phi_n(t)\|_\beta \le q(\alpha,\beta)M(\alpha,\beta,q,\gamma)\|\phi\|_{S^p_{\alpha-1}}. \tag{3.8}$$

We show that for all $n=1,2,3$, $\phi_n\in AA(X_\beta)$. Since $g\in AS^p(X_{\alpha-1})$, which implies that for every sequence $(s_n')_{n\in\mathbb{N}}$ of real numbers, there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function g' such that

$$\int_t^{t+1}\|g(\sigma+s_n)-g'(\sigma)\|^p_{\alpha-1}d\sigma\to 0. \tag{3.9}$$

Let us define another sequence of integral operators

$$\widehat{\phi_n}(t) = \int_{n-1}^{n} T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}g'(\sigma)d\sigma \quad \text{for } n=1,2,3,\dots. \tag{3.10}$$

Now we show for $n=1,2,3,\dots$ that $\phi_n\in AA(X_\beta)$. Since $g\in AS^p(X_{\alpha-1})$, for every sequence $(s_n')_{n\in\mathbb{N}}$ of real numbers, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function g' such that

$$\int_t^{t+1}\|g(\sigma+s_n)-g'(\sigma)\|^p_{\alpha-1}d\sigma\to 0. \tag{3.11}$$

Define for all $n=1,2,3,\dots$ another sequence of integral operators

$$\widehat{\phi_n}(t) = \int_{n-1}^{n} T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}g'(\sigma)d\sigma, \tag{3.12}$$

for all $t\in\mathbb{R}$. Consider

$$\phi_n(t+s_{n_k})-\widehat{\phi_n}(t)$$

$$
\begin{aligned}
&= \int_{n-1}^{n} T_{\alpha-1}(t+s_{n_k}-\sigma)P_{s,\alpha-1}g(\sigma)d\sigma - \int_{n-1}^{n} T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}g'(\sigma)d\sigma,\\
&= \int_{n-1}^{n} T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}g(\sigma+s_{n_k})d\sigma - \int_{n-1}^{n} T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}g'(\sigma)d\sigma,\\
&= \int_{n-1}^{n} T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}\big[g(\sigma+s_{n_k})-g'(\sigma)\big]d\sigma.
\end{aligned}
$$

Using Proposition 2.5, we have

$$
\begin{aligned}
&\|\phi_n(t+s_{n_k}) - \widehat{\phi_n}(t)\|_\beta\\
&\le \int_{n-1}^{n} m(\alpha,\beta)e^{-\gamma(t-\sigma)}(t-\sigma)^{-(\beta-\alpha+\tilde\epsilon+1)}\|g(\sigma+s_{n_k})-g'(\sigma)\|_{S^p_{\alpha-1}}d\sigma\\
&\to 0, \quad \text{as } k\to\infty,\ t\in\mathbb{R}, \quad (\text{since } g\in AS^p(X_{\alpha-1})).
\end{aligned}
$$

This implies that $\widehat{\phi_n}(t) = \lim_{k\to\infty}\phi_n(t+s_{n_k})$, $n=1,2,3,\dots$ and $t\in\mathbb{R}$.

In a similar way, one can show that $\phi_n(t) = \lim_{k\to\infty}\widehat{\phi}_n(t-s_{n_k})$, for all $t\in\mathbb{R}$ and $n=1,2,3,\dots$. Therefore for each $n=1,2,3,\dots$, the sequence $\phi_n\in AA(X_\beta)$. ☐

Now we state the main result of this Section.

Theorem 3.4. *Let $0\le\beta<\alpha$, $\tilde\epsilon>0$ such that $0<\alpha-\tilde\epsilon<1$ and $0<\beta+\tilde\epsilon<\alpha$, moreover assume that the constant*

$$
K := k.m(\alpha,\beta)\gamma^{\beta-\alpha+\tilde\epsilon}\Gamma(\alpha-\beta-\tilde\epsilon) + c(\alpha,\beta)\delta^{-1} < 1
$$

and equation (3.5) hold. Then under assumptions (A1)–(A3) and for $f\in AS^p(\mathbb{R}\times X_\beta, X_{\alpha-1})$, equation (3.1) has unique almost automorphic solution $u\in AA(X_\beta)$, satisfying the following variation of constants formula.

$$
u(t) = \int_{-\infty}^{t} T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}f(\sigma,u(\sigma))d\sigma - \int_{t}^{\infty} T_{\alpha-1}(t-\sigma)P_{u,\alpha-1}f(\sigma,u(\sigma))d\sigma,
$$

$t\in\mathbb{R}$.

Proof. We first show that $\mathcal{H}$ is a contraction. Let $v,w\in AA(X_\beta)$ and consider the following

$$
\begin{aligned}
&\|H_1v(t)-H_1w(t)\|_\beta\\
&\le \int_{-\infty}^{t} m(\alpha,\beta)(t-s)^{\alpha-\beta-1-\tilde\epsilon}e^{-\gamma(t-s)}\|f(s,v(s))-f(s,w(s))\|_{\alpha-1}ds\\
&\le \int_{-\infty}^{t} km(\alpha,\beta)(t-s)^{\alpha-\beta-1-\tilde\epsilon}e^{-\gamma(t-s)}\|v(s)-w(s)\|_\beta ds\\
&\le k.m(\alpha,\beta)\gamma^{\beta-\alpha+\tilde\epsilon}\Gamma(\alpha-\beta-\tilde\epsilon)\|v-w\|_\beta,
\end{aligned}
$$

where $\Gamma(\alpha) := \int_0^\infty t^{\alpha-1}e^{-t}dt$. Similarly we have

$$
\begin{aligned}
\|H_2v(t)-H_2w(t)\|_\beta &\le \int_t^\infty c(\alpha,\beta)e^{-\delta(t-s)}\|v(s)-w(s)\|_\beta ds\\
&\le c(\alpha,\beta)\delta^{-1}\|v-w\|_\beta.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\|\mathcal{H}v(t)-\mathcal{H}v(t)\|_\beta &\le \Big(k.m(\alpha,\beta)\gamma^{\beta-\alpha+\tilde\epsilon}\Gamma(\alpha-\beta-\tilde\epsilon)+c(\alpha,\beta)\delta^{-1}\Big)\|v-w\|_\beta\\
&< \|v-w\|_\beta.
\end{aligned}
$$

Hence by the well-known Banach contraction principle, $\mathcal{H}$ has unique fixed point u in $AA(X_\beta)$ satisfying $\mathcal{H}u = u$ (cf. Lemma 3.3 for almost automorphy of solution). □

4. Semilinear boundary differential equations

Consider the semilinear boundary differential equation

$$\begin{aligned} x'(t) &= A_m x(t) + h(t, x(t)), \quad t \in \mathbb{R}, \\ Lx(t) &= \phi(t, x(t)), \quad t \in \mathbb{R}, \end{aligned} \tag{4.1}$$

where $(A_m, D(A_m))$ is a densely defined linear operator on a Banach space X and $L : D(A_m) \to \partial X$, the boundary Banach space and the functions $h : \mathbb{R} \times X_m \to \partial X$ and $\phi : \mathbb{R} \times X_m \to \partial X$ are continuous.

Likewise [1, 2] here we assume the assumptions introduced by Greiner [8] which are given as follows

- (H1) There exists a new norm $|\cdot|$ which makes the domain $D(A_m)$ complete and then denoted by X_m. The space X_m is continuously embedded in X and $A_m \in \mathcal{L}(X_m, X)$.
- (H2) The restriction operator $A := A_m|_{ker(L)}$ is a sectorial operator such that $\sigma(A) \cap i\mathbb{R} = \phi$.
- (H3) The operator $L : X_m \to \partial X$ is bounded and surjective.
- (H4) $X_m \hookrightarrow X_\alpha$ for some $0 < \alpha < 1$.
- (H5) $h : \mathbb{R} \times X_\beta \to X$ and $\phi : \mathbb{R} \times X_\beta \to \partial X$ are continuous for $0 \le \beta < \alpha$.

A function $x : \mathbb{R} \to X_\beta$ is a mild solution of (1.1) if we have the following

- (i) $\int_s^t x(\tau)d\tau \in X_m$,
- (ii) $x(t) - x(s) = A_m \int_s^t x(\tau)d\tau + \int_s^t h(\tau, x(\tau))d\tau$,
- (iii) $L \int_s^t x(\tau)d\tau = \int_s^t \phi(\tau, x(\tau))d\tau$,

for all $t \ge s,\ t, s \in \mathbb{R}$.

Now we transform (1.1) to the equivalent semilinear evolution equation

$$x'(t) = A_{\alpha-1}x(t) + h(t, x(t)) - A_{\alpha-1}L_0\phi(t, x(t)), \quad t \in \mathbb{R}, \tag{4.2}$$

where $L_0 := (L|Ker(A_m))^{-1}$.

Theorem 4.1. *Assume that functions $\phi \in AS^p(\mathbb{R} \times X_\beta, \partial X)$ and $h \in AS^p(\mathbb{R} \times X_\beta, X)$, are globally Lipschitzian with small lipschitz constants. Then under the assumptions (H1)-(H5), the semilinear boundary differential equation* (1.1) *has a unique mild solution $x \in AA(X_\beta)$, satisfying the following formula for all $t \in \mathbb{R}$.*

$$\begin{aligned} x(t) &= \int_{-\infty}^t T(t-s)P_s h(s, x(s))ds - \int_t^\infty T(t-s)P_u h(s, x(s))ds \\ &\quad - A\Big[\int_{-\infty}^t T(t-s)P_s L_0\phi(s, x(s))ds - \int_t^\infty T(t-s)P_u L_0\phi(s, x(s))ds\Big]. \end{aligned} \tag{4.3}$$

Proof. It is clear that $A_{\alpha-1}L_0$ is a bounded operator from $\partial X \to X_{\alpha-1}$. Since $\phi \in AS^p(\mathbb{R} \times X_\beta, \partial X)$ and $h \in AS^p(\mathbb{R} \times X_\beta, X)$ and from the injection $X \hookrightarrow X_{\alpha-1}$, the function $f(t, x) := h(t, x) - A_{\alpha-1}L_0\phi(t, x) \in AS^p(\mathbb{R} \times X_\beta, X_{\alpha-1})$. This function is also globally Lipschitzian with a small constant. Hence by Theorem 3.4, there is a unique mild solution $x \in AA(X_\beta)$ of (4.2), satisfying

$$x(t) = \int_{-\infty}^{t} P_{s,\alpha-1} T_{\alpha-1}(t-s) f(s, x(s)) ds - \int_{t}^{\infty} P_{u,\alpha-1} T_{\alpha-1}(t-s) f(s, x(s)) ds,$$

from which we deduce the variation of constants formula (4.3) and $x \in AA(X_\beta)$ is the unique mild solution. □

Example 4.2. Consider the partial differential equation

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= \Delta u(t,x) + a u(t,x), \quad t \in \mathbb{R},\ x \in \Omega \\ \frac{\partial}{\partial n} u(t,x) &= \Gamma(t, m(x) u(t,x)), \quad t \in \mathbb{R},\ x \in \partial\Omega. \end{aligned} \tag{4.4}$$

Where $a \in \mathbb{R}_+$ and m is a $\mathbb{C}^1$ function and $\Omega \subset \mathbb{R}^n$ is a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Here we use the following notation/conventions: $X = L^2(\Omega)$, $X_m = H^2(\Omega)$ and the boundary space $\partial X = H^{1/2}(\partial\Omega)$. The operators $A_m : X_m \to X$, given by $A_m \varphi = \Delta\varphi + a\phi$ and $L : X_m \to \partial X$, given by $L\varphi := \frac{\partial \varphi}{\partial n}$. The operator L is bounded and surjective, follows from Sections [15, 4.3.3, 4.6.1]. It is also known that the operator $A := A_m|_{\ker L}$ generates an analytic semigroup, moreover we also have $X_m \hookrightarrow X_\alpha$ for $\alpha < 3/4$ (cf. [15, Sections 4.3.3, 4.6.1]). The eigenvalues of Neumann Laplacian A is a decreasing sequence (λ_n) with $\lambda_0 = 0$, $\lambda_1 < 0$, taking $a = -\lambda_1/2$, we have $\sigma(A) \cap i\mathbb{R} = \phi$. Hence the analytic semigroup generated by A is hyperbolic.

$$\phi(t,\varphi)(x) = \Gamma(t, m(x)\varphi(x)) = \frac{k b(t)}{1 + |m(x)\varphi(x)|}, \quad t \in \mathbb{R},\ x \in \partial\Omega$$

where $b(t)$ is S^p Stepanov-like almost automorphic function and $b(\cdot)$ has relatively compact range. It can be easily seen that ϕ is continuous on $\mathbb{R} \times H^{2\beta'}(\Omega)$ for some $\frac{1}{2} < \beta < \beta' < \frac{3}{4}$, which is embedded in $\mathbb{R} \times X_\beta$ (cf. [15]). Using the definitions of fractional Sobolev spaces, one can easily show that $\phi(t,\varphi)(.) \in H^{1/2}(\partial\Omega)$ for all $\varphi \in H^{2\beta'} \hookrightarrow H^1(\Omega)$. Moreover ϕ is globally Lipschitzian for each $\varphi \in X_\beta$. Now for a small constant k, all assumptions of Theorem 4.1 are satisfied. Hence (4.4) admits a unique almost automorphic mild solution u with values in X_β.

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