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EXISTENCE OF SOLUTIONS FOR NON-UNIFORMLY NONLINEAR ELLIPTIC SYSTEMS

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Abstract. Using a variational approach, we prove the existence of solutions for the degenerate quasilinear elliptic system

$$\begin{gathered}
-\operatorname{div}(\nu_1(x)|\nabla u|^{p-2}\nabla u) = \lambda F_u(x,u,v) + \mu G_u(x,u,v),\\
-\operatorname{div}(\nu_2(x)|\nabla v|^{q-2}\nabla v) = \lambda F_v(x,u,v) + \mu G_v(x,u,v),
\end{gathered}$$

with Dirichlet boundary conditions.

1. Introduction

In this article, we study the degenerate quasilinear elliptic system

$$\begin{gathered}
-\operatorname{div}(\nu_1(x)|\nabla u|^{p-2}\nabla u) = \lambda F_u(x,u,v) + \mu G_u(x,u,v), \quad \text{in } \Omega,\\
-\operatorname{div}(\nu_2(x)|\nabla v|^{q-2}\nabla v) = \lambda F_v(x,u,v) + \mu G_v(x,u,v), \quad \text{in } \Omega,\\
u = v = 0, \quad \text{on } \partial\Omega.
\end{gathered} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$ and $1 < p,q < N$. The parameters λ and μ are nonnegative real numbers.

Throughout this work we assume that

$$(F_u, F_v) = \nabla F \quad \text{and} \quad (G_u, G_v) = \nabla G \tag{1.2}$$

which stand for gradient of F and G, respectively, in the variables $w = (u,v) \in \mathbb{R}^2$. Systems of form (1.1), where hypothesis (1.2) is satisfied, are called potential systems. In recent years, more and more attention have been paid to the existence and multiplicity of positive solutions for potential systems. For more details about this kind of systems see [1, 2, 3, 4, 7, 8, 12, 13, 14, 17, 20] and references therein.

The degeneracy of this system is considered in the sense that the measurable, non-negative diffusion coefficients ν_1, ν_2 are allowed to vanish in Ω, (as well as at the boundary $\partial\Omega$) and/or to blow up in $\bar{\Omega}$. The point of departure for the consideration of suitable assumptions on the diffusion coefficients is the work [9], where the degenerate scalar equation was studied.

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We introduce the space $(\mathcal{H})_p$ consisting of functions $\nu:\Omega\subset\mathbb{R}^N\to\mathbb{R}$, such that $\nu\in L^1(\Omega)$, $\nu^{-1/(p-1)}\in L^1(\Omega)$ and $\nu^{-s}\in L^1(\Omega)$, for some $p>1$, $s>\max\{\frac{N}{p},\frac{1}{p-1}\}$ satisfying $ps\le N(s+1)$.

Then for the weight functions ν_1, ν_2 we assume the hypothesis:

(H1) There exist μ_1 in the space $(\mathcal{H})_p$ for some s_p, and there exists μ_2 in the spaces $(\mathcal{H})_q$ for some s_p, such that

$$\frac{\mu_1(x)}{c_1}\le\nu_1(x)\le c_1\mu_1(x),\quad \frac{\mu_2(x)}{c_2}\le\nu_2(x)\le c_2\mu_2(x), \tag{1.3}$$

a.e. in Ω, for some constants $c_1>1$ and $c_2>1$.

There exists a vast literature on non-uniformly nonlinear elliptic problems in bounded or unbounded domains. Many authors studied the existence of solutions for such problems (equations or systems); see for example [5, 6, 11, 15, 16, 18, 19]. Recently in [6], the authors considered the system

$$\begin{gathered}
-\operatorname{div}(h_1(x)\nabla u)=\lambda F_u(x,u,v),\quad \text{in } \Omega,\\
-\operatorname{div}(h_2(x)\nabla v)=\lambda F_v(x,u,v),\quad \text{in } \Omega,\\
u=v=0,\quad \text{on } \partial\Omega.
\end{gathered}$$

They are concerned with the nonexistence and multiplicity of nonnegative, nontrivial solutions. In [19], the author studied the principal eigenvalue of the system

$$\begin{gathered}
-\nabla(\nu_1(x)|\nabla u|^{p-2}\nabla u)=\lambda a(x)|u|^{p-2}u+\lambda b(x)|u|^{\alpha}|v|^{\beta}v,\quad \text{in } \Omega,\\
-\nabla(\nu_2(x)|\nabla v|^{q-2}\nabla v)=\lambda d(x)|v|^{q-2}v+\lambda b(x)|u|^{\alpha}|v|^{\beta}u,\quad \text{in } \Omega,\\
u=v=0,\quad \text{on } \partial\Omega.
\end{gathered}$$

While in [11] the following system was considered

$$\begin{gathered}
-\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u)=\lambda g_1(x,u,v),\quad \text{in } \Omega,\\
-\operatorname{div}(|x|^{-bq}|\nabla v|^{q-2}\nabla v)=\lambda g_2(x,u,v),\quad \text{in } \Omega,\\
u=v=0,\quad \text{on } \partial\Omega,
\end{gathered}$$

where $g_1,g_2:\Omega\times\mathbb{R} times\mathbb{R}$ are continuous and monotone functions.

The aim of this work is to extend or complete some of the above results for system (1.1). Our assumptions are as follows: $F(x,t,s)$ and $G(x,t,s)$ are C^1-functions satisfying the hypotheses below:

(F1) There exist positive constants $c_1,c_2>0$ such that

$$|F_u(x,t,s)|\le c_1|t|^{\theta}|s|^{\delta+1},\quad |F_v(x,t,s)|\le c_2|t|^{\theta+1}|s|^{\delta}$$

for all $(t,s)\in\mathbb{R}^2$, a.e. $x\in\Omega$ and some $\theta,\delta>0$ with

$$\frac{\theta+1}{p}+\frac{\delta+1}{q}=1. \tag{1.4}$$

(F2)

$$\lim_{|(s,t)|\to\infty}\frac{1}{p}F_u(x,s,t)+\frac{1}{q}F_v(x,s,t)-F(x,s,t)=\infty$$

(G1) There exist positive constants c_1',c_2'

$$G_u(x,t,s)\le c_1'|t|^{\alpha}|s|^{\gamma+1},\quad G_v(x,t,s)\le c_2'|t|^{\alpha+1}|s|^{\gamma};$$

for all $(t,s) \in \mathbb{R}^2$, a.e. $x \in \Omega$ and for some $\alpha, \gamma > 0$. We will distinguish the following cases:

$$\frac{\alpha+1}{p} + \frac{\gamma+1}{q} < 1; \tag{1.5}$$

$$\frac{\alpha+1}{p} + \frac{\gamma+1}{q} > 1 \quad \text{and} \quad \frac{\alpha+1}{p^*} + \frac{\gamma+1}{q^*} < 1; \tag{1.6}$$

(G2)

$$\lim_{|(s,t)|\to\infty} \frac{1}{p}G_u(x,s,t) + \frac{1}{q}G_v(x,s,t) - G(x,s,t) = \infty$$

The main results of this paper are the following two theorems.

Theorem 1.1. *In addition to* (F1), (G1) *and* (1.5), *assume that there exist* $p_1 \in (1,p)$ *and* $q_1 \in (1,q)$, *such that* $\frac{\alpha+1}{p_1} + \frac{\gamma+1}{q_1} = 1$. *Then there exists* $\lambda_0 > 0$, *such that* (1.1) *possesses a weak solution for all* $\mu > 0$ *and* $0 \le \lambda < \lambda_0$.

Theorem 1.2. *In addition to* (F1), (G1), (F2) *or* (G2) *and* (1.6), *assume that there exist* $p_2 \in (p,p^*)$ *and* $q_2 \in (q,q^*)$, *such that* $\frac{\alpha+1}{p_2} + \frac{\gamma+1}{q_2} = 1$. *Then there exists* $\lambda_0 > 0$ *such that system* (1.1) *possesses a weak solution for all* $\mu > 0$ *and* $0 \le \lambda < \lambda_0$.

The quantities p^* and q^* are defined in the next section.

2. Preliminaries

Let $\nu(x)$ be a nonnegative weight function in Ω which satisfies condition $\mathcal{H}_p$. We consider the weighted Sobolev space $\mathcal{D}_0^{1,p}(\Omega,\nu)$ defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mathcal{D}_0^{1,p}(\Omega,\nu)} := \Big(\int_\Omega \nu(x)|\nabla u|^p\Big)^{1/p}.$$

The space $\mathcal{D}_0^{1,p}(\Omega,\nu)$ is a reflexive Banach space. For a discussion about the space setting we refer the reader to [9] and the references therein. Let

$$p_s^* := \frac{Nps}{N(s+1)-ps}. \tag{2.1}$$

Lemma 2.1. *Assume that* Ω *is a bounded domain in* $\mathbb{R}^N$ *and the weight* ν *satisfies* $(\mathcal{N})_p$. *Then the following embeddings hold:*

(i) $\mathcal{D}_0^{1,p}(\Omega,\nu) \hookrightarrow L^{p_s^*}(\Omega)$ *continuously for* $1 < p_s^* < N$,
(ii) $\mathcal{D}_0^{1,p}(\Omega,\nu) \hookrightarrow L^r(\Omega)$ *compactly for any* $r \in [1,p_s^*)$.

In the sequel we denote by p^* and q^* the quantities $p^*_{s_p}$ and $p^*_{s_q}$, respectively, where s_p and s_q are induced by condition $(\mathcal{H})$, recall that ν_1, ν_2 satisfy $(\mathcal{H})$.

The space setting for our problem is the product space $H := \mathcal{D}_0^{1,p}(\Omega,\nu_1) \times \mathcal{D}_0^{1,q}(\Omega,\nu_2)$ equipped with the norm

$$\|h\|_H := \|u\|_{\mathcal{D}_0^{1,p}(\Omega,\nu_1)} + \|v\|_{\mathcal{D}_0^{1,q}(\Omega,\nu_2)}, \quad h=(u,v)\in H.$$

Observe that (1.3) in condition $(\mathcal{H})$ implies that the spaces $\mathcal{D}_0^{1,p}(\Omega,\nu_1)\times\mathcal{D}_0^{1,q}(\Omega,\nu_2)$ and $\mathcal{D}_0^{1,p}(\Omega,\mu_1)\times\mathcal{D}_0^{1,q}(\Omega,\mu_2)$ are equivalent. Next, we introduce the functionals

$I, J, \tilde{J} : H \to \mathbb{R}$ as follows:

$$\begin{gathered}
I(u,v) := \frac{1}{p}\int_\Omega \nu_1(x)|\nabla u|^p\,dx + \frac{1}{q}\int_\Omega \nu_2(x)|\nabla v|^q\,dx,\\
J(u,v) := \int_\Omega F(x,u,v)\,dx,\\
\tilde{J}(u,v) := \int_\Omega G(x,u,v)\,dx.
\end{gathered}$$

It is a standard procedure (see [10, 13]) to prove the following properties of these functionals.

Lemma 2.2. *The functionals $I, J, \tilde{J}$ are well defined. Moreover, I is continuous and $J, \tilde{J}$ are compact.*

We say that (u,v) is a *weak solution* of problem (1.1) if (u,v) is a critical point of the functional $\Phi(u,v) := I(u,v) - \lambda J(u,v) - \mu\tilde{J}(u,v)$; i.e.,

$$\int_\Omega \nu_1(x)|\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,dx = \lambda\int_\Omega F_u(x,u,v)\phi\,dx + \mu\int_\Omega G_u(x,u,v)\phi\,dx, \quad (2.2)$$
$$\int_\Omega \nu_2(x)|\nabla v|^{q-2}\nabla v\cdot\nabla\psi\,dx = \lambda\int_\Omega F_v(x,u,v)\psi\,dx + \mu\int_\Omega G_v(x,u,v)\psi\,dx, \quad (2.3)$$

for any $(\phi,\psi) \in H$.

Also, we mention some results concerning the associated eigenvalue problem. Let λ_1 be the first eigenvalue of the Dirichlet problem

$$\begin{gathered}
-\operatorname{div}(\nu_1(x)|\nabla u|^{p-2}\nabla u) = \lambda|u|^{\theta-1}|v|^{\delta+1}u, \quad \text{in } \Omega,\\
-\operatorname{div}(\nu_2(x)|\nabla v|^{q-2}\nabla v) = \lambda|u|^{\theta+1}|v|^{\delta-1}v, \quad \text{in } \Omega,\\
u = v = 0, \quad \text{on } \partial\Omega.
\end{gathered} \quad (2.4)$$

where the functions $\nu_1(x)$ and $\nu_2(x)$ satisfy (H1), and the exponents θ, δ satisfy (1.4). Then, we have that λ_1 is a positive number, which is characterized variationally by

$$\lambda_1 = \inf_{(u,v)\in H-\{(0,0)\}} \frac{\int(\frac{\theta+1}{p}\nu_1(x)|\nabla u|^p + \frac{\delta+1}{q}\,\nu_2(x)|\nabla v|^q)\,dx}{\int|u|^{\theta+1}|v|^{\delta+1}\,dx}.$$

Moreover, λ_1 is isolated, the associated eigenfunction (φ_1,φ_2) is componentwise nonnegative and λ_1 is the only eigenvalue of (2.4) to which corresponds a componentwise nonnegative eigenfunction. In addition, the set of all eigenfunctions corresponding to the principal eigenvalue λ_1 forms a one-dimensional manifold $E_1 \subset H$, which is defined by

$$E_1 = \{(t_1\varphi_1, t_1^{p/q}\varphi_2); t_1 \in \mathbb{R}\}.$$

In the rest of this article, the following assumption is required.

$$\lambda_1 \le \liminf_{|(t,s)|\to\infty} \frac{\lambda F(x,t,s) + \mu G(x,t,s)}{|t|^{\theta+1}|s|^{\delta+1}} \quad (2.5)$$

3. Proof of main theorems

To prove Theorem 1.1 we need following two Lemmas.

Lemma 3.1. *Let $\{w_m\}$ be a sequence weakly converging to w in H. Then we have*
- (i) $\Phi(w) \leq \liminf_{m\to\infty} \Phi(w_m)$
- (ii) $\lim_{m\to\infty} J(w_m) = J(w)$
- (iii) $\lim_{m\to\infty} \tilde{J}(w_m) = \tilde{J}(w)$

Proof. (i) Let $\{w_m\} = \{(u_m, v_m)\}$ be a sequence that converges weakly to $w = (u, v) \in H$. By the weak lower semicontinuity of the norm in the space $\mathcal{D}_0^{1,p}(\Omega, \nu_1)$ and $\mathcal{D}_0^{1,q}(\Omega, \nu_2)$, we deduce that

$$\liminf_{m\to\infty} \int_\Omega \nu_1(x)|\nabla u_m|^p + \int_\Omega \nu_2(x)|\nabla v_m|^q \geq \int_\Omega \nu_1(x)|\nabla u|^p + \int_\Omega \nu_2(x)|\nabla v|^q.$$

The compactness of operators J and $\tilde{J}$, by Lemma(2.2), imply the conclusion. $\square$

Lemma 3.2. *The functional ϕ is coercive and bounded from below.*

Proof. By (F1) and (G1), there exists c_3, c_3', such that for all $(t, s) \in \mathbb{R}^2$ and a. e. $x \in \Omega$, we deduce that

$$F(x, t, s) \leq c_3|t|^{\theta+1}|s|^{\delta+1}, \quad G(x, t, s) \leq c_3'|t|^{\alpha+1}|s|^{\gamma+1}.$$

By taking $p_1 \in (1, p$, $q_1 \in (1, q)$ such that $\frac{\alpha+1}{p_1} + \frac{\gamma+1}{q_1} = 1$ and applying Young's inequality, we obtain

$$\begin{aligned}
\int F(x, u, v)dx &\leq c_3 \int |u|^{\theta+1}|v|^{\delta+1}dx \\
&\leq c_3\Big(\frac{\theta+1}{p}\int |u|^p dx + \frac{\delta+1}{q}\int |v|^q dx\Big) \\
&\leq c_3\Big(\frac{\theta+1}{p} s_1 \int \nu_1(x)|\nabla u|^p dx + \frac{\delta+1}{q} s_2 \int \nu_2(x)|\nabla v|^q dx\Big) \\
&\leq c\Big(\frac{\theta+1}{p}\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\nu_1)} + \frac{\delta+1}{q}\|v\|^q_{\mathcal{D}_0^{1,q}(\Omega,\nu_2)}\Big)
\end{aligned} \tag{3.1}$$

where s_1, s_2 are the embedding constants of $\mathcal{D}_0^{1,p}(\Omega, \nu_1) \hookrightarrow L^p(\Omega)$, $\mathcal{D}_0^{1,q}(\Omega, \nu_2) \hookrightarrow L^q(\Omega)$ and $c = \max\{c_3 s_1, c_3 s_2\}$, while

$$\begin{aligned}
\int G(x, u, v)dx &\leq c_3' \int |u|^{\alpha+1}|v|^{\gamma+1}dx \\
&\leq c_3' \frac{\alpha+1}{p_1}\int |u|^{p_1}dx + c_3'\frac{\gamma+1}{q_1}\int |v|^{q_1}dx \\
&\leq c'\Big(\frac{\alpha+1}{p_1}\|u\|^{p_1}_{\mathcal{D}_0^{1,p}(\Omega,\nu_1)} + \frac{\gamma+1}{q_1}\|v\|^{q_1}_{\mathcal{D}_0^{1,q}(\Omega,\nu_2)}\Big)
\end{aligned} \tag{3.2}$$

Consequently, using (3.1), (3.2), we obtain the estimate

$$\begin{aligned}
\Phi(u, v) \geq{} & \Big(\frac{1}{p} - \lambda c\frac{\theta+1}{p}\Big)\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\nu_1)} + \Big(\frac{1}{q} - \lambda c\frac{\delta+1}{q}\Big)\|v\|^q_{\mathcal{D}_0^{1,q}(\Omega,\nu_2)} \\
& - \mu c'\frac{\alpha+1}{p_1}\|u\|^{p_1}_{\mathcal{D}_0^{1,p}(\Omega,\nu_1)} - \mu c'\frac{\gamma+1}{q_1}\|v\|^{q_1}_{\mathcal{D}_0^{1,q}(\Omega,\nu_2)}.
\end{aligned}$$

Taking $\lambda_0 > 0$ such that $\min\{1-\lambda(\theta+1)c, 1-\lambda(\delta+1)c\} > 0$ for all $0 \le \lambda < \lambda_0$, it follows that for $\mu > 0$ and $0 \le \lambda < \lambda_0$, ϕ is coercive, indeed $\phi(u,v) \to \infty$ as $\|(u,v)\|_H \to \infty$. □

Proof of Theorem 1.1. The coerciveness of Φ and the weak sequential lower semicontinuity are enough in order to prove that Φ attains its infimum, so the system (1.1) has at least one weak solution. □

Proof of theorem 1.2. To prove the existence of a weak solution we apply a version of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [1]. For this purpose we verify that Φ satisfies:

(i) the mountain pass type geometry,
(ii) the $(PS)_c$ condition.

(i) By choosing $p_2 \in (p, p^*)$ and $q_2 \in (q, q^*)$ such that $\frac{\alpha+1}{p_2} + \frac{\gamma+1}{q_2} = 1$ and applying the Young's inequality, we obtain

$$\begin{aligned}\int G(x,u,v)dx &\le c_3' \int |u|^{\alpha+1}|v|^{\gamma+1}dx \\ &\le c_3'\Big(\frac{\alpha+1}{p_2}\int |u|^{p_2}dx + \frac{\gamma+1}{q_2}\int |v|^{q_1}dx\Big) \\ &\le c\Big(\frac{\alpha+1}{p_2}\|u\|^{p_2}_{\mathcal{D}_0^{1,p}(\Omega,\nu_1)} + \frac{\gamma+1}{q_2}\|v\|^{q_2}_{\mathcal{D}_0^{1,q}(\Omega,\nu_2)}\Big),\end{aligned}$$

which implies

$$\begin{aligned}\Phi(u,v) \ge{}& \Big(\frac{1}{p} - \lambda c\frac{\theta+1}{p}\Big)\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\nu_1)} + \Big(\frac{1}{q} - \lambda c\frac{\delta+1}{q}\Big)\|v\|^q_{\mathcal{D}_0^{1,q}(\Omega,\nu_2)} \\ &- \mu c'\frac{\alpha+1}{p_2}\|u\|^{p_2}_{\mathcal{D}_0^{1,p}(\Omega,\nu_1)} - \mu c'\frac{\gamma+1}{q_2}\|v\|^{q_2}_{\mathcal{D}_0^{1,q}(\Omega,\nu_2)}.\end{aligned}$$

Hence, there exists $r > 0$, small enough, such that

$$\inf_{\|(u,v)\|=r} \Phi(u,v) > 0 = \Phi(0,0).$$

On the other hand by using (2.5) we have

$$\begin{aligned}&\Phi(t^{1/p}\varphi_1, t^{1/q}\varphi_2) \\ &\le \frac{t}{p}\int \nu_1|\nabla\varphi_1|^p dx + \frac{t}{q}\int \nu_2|\nabla\varphi_2|^q dx - (\lambda_1+\varepsilon)\int (|t^{1/p}\varphi_1|^{\theta+1}|t^{\frac{1}{q}}\varphi_2|^{\delta+1})dx \\ &= -t\varepsilon\int (|\varphi_1|^{\theta+1}|\varphi_2|^{\delta+1})dx.\end{aligned}$$

Thus, we conclude that there exists $t > 0$, large enough, such that for $e = (t^{1/p}\varphi_1, t^{1/q}\varphi_2)$, we have $\|e\| > r$ and $\Phi(e) < 0$.

(ii) Let $\{w_n\}_{n=1}^{\infty} \in H$ be such that there exists $c > 0$, with

$$|\Phi(w_n)| \le c, \quad \forall n \in \mathbb{N}, \tag{3.3}$$

and there exists a strictly decreasing sequence $\{\varepsilon_n\}_{n=1}^{\infty}, \lim_{n\to\infty}\varepsilon_n = 0$, such that

$$|\langle \Phi'(w_n), z\rangle| \le \varepsilon_n\|z\|_H, \quad \forall n \in N, z \in H. \tag{3.4}$$

We will prove that $\{w_n\}$ contains a subsequence which converges strongly in H. Let us begin by proving that $\{w_n\}$ is bounded in H. Suppose, by contradiction, that $\|w_n\|_H \to \infty$. We have

$$
\begin{aligned}
&|\langle \Phi'(u_n, v_n), (u_n, v_n)\rangle| \\
&= \Big|\int \nu_1(x)|\nabla u_n|^p dx + \int \nu_2(x)|\nabla v_n|^q dx - \lambda \int F_u(x, u_n, v_n)u_n dx \\
&\quad - \lambda \int F_v(x, u_n, v_n)v_n dx - \mu \int G_u(x, u_n, v_n)u_n dx - \mu \int G_v(x, u_n, v_n)v_n dx\Big| \\
&\le \varepsilon_n \|(u_n, v_n)\|_H .
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
|\Phi(u_n, v_n)| = \Big|\frac{1}{p} \int \nu_1(x)|\nabla u_n|^p dx + \frac{1}{q} \int \nu_2(x)|\nabla v_n|^q dx \\
- \lambda \int F(x, u_n, v_n)dx - \mu \int G(x, u_n, v_n)dx\Big| \le c.
\end{aligned}
$$

Thus one has

$$
\begin{aligned}
&c + \varepsilon_n \|(u_n, v_n)\|_H \\
&\ge \Phi(u_n, v_n) - \langle \Phi'(u_n, v_n), (\frac{u_n}{p}, \frac{v_n}{q})\rangle \\
&= \lambda \int \Big(\frac{1}{p} F_u(x, u_n, v_n)u_n + \frac{1}{q} F_v(x, u_n, v_n)v_n - F(x, u_n, v_n)\Big)dx \\
&\quad \mu \int \Big(\frac{1}{p} G_u(x, u_n, v_n)u_n + \frac{1}{q} G_v(x, u_n, v_n)v_n - G(x, u_n, v_n)\Big)dx,
\end{aligned}
$$

which contradicts both (F2) and (G2). So $\{w_n\}$ is bounded. This imply that there exists $(u, v) \in H$ such that at least its subsequence, w_n converges and strongly in $L^p(\Omega) \times L^q(\Omega)$. Choosing $z = (u_n - u, 0)$ in (3.4), we obtain

$$
\begin{aligned}
&\Big|\int \nu_1(x)|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)dx - \lambda \int F_u(x, u_n, v_n)(u_n - u)dx \\
&- \mu \int G_u(x, u_n, v_n)(u_n - u)dx\Big| \\
&\le \varepsilon_n \|u_n - u\|_{\mathcal{D}_0^{1,p}(\Omega,\nu_1)},
\end{aligned}
$$

$$
\begin{aligned}
\Big|\int F_u(x, u_n, v_n)(u_n - u)dx\Big| &\le \int |F_u(x, u_n, v_n)||(u_n - u)|dx \\
&\le \int |u_n|^\theta |v_n|^{\gamma+1}|u_n - u|dx \\
&\le \|u_n\|^\theta_{L^p} \|v_n\|^{\gamma+1}_{L^q} \|u_n - u\|_{L^p},
\end{aligned}
$$

and

$$
\begin{aligned}
\Big|\int G_u(x, u_n, v_n)(u_n - u)dx\Big| &\le \int |G_u(x, u_n, v_n)||(u_n - u)|dx \\
&\le \int |u_n|^\alpha |v_n|^{\delta+1}|u_n - u|dx \\
&\le \|u_n\|^\alpha_{L^p} \|v_n\|^{\delta+1}_{L^q} \|u_n - u\|_{L^p}.
\end{aligned}
$$

Thus, we obtain

$$
\int \nu_1(x)|\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u)dx \to 0,
$$

as $n \to \infty$. In the same way we obtain

$$\int \nu_1(x)|\nabla u|^{p-2}\nabla u(\nabla u_n - \nabla u)dx,$$

as $n \to \infty$. Finally, we conclude that

$$\lim_{n\to\infty}\int \nu_1(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u)dx = 0. \tag{3.5}$$

Observe now that for all $\xi, \eta \in \mathbb{R}^N$, there exists constant $c_3 > 0$, such that

$$\begin{gathered}(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta) \geq c(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2 \quad \text{if } 1 < p < 2\\ (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta) \geq c|\xi - \eta|^p \quad \text{if } p \geq 2.\end{gathered} \tag{3.6}$$

where $(\cdot,\cdot)$ denotes the usual product in $\mathbb{R}^N$.

So, for $1 < p < 2$, by Hölder's inequality and substituting $z_n = \nu_1^{1/p}u_n, z = \nu_1^{1/p}u$ in (3.6), there exists $c^* > 0$, such that

$$\begin{aligned}
0 &\leq \int |\nabla z_n - \nabla z|^p dx\\
&= \int |\nabla z_n - \nabla z|^p(|\nabla z_n| + |\nabla z|)^{p(p-2)/2}(|\nabla z_n| + |\nabla z|)^{p(2-p)/2}dx\\
&\leq \Big(\int |\nabla z_n - \nabla z|^2(|\nabla z_n| + |\nabla z|)^{p-2}dx\Big)^{p/2}\Big(\int(|\nabla z_n| + |\nabla z|)^p dx\Big)^{(2-p)/2}\\
&\leq \frac{1}{c^*}\Big(\int(|\nabla z_n|^{p-2}\nabla z_n - |\nabla z|^{p-2}\nabla z, (\nabla z_n - \nabla z)dx\Big)^{p/2}\\
&\quad\times\Big(\int(|\nabla z_n| + |\nabla z|)^p dx\Big)^{(2-p)/2}\\
&\leq \frac{c}{c^*}\Big(\int(|\nabla z_n|^{p-2}\nabla z_n - |\nabla z|^{p-2}\nabla z, (\nabla z_n - \nabla z)dx\Big)^{p/2},
\end{aligned}$$

which implies $\|u_n - u\|_{\mathcal{D}_0^{1,p}(\Omega,\nu_1)} \to 0$, by (3.5), as $n \to \infty$. While, for $p \geq 2$, by (3.6), one has

$$0 \leq \|u_n - u\|_{\mathcal{D}_0^{1,p}(\Omega,\nu_1)} \leq \frac{1}{c^*}\Big(\int(|\nabla z_n|^{p-2}\nabla z_n - |\nabla z|^{p-2}\nabla z, (\nabla z_n - \nabla z)dx\Big),$$

so we have $\|u_n - u\|_{\mathcal{D}_0^{1,p}(\Omega,\nu_1)} \to 0$, by (3.5), as $n \to \infty$. Therefore, $\|u_n - u\|_{\mathcal{D}_0^{1,p}(\Omega,\nu_1)} \to 0$ for $p > 1$, as $n \to \infty$, that is, $u_n \to u$ in $\mathcal{D}_0^{1,p}(\Omega,\nu_1)$ as $n \to \infty$. Similarly , we obtain $v_n \to v$ in $\mathcal{D}_0^{1,q}(\Omega,\nu_2)$ as $n \to \infty$. Consequently, Φ satisfies the $(PS)_c$ condition and the proof of is completed. □

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