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AN OSCILLATION THEOREM FOR A SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE POTENTIAL

JAGMOHAN TYAGI

Abstract. We obtain a new oscillation theorem for the nonlinear second-order differential equation

$$(a(t)x'(t))' + p(t)f(t, x(t), x'(t)) + q(t)g(x(t)) = 0, \quad t \in [0, \infty),$$

via the generalization of Leighton's variational theorem.

1. Introduction

The purpose of this study is to establish a new oscillation criteria for the nonlinear differential equation

$$(a(t)x'(t))' + p(t)f(t, x(t), x'(t)) + q(t)g(x(t)) = 0, \tag{1.1}$$

where $a, p, q \in C(\mathbb{R}^+, \mathbb{R})$, $f \in C(\mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R})$, $g \in C(\mathbb{R}, \mathbb{R})$, $a(t) > 0$ and $p(t) \geq 0$.

Komkov [5] generalized a well-known variational theorem of Leighton [7]. In this note, we establish a new oscillation theorem for (1.1) via Komkov's result. Also, we do not impose restriction on the sign of the potential q. Here, we consider only solution of (1.1) which are defined for all large t. A solution of (1.1) is called *oscillatory* if it has arbitrarily large zeros, otherwise it is called *nonoscillatory.* Oscillation criteria for the special cases of (1.1)

$$x''(t) + q(t)g(x(t)) = 0, \tag{1.2}$$

$$x''(t) + q(t)x(t) = 0, \tag{1.3}$$

have been extensively investigated; (see, e.g., [1, 2, 3, 4, 6], [8]–[13] for an excellent bibliography). The most important simple oscillation criterion for linear differential equations is the well-known Leighton's theorem [6], which states that if $q(t) \geq 0$ and satisfies

$$\lim_{t\to\infty} \int_0^t q(s)ds = \infty, \tag{1.4}$$

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then (1.3) is oscillatory. Wintner [11] modified the Leighton's criteria and proved a stronger result which required a weaker condition

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \int_0^s q(\tau)d\tau ds = \infty. \tag{1.5}$$

Also, Wintner did not impose any condition on the sign of $q(t)$. Wintner's result was further improved by Hartman [3] who proved that (1.5) can be substituted by the weaker condition

$$-\infty < \liminf_{t\to\infty} \frac{1}{t}\int_0^t \int_0^s q(\tau)d\tau ds < \limsup_{t\to\infty} \frac{1}{t}\int_0^t \int_0^s q(\tau)d\tau ds \le \infty. \tag{1.6}$$

Later in 1978, Kamenev [4] showed that if for some positive integer $n > 2$,

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}}\int_0^t (t-s)^{n-1} q(s)ds = \infty, \tag{1.7}$$

then (1.3) is oscillatory. Also, there is a good amount of literature on oscillation of (1.2) (see [1, 2, 8, 9, 10, 12, 13] and the literature cited therein). In 1992, James S. W. Wong [12] proved the following extension of Cole's result [1] to the more general equation (1.2).

Theorem 1.1. *Let $g(x)$ satisfy the superlinearity condition*

$$0 < \int_x^\infty \frac{du}{g(u)} < \infty, \quad 0 < \int_{-x}^{-\infty} \frac{du}{g(u)} < \infty, \quad \forall 0 < x \in \mathbb{R}.$$

Also, let $A(t) = \int_t^\infty q(s)ds$ exists for each $t \ge 0$ and satisfy

$$\lim_{T\to\infty} \int_0^T A(t)dt = \infty.$$

Then (1.2) is oscillatory.

The above cited results do not include a damping term. The main result is stated and proved in section 2 which includes a nonlinear damping term.

2. Main Result

In this section, we state and prove the main theorem of the paper.

Theorem 2.1. *Let there exist two divergent sequences $\{\tau_n\}$, $\{\eta_n\} \subset \mathbb{R}^+$ such that $0 < \tau_n < \eta_n \le \tau_{n+1} < \eta_{n+1} \le \dots$, for all $n \in \mathbb{N}$. Let there exist a C^1 function y defined on $[\tau_n, \eta_n]$ such that $y(\tau_n) = 0 = y(\eta_n)$, for all $n \in \mathbb{N}$. Let $g'(u)$ exist and there exist $\mu > 0$ such that $g'(u) \ge \mu^2 > 0$, $ug(u) > 0$, for all $0 \ne u \in \mathbb{R}$ and $xf(t,x,u) \ge 0$, for all $(t,x,u) \in \mathbb{R}^+ \times \mathbb{R}^2$, $x \ne 0$. Assume that there exist a C^1 function F defined on $\mathbb{R}$ and a continuous function h on $\mathbb{R}$ such that $F(0) = 0$, $F(y(t))$ is not constant on $[\tau_n, \eta_n]$, for all $n \in \mathbb{N}$, $F'(y) = \mu h(y)$ with $[h(y(t))]^2 \le 4F(y(t))$ and*

$$\int_{\tau_n}^{\eta_n} [a(t)(y'(t))^2 - q(t)F(y(t))]dt < 0, \forall\, t \in [\tau_n, \eta_n], \quad \forall\, n \in \mathbb{N}. \tag{2.1}$$

Then every solution of (1.1) will vanish on $[\tau_n, \eta_n]$, for all $n \in \mathbb{N}$, and hence (1.1) is oscillatory.

Proof. Suppose on the contrary, there exist a solution x of (1.1) such that $x(t) \neq 0$, for all $t \in [\tau_p, \eta_p]$ for some $p \in \mathbb{N}$. Now there are two cases.
Case 1. $x(t) > 0$, for all $t \in [\tau_p, \eta_p]$. We observe that the following is valid on $[\tau_p, \eta_p]$:

$$
\begin{aligned}
&a(t)(y'(t))^2 - q(t)F(y(t)) + \frac{F(y(t))}{g(x(t))}[(a(t)x'(t))' + p(t)f(t,x(t),x'(t)) + q(t)g(x(t))] \\
&= a(t)(x(t))^2\big[\big(\frac{y(t)}{x(t)}\big)'\big]^2 + \Big(\frac{a(t)x'(t)F(y(t))}{g(x(t))}\Big)' - \Big(\frac{a(t)x'(t)F'(y(t))y'(t)}{g(x(t))}\Big) \\
&\quad - \Big(\frac{a(t)(x'(t))^2(y(t))^2}{(x(t))^2}\Big) + \Big(\frac{a(t)(x'(t))^2g'(x(t))F(y(t))}{(g(x(t)))^2}\Big) + \Big(\frac{2a(t)y'(t)y(t)x'(t)}{(x(t))}\Big) \\
&\quad + \frac{F(y(t))}{g(x(t))}p(t)f(t,x(t),x'(t)) \\
&\geq a(t)(x(t))^2\big[\big(\frac{y(t)}{x(t)}\big)'\big]^2 + \Big(\frac{a(t)x'(t)F(y(t))}{g(x(t))}\Big)' - \Big(\frac{a(t)x'(t)\mu h(y(t))y'(t)}{g(x(t))}\Big) \\
&\quad - \Big(\frac{a(t)(x'(t))^2(y(t))^2}{(x(t))^2}\Big) + \Big(\frac{a(t)(x'(t))^2\mu^2(h(y(t)))^2}{4(g(x(t)))^2}\Big) + \Big(\frac{2a(t)y'(t)y(t)x'(t)}{(x(t))}\Big) \\
&\quad + \frac{F(y(t))}{g(x(t))}p(t)f(t,x(t),x'(t)) \\
&\geq \Big(\frac{a(t)x'(t)F(y(t))}{g(x(t))}\Big)' + a(t)\big[y'(t) - \frac{x'(t)\mu h(y(t))}{2g(x(t))}\big]^2 \\
&\quad + \frac{F(y(t))}{g(x(t))}p(t)f(t,x(t),x'(t)).
\end{aligned}
$$

Since x is a solution of (1.1), so, we have

$$
\begin{aligned}
&a(t)(y'(t))^2 - q(t)F(y(t)) \\
&\geq \Big(\frac{a(t)x'(t)F(y(t))}{g(x(t))}\Big)' + a(t)\big[y'(t) - \frac{x'(t)\mu h(y(t))}{2g(x(t))}\big]^2 \\
&\quad + \frac{F(y(t))}{g(x(t))}p(t)f(t,x(t),x'(t)).
\end{aligned} \tag{2.2}
$$

An integration of (2.2) on $[\tau_p, \eta_p]$ yields

$$
\begin{aligned}
&\int_{\tau_p}^{\eta_p} [a(t)(y'(t))^2 - q(t)F(y(t))]dt \\
&\geq \Big(\frac{a(t)x'(t)F(y(t))}{g(x(t))}\Big)_{\tau_p}^{\eta_p} + \int_{\tau_p}^{\eta_p} a(t)\big[y'(t) - \frac{x'(t)\mu h(y(t))}{2g(x(t))}\big]^2 dt \\
&\quad + \int_{\tau_p}^{\eta_p} \frac{F(y(t))}{g(x(t))}p(t)f(t,x(t),x'(t))dt.
\end{aligned} \tag{2.3}
$$

From this inequality, it follows that

$$
\int_{\tau_p}^{\eta_p} [a(t)(y'(t))^2 - q(t)F(y(t))]dt \geq 0,
$$

which contradicts (2.1).
Case 2. $x(t) < 0$ for all $t \in [\tau_p, \eta_p]$. The proof of case 2 is similar to that of case 1 and is omitted for the sake of brevity. This completes the proof. □

Remark 2.2. Consider the differential equation

$$(a(t)x'(t))' + p(t)f(t, x(t), x'(t))x'(t) + q(t)g(x(t)) = 0, \tag{2.4}$$

where $a, p, q \in C(\mathbb{R}^+, \mathbb{R})$, $f \in C(\mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R})$, $g \in C(\mathbb{R}, \mathbb{R})$, $a(t) > 0$ and $p(t) \geq 0$. With the hypotheses of Theorem 2.1, if we replace the condition $xf(t, x, u) \geq 0$ for all $(t, x, u) \in \mathbb{R}^+ \times \mathbb{R}^2$, $x \neq 0$ in Theorem 2.1 by $xuf(t, x, u) \geq 0$ for all $(t, x, u) \in \mathbb{R}^+ \times \mathbb{R}^2$, $x \neq 0$, then (2.4) is oscillatory.

3. Examples

In this section, we construct some examples for illustration.

Example 3.1. Consider the differential equation

$$(a(t)x'(t))' + p(t)f(t, x(t), x'(t)) + q(t)g(x(t)) = 0, \tag{3.1}$$

where $a(t) \equiv 1$, $p(t) \equiv 1$, $f(t, x, y) = x^3 e^y$, $q(t) = t^2 \sin t$ and $g(x) = x + x^{2n+1}, n \in \mathbb{N}$. With the choice of $y(t) = \sin t$, $\tau_n = (2n-1)\pi$, $\eta_n = (2n+1)\pi$, $F(y) = y^2$, $\mu = 1$, it is easy to see that the hypotheses of Theorem 2.1 are satisfied. Also, it is easy to verify

$$\int_{(2n-1)\pi}^{(2n+1)\pi} [\cos^2 t - t^2 \sin t \, \sin^2 t] dt < 0, \quad \forall \, n \in \mathbb{N}.$$

An application of Theorem 2.1 implies that (3.1) is oscillatory.

Remark 3.2. Let $a(t) \equiv 1$, $p(t) \equiv 0$, $q(t) = t^2 \sin t$ and $g(x) = x$ in (3.1). Then none of the known criteria (see, [3, 6, 11], [9, Thms. 3.3, 3.5], [10, Thm. 3.1]) can be applied to (3.1).

Remark 3.3. Let $a(t) \equiv 1$, $p(t) \equiv 0$, $g(x) = x + x^3$ in (3.1). Then [2, Thm. 3] cannot be applied to (3.1).

Example 3.4. Let $a, b \in \mathbb{R}$ and $a > 4$. Consider the damped Mathieu's equation

$$x''(t) + e^t x(t)(x'(t))^2 + (a + b\cos 2t)x(t) = 0. \tag{3.2}$$

This equation can be viewed as (3.1) with $a(t) \equiv 1$, $p(t) = e^t$, $f(t, x, y) = xy^2$, $q(t) = a + b\cos 2t$ and $g(x) = x$. With the selection of $y(t) = \sin 2t$, $\tau_n = \frac{(n-1)\pi}{2}$, $\eta_n = \frac{(n+1)\pi}{2}$, $F(y) = y^2$, $\mu = 1$, it is easy to verify the hypotheses of Theorem 2.1. Also, the condition

$$\int_{\frac{(n-1)\pi}{2}}^{\frac{(n+1)\pi}{2}} [4\cos^2 2t - (a + b\cos 2t)\sin^2 2t] dt < 0, \forall \, a > 4, \quad \forall \, n \in \mathbb{N}$$

holds. Thus, from Theorem 2.1, (3.2) is oscillatory.

Example 3.5. Consider the equation

$$x''(t) + \cos t \, x'(t) + \sin t \, x(t) = 0. \tag{3.3}$$

This equation is oscillatory; see [13, Cor. 3]. Here, we give another alternative which is simple. (3.3) can be converted into

$$u''(t) + \Big(\frac{3\sin t}{2} - \frac{\cos^2 t}{4}\Big)u(t) = 0, \tag{3.4}$$

where $u(t) = x(t)e^{(\sin t)/2}$. (3.4) can be viewed as (3.1) with $a(t) \equiv 1$, $p(t) = 0$, $q(t) = \big(\frac{3\sin t}{2} - \frac{\cos^2 t}{4}\big)$ and $g(x) = x$. After setting $y(t) = \sin t$, $\tau_n = 2n\pi$, $\eta_n =$

$(2n+1)\pi$, $F(y)=y^2$, $\mu=1$, it is not difficult to satisfy the hypotheses of Theorem 2.1 with

$$\int_{2n\pi}^{(2n+1)\pi}\big[\cos^2 t-\big(\frac{3\sin t}{2}-\frac{\cos^2 t}{4}\big)\sin^2 t\big]dt<0,\quad \forall\, n\in\mathbb{N}.$$

It follows from Theorem 2.1 that (3.4) is oscillatory. Since $u(t)=x(t)e^{(\sin t)/2}$ is an oscillation preserving substitution, so, (3.3) is oscillatory.

Remark 3.6. The results of Li and Agarwal [8] cannot be applied to (3.3).

Finally, it remains an open question if the result of this note can be modified for (1.1) with linear damping and variable potential.

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Jagmohan Tyagi
Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur - 208016, India
E-mail address: `jagmohan.iitk@gmail.com`