Electronic Journal of Differential Equations, Vol. 2009(2009), No. 16, pp. 1–17.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

A CELL COMPLEX STRUCTURE FOR THE SPACE OF HETEROCLINES FOR A SEMILINEAR PARABOLIC EQUATION

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Abstract. It is well known that for many semilinear parabolic equations there is a global attractor which has a cell complex structure with finite dimensional cells. Additionally, many semilinear parabolic equations have equilibria with finite dimensional unstable manifolds. In this article, these results are unified to show that for a specific parabolic equation on an unbounded domain, the space of heteroclinic orbits has a cell complex structure with finite dimensional cells. The result depends crucially on the choice of spatial dimension and the degree of the nonlinearity in the parabolic equation, and thereby requires some delicate treatment.

1. Introduction

In this article, the space of heteroclinic orbits of
$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - u^2(t,x) + \phi(x) \tag{1.1}$$
is shown to have the structure of a cell complex with finite-dimensional cells, where $u \in C^1(\mathbb{R}, C^{0,\alpha}(\mathbb{R}))$, $\phi \in L^1 \cap C^{0,\alpha}(\mathbb{R})$, and $|\phi| \to 0$ as $|x| \to \infty$. This result makes precise the intuition that there are relatively few eternal solutions (those that exist for all time t), and fewer still that are heteroclines. Moreover, the cell complex structure provides a helpful framework for understanding the bifurcations that occur in solutions to (1.1) when ϕ is varied. As will be clear from the analysis, bifurcations occur when the number of cells or the attaching maps in the cell complex of heteroclines change as a result of changes in the spectrum of a certain operator involving ϕ. One should note that the bifurcations are rather delicate. The decay condition on ϕ ensures that even small changes in ϕ as measured by L^p-norms can result in vastly different cell complex structures. Perhaps more importantly, this result is a key step in the programme of constructing a Floer homology theory for (1.1). In particular, it is relatively easy to show that (1.1) is a gradient differential equation [16]. The right side of (1.1) is the L^2-gradient of the following functional defined for all $f \in C^1(\mathbb{R})$:
$$A(f) = \int_{-\infty}^{\infty} -\frac{1}{2}\|\nabla f(x)\|^2 - \frac{1}{3} f^3(x) + f(x)\phi(x) dx. \tag{1.2}$$

2000 *Mathematics Subject Classification.* 35B40, 35K55.
Key words and phrases. Eternal solution; heteroclinic connection; cell complex; semilinear parabolic equation; equilibrium.

Submitted January 5, 2009. Published January 16, 2009.

This article provides a demonstration that the linearization of (1.1) about a heterocline is a Fredholm operator, much as is done in Floer's original work [8]. Although a Floer theory for (1.1) is not completed yet, we hope that the cell complex described in this article is an analogue of the usual Morse complex.

The result of this article is a generalization of the well-known result that the unstable manifolds of (1.1) are finite dimensional. Indeed, a standard proof of the finite dimensionality of unstable manifolds (for instance, Theorem 5.2.1 in [10]) can easily be made to apply with the Banach spaces we shall choose. One can then use the iterated time-1 map of the flow for (1.1) to extend this local manifold to a maximal unstable manifold. On the other hand, there are also finite Hausdorff dimensional attractors for the forward Cauchy problem on bounded domains [15]. We shall exhibit a more global approach to the finite dimensionality of the unstable manifolds than is usual, which allows us to examine the finite dimensionality of the space of heteroclinic orbits connecting a pair of equilibria. In essence, the result that is obtained here shows that the intersection of the stable and unstable manifolds of (1.1) is relatively benign, and in any event is a finite-dimensional submanifold of both the stable and unstable manifolds. (We note in passing that no transversality results for stable and unstable manifolds are obtained in this article.)

The techniques used here depend rather delicately on both the degree of the nonlinearity (which is quadratic) and the spatial dimension (which is 1). Both of these are important in the standard methodology as well, as the portion of the spectrum of the linearization in the right half-plane needs to be bounded away from zero. In the case of (1.1), the spectrum in the right-half plane is discrete and consists of a finite number of points.

Of an immediate and important concern is that there may not be any solutions to (1.1) which are defined in $C^1(\mathbb{R}, C^{0,\alpha}(\mathbb{R}))$. More particularly, are there solutions to (1.1) which are defined for all time? This question can be answered in the affirmative [18], so this article makes the assumption that the space of heteroclines is nonempty and draws heavily on their properties as explained in [16].

2. Applications

Equation (1.1) is a very simple model of combustion. If ϕ is a positive constant, then the equation supports traveling waves. Such traveling waves can model the propagation of a flame through a fuel source [21]. In addition to a model of combustion, (1.1) can also be a simple model of the population of a single species, with a spatially-varying carrying capacity, ϕ. Indeed, one easily finds that under certain conditions the behavior of solutions to (1.1) is reminiscent of the growth and (admittedly tenuous) control of invasive species [2]. It is the control of invasive species that is of most interest, and it is also what the structure of the attaching maps of the cell complex reveals. In the example given in Section 6, there is one more stable equilibrium, and several other less stable ones. The more stable equilibrium can be thought of as the situation where an invasive species dominates. The task, then, is to try to perturb the system so that it no longer is attracted to that equilibrium. An optimal control approach is to perturb the system so that it barely crosses the boundary of the stable manifold of the the undesired equilibrium, and thereby the invasive species is eventually brought under control with minimal disturbance to the rest of the environment.

3. Prior work

Equations of the form (1.1) have been of interest to researchers for quite some time. Existence and uniqueness of solutions on short time intervals (on strips $(0, t_0) \times \mathbb{R}$) can been shown using semigroup methods and are entirely standard [24]. However, there are obstructions to the existence of eternal solutions, those which exist for all time. Aside from the typical loss of regularity due to solving the backwards heat equation, there is also a blow-up phenomenon which can spoil existence in the forward-time solution to (1.1). Blow-up phenonmena in the forward time Cauchy problem (where one does not consider $t < 0$) have been studied by a number of authors [9, 6, 22, 13, 3, 26, 27]. More recently, Zhang *et al.* [25, 20, 23] studied global existence for the forward Cauchy problem for

$$\frac{\partial u}{\partial t} = \Delta u + u^p - V(x)u$$

for positive u, V. Du and Ma studied a related problem in [5] under more restricted conditions on the coefficients but they obtained stronger existence results. In fact, they found that all of the solutions which were defined for all $t > 0$ tended to equilibrium solutions.

The boundary value problem that results from taking $x \in \Omega \subset \mathbb{R}^n$ for some bounded Ω (instead of $x \in \mathbb{R}^n$) has also been discussed extensively in the literature [10, 11, 4].

Much of the literature (including this article) describing eternal solutions to (1.1) is restricted to discussing heteroclines. For unbounded domains and certain choices of ϕ, one can find traveling waves. Since the propagation of waves in nonlinear models is of great interest in applications, there is much written on the subject. The general idea is that one makes a change of variables $(t, x) \mapsto \xi = x - ct$ which reduces (1.1) to an ordinary differential equation. This ordinary differential equation describes the profile of a traveling wave. Powerful topologically-motivated techniques, such as the Leray-Schauder degree, can be used to prove existence of wave solutions to (1.1). Asymptotic methods can be used to determine the wave speed c, which is often of interest in applications. See [21] for a very thorough introduction to the subject of traveling waves in (1.1).

4. The linearization and its kernel

We begin by considering an equilibrium solution f to (1.1). As discussed in [19], this solution has asymptotic behavior which places it in $C^2 \cap L^1 \cap L^\infty(\mathbb{R})$, which is a consequence of the decay condition on ϕ. Moreover, we have that $|A(f)| < \infty$ in (1.2). We are particularly interested in solutions which lie in the α-limit set of f, those solutions which are defined for all $t < 0$ and tend to f. Center attention on this equilibrium by applying the change of variables $u(t, x) \mapsto u(t, x) - f(x)$ to obtain

$$\begin{gathered}
\frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) - 2f(x)u(t,x) - u^2(t,x) \\
u(0,x) = h(x) \in C^2(\mathbb{R}) \\
\lim_{t \to -\infty} u(t,x) = f(x) \\
t < 0, x \in \mathbb{R}.
\end{gathered} \tag{4.1}$$

Thus we have a final value problem for our nonlinear equation. All solutions to (4.1) (which exist at all) will tend to zero as $t \to -\infty$ uniformly, which is a result of Lemma 6 of [16]. Although this result is somewhat nontrivial, it is a consequence of parabolic regularity and the fact that the function space $C^{0,\alpha}(\mathbb{R})$ is a Banach algebra. Of course, (4.1) is ill-posed. We show that there is only a finite dimensional manifold of choices of h for which a solution exists.

4.1. **Backward time decay.** The decay of solutions to zero is a crucial part of the analysis, as it provides the ability to perform Laplace transforms. In the forward time direction, one obtains upper bounds for solutions by way of maximum principles, and lower bounds for the upper bounds by way of Harnack estimates. In the backward time direction, these tools reverse roles. Harnack estimates provide upper bounds, while the maximum principle provides lower bounds for the upper bound. In this section, we briefly apply a standard Harnack estimate to obtain an exponentially decaying upper bound.

Harnack estimates for a very general class of parabolic equations are discussed in [14] and [1]. In those articles, the authors examine positive solutions to

$$\operatorname{div} \mathbf{A}(x,t,u,\nabla u) - \frac{\partial u}{\partial t} = B(x,t,u,\nabla u),$$

where $x \in \mathbb{R}^n$, and $\mathbf{A} : \mathbb{R}^{2n+2} \to \mathbb{R}^n$ and $B : \mathbb{R}^{2n+2} \to \mathbb{R}$ satisfy

$$\begin{gathered}
|\mathbf{A}(x,t,u,p)| \le a|p| + c|u| + e \\
|B(x,t,u,p)| \le b|p| + d|u| + f \\
p \cdot \mathbf{A}(x,t,u,p) \ge \frac{1}{a}|p|^2 - d|u|^2 - g,
\end{gathered}$$

for some $a > 0$ and $b, \dots g$ are measurable functions. For a solution u defined on a rectangle R, the authors define a pair of congruent, disjoint closed rectangles $R^+, R^- \subset R$ with R^- being a backward time translation of R^+. The main result is the Harnack inequality

$$\max_{R^-} u \le \gamma\Big(\min_{R^+} u + L\Big), \tag{4.2}$$

where $\gamma > 0$ depends only on geometry and a (but not $b, \dots g$) and L is a linear combination of e, f, g whose coefficients depend on geometry.

In the case of (4.1), we have that (4.2) will apply with $L = 0$, since the e, f, g can all be chosen to be zero. Notice that the conditions on A, B are satisfied because any solution to (4.1) is automatically a finite energy solution in the sense of [16] (the functional A in (1.2) remains finite along time-slices of the solution), and therefore is bounded and has bounded first derivatives. This can also be viewed as a consequence of parabolic regularity. The only difficulty is that (4.2) applies for *positive* solutions, while (4.1) may have solutions with negative portions. However, one can pose the problem for the (weak) solution of

$$\begin{aligned}
\frac{\partial |u|}{\partial t} &= \operatorname{sgn}(u)\big(\Delta u - u^2 - 2fu\big) \\
&= \Delta|u| - u|u| - 2f|u| \\
&\ge \Delta|u| - |u|^2 - 2|f||u|
\end{aligned}$$

for which we only get positive solutions. By iterating (4.2) over a sequence of rectangles $R_k = \{(t,x) \in \mathbb{R}^2 | -k+1 \le t \le -k \text{ and } a \le x \le b\}$ for $k = 1, 2, \dots$

and fixed a, b, we have that solutions to (4.1) decay exponentially (uniformly on compact spatial subsets) as $t \to -\infty$. However, Lemma 6 of [16] asserts that this decay is stronger: in fact, it is *uniform* as $t \to -\infty$.

4.2. **Topological considerations.**

Definition 4.1. Let $Y_a(X)$ be the subspace of $C^1(X, C^{0,\alpha}(\mathbb{R}))$ which consists of functions which decay exponentially to zero like e^{at}, where $0 < \alpha \leq 1$. We define the weighted norm

$$\|u\|_{Y_a} = \|e^{-at}\|u(t)\|_{C^{0,\alpha}(\mathbb{R})}\|_{C^1}$$

and the space

$$Y_a(X) = \{u = u(t,x) \in C^1(X, C^{0,\alpha}(\mathbb{R})) | \|u\|_{Y_a} < \infty\}.$$

In a similar way, we can define the weighted Banach space $Z_a(X)$ as a subspace of $C^0(X, C^{0,\alpha}(\mathbb{R}))$. It is quite important that Y_a and Z_a are Banach algebras under pointwise multiplication.

Eternal solutions to (4.1) are zeros of the densely defined nonlinear operator $N : Y_a((-\infty, 0]) \to Z_a((-\infty, 0])$ given by

$$N(u) = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u^2 + 2fu. \tag{4.3}$$

About the zero function, the linearization of N is the densely defined linear map $L : Y_a((-\infty, 0]) \to Z_a((-\infty, 0])$ given by

$$L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + 2f = \frac{\partial}{\partial t} - H, \tag{4.4}$$

where we define $H = \frac{\partial^2}{\partial x^2} - 2f$. Also note that L is the Frechét derivative of N, which follows from the fact that Y_a and Z_a are Banach algebras.

Remark 4.2. We are using $C^{0,\alpha}(\mathbb{R})$ instead of $C^0(\mathbb{R})$ to ensure that N and L be densely defined. We could use space of continuous functions which decay to zero, or the space of uniformly continuous functions equally well.

Convention 4.3. We shall conventionally take $a > 0$ to be smaller than the smallest eigenvalue of H.

We show two things: that the kernel of L is finite dimensional, and that L is surjective. These two facts enable us to use the implicit function theorem to conclude that the space of solutions comprising the α-limit set of an equilibrium is a finite dimensional submanifold of $Y_a((-\infty, 0])$.

4.3. **Dimension of the kernel.**

Lemma 4.4. *If f is an equilibrium solution, then the operator $L : Y_a((-\infty, 0]) \to Z_a((-\infty, 0])$ in (4.4) has a finite dimensional kernel.*

Proof. Notice that the operator L is separable, so we try the usual separation $h(t,x) = T(t)X(x)$. Substituting into (4.4) gives

$$\begin{aligned} 0 = Lh &= \Big(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + 2f\Big)h \\ &= T'X + T\Big(-\frac{\partial^2}{\partial x^2} + 2f\Big)X \end{aligned}$$

$$\frac{T'}{T} = \frac{\left(\frac{\partial^2}{\partial x^2} - 2f\right)X}{X} = \lambda$$

for some $\lambda \in \mathbb{C}$. The separated equation for T yields $T = C_x e^{\lambda t}$. Since we are looking for the kernel of L in $Y_a \subset L^\infty(\mathbb{R}^2)$, we must conclude that λ must have nonnegative real part. On the other hand, the spectrum of $H = (\frac{\partial^2}{\partial x^2} - 2f)$ is strictly real, so $\lambda \geq 0$. Indeed, there are finitely many positive possibilities for λ each with finite-dimensional eigenspace. This is a standard fact about the Schrödinger operator H since f is an equilibrium. Thus L has a finite dimensional kernel. $\square$

4.4. **Surjectivity of the linearization.** In order to show the surjectivity of L, we will construct a map $\Gamma : Z_a((-\infty, 0]) \to Y_a((-\infty, 0])$ for which $L \circ \Gamma = \mathrm{id}_{Z_a}$. That is, we construct a right-inverse to L, noting of course that L is typically not injective. We shall derive a formula for Γ using the Laplace transform $v \mapsto \overline{v}$

$$\overline{v}(s, x) = \int_{-\infty}^{0} e^{st} v(t, x) dt,$$

where $\Re(s) > -a$ and $v \in Z_a((-\infty, 0])$.

Since Lemma 4.4 essentially solves (4.1), we will be solving the inhomogeneous problem with zero final condition

$$\begin{gathered} \frac{\partial v(t,x)}{\partial t} - \frac{\partial^2 v(t,x)}{\partial x^2} + 2f(x)v(t,x) = -w(t,x) \in Z_a((-\infty,0]) \\ v(0,x) = 0 \end{gathered} \tag{4.5}$$

for $t < 0$. The Laplace transform of this problem is

$$\begin{gathered} s\overline{v}(s,x) + \frac{\partial^2 \overline{v}(s,x)}{\partial x^2} - 2f(x)\overline{v}(s,x) = \overline{w}(s,x) \\ (H + s)\overline{v}(s,x) = \overline{w}(s,x). \end{gathered}$$

Choose a vertical contour C with $0 > \Re(s) > -a$, so that the Laplace transforms are well-defined, and that the contour remains entirely in the resolvent set of $-H$. Then we can invert to obtain

$$\overline{v}(s,x) = (H + s)^{-1}\overline{w}(s,x).$$

Using the inversion formula for the Laplace transform yields

$$\begin{aligned} v(t,x) &= \frac{1}{2\pi i}\int_C e^{-st}(H+s)^{-1}\overline{w}(s,x)ds \\ &= \frac{1}{2\pi i}\int_C e^{-st}(H+s)^{-1}\int_t^0 e^{s\tau}w(\tau,x)d\tau\, ds \\ &= \int_t^0 \Big(\frac{1}{2\pi i}\int_C e^{s(\tau-t)}(H+s)^{-1}ds\Big)w(\tau,x)d\tau. \end{aligned}$$

We can obtain operator convergence of the operator-valued integral in parentheses if we deflect the contour C. Choose instead the portion C' of the hyperbola (See Figure 1)

$$(\Re(s))^2 - (\Im(s))^2 = \frac{1}{4}(\lambda - a)^2 \tag{4.6}$$

(where λ is the smallest magnitude eigenvalue of $-H$) which lies in the left half-plane as our new contour. Then, since $-H : C^{0,\alpha} \to C^{0,\alpha}$ is sectorial about

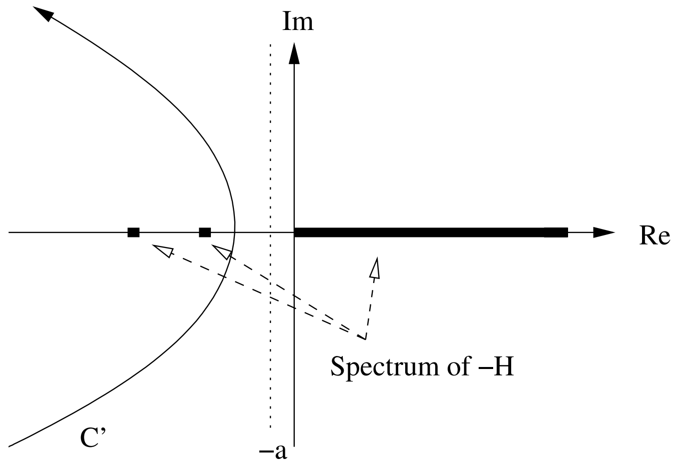

FIGURE 1. Definition of the contour C'

$(\lambda - a)/2$, in [10, Theorem 1.3.4] implies that the integral

$$\Big(\frac{1}{2\pi i}\int_{C'} e^{s(\tau-t)}(H+s)^{-1}ds\Big)$$

defines an operator-valued semigroup $e^{-H(\tau-t)}$, so the formula for Γ is given by

$$\Gamma(w)(t,x) = \int_t^0 e^{-H(\tau-t)}w(\tau,x)d\tau. \tag{4.7}$$

It remains to show that the image of Γ is in fact Y_a, as it is easy to see that its image is in L^∞. That the image is as advertised is not immediately obvious because the contour deflection $C \to C'$ changes the domain of the Laplace transform. In particular, the derivation given above is no longer valid with the new contour.

Therefore, we must estimate $\|v\|_{Z_a}$ (recall that λ is the smallest magnitude eigenvalue of $-H$)

$$\begin{aligned}
\|e^{-at}v(t,x)\|_{C^0} &= \Big\|\frac{1}{2\pi i}\int_{C'}(s+H)^{-1}\int_t^0 e^{-(s+a)(t-\tau)}e^{a\tau}w(\tau,x)d\tau\, ds\Big\|_{C^0}\\
&\le \frac{1}{2\pi}\int_{C'}\frac{K_1}{|s-\lambda|}e^{-\Re(s+a)t}\int_t^0 e^{\Re(s+a)\tau}\|w\|_{Z_a}d\tau\, ds\\
&\le \frac{K_1\|w\|_{Z_a}}{2\pi}\int_{C'}\frac{1}{|s-\lambda|}e^{-\Re(s+a)t}\frac{1}{\Re(s+a)}\big(1-e^{\Re(s+a)t}\big)ds\\
&\le \frac{K_1\|w\|_{Z_a}}{\pi}\int_{C'}\frac{ds}{|s-\lambda||\Re(s+a)|}\\
&\le K_2\|w\|_{Z_a},
\end{aligned}$$

where $0 < K_1, K_2 < \infty$ are independent of t and w. We have made use of the usual estimate of the norm of $(H+s)^{-1} : C^{0,\alpha} \to C^{0,\alpha}$ when s is in the resolvent set of $-H$. In particular, note that the choice of C' being to the left of $-a$ is crucial to the convergence of the integrals. Thus the image of Γ lies in Z_a. The backward-time decay of $\frac{\partial v}{\partial t}$ is immediate from the Harnack inequality (4.2), so in fact the image of Γ lies in Y_a.

Theorem 4.5. *The linear map $L : Y_a((-\infty, 0]) \to Z_a((-\infty, 0])$ is surjective and has a finite dimensional kernel. Therefore the set $N^{-1}(0)$ is a finite dimensional manifold, which is the unstable manifold of the equilibrium f. The dimension of $N^{-1}(0)$ is precisely the dimension of the positive eigenspace of H.*

Proof. The only thing which remains to be shown is that the domain Y_a splits into a pair of closed complementary subspaces: the kernel of L and its complement. That its complement is closed follows immediately from a standard application of the Hahn-Banach theorem. (Extend $\mathrm{id}_{\ker L}$ to all of Y_a.) □

Combining the fact that an equilibrium solution can have an empty unstable manifold (a numerical computation of the dimension of the eigenspaces of L can be found in [19]) and is yet unstable, we have proven the following result.

Theorem 4.6. *All equilbrium solutions to* (1.1) *are degenerate critical points in the sense of Morse.*

5. Linearization about heteroclinic orbits

We can extend the technique of the previous section to the linearization about a heteroclinic orbit. The resulting generalization of Theorem 4.5 is that the connecting manifolds of (1.1) are all finite dimensional.

Suppose that u is a heteroclinic orbit of (1.1). Let f_-, f_+ be the equilibrium solutions of (1.1) to which u converges as $t \to -\infty$ and $t \to +\infty$ respectively.

Suppose that $\lambda_0 : \mathbb{R} \to (0, \infty)$ is the smallest positive eigenvalue of $H(t)$. It is easy to see that λ_0 is piecewise C^1, for instance, see Proposition I.7.2 in [12]. The fact that the the spectrum of H lies entirely to the left of $\max\{2\|f_+\|_\infty, 2\|f_-\|_\infty\}$ ensures that λ_0 is a bounded function. We will define a pair of bounded, piecewise C^1 functions λ_1 and λ_2 which will aid us in defining a two more pairs of function spaces. Let $\lambda_1 : \mathbb{R} \to (0, \infty)$ be a bounded, piecewise C^1 function with bounded derivative which has the following properties:

- $\lambda_1(t)$ is never an eigenvalue of $H(t)$,
- $\lim_{t\to\infty} \frac{\lambda_1(t)}{\lambda_0(t)} < 1$,
- $\lim_{t\to-\infty} \frac{\lambda_1(t)}{\lambda_0(t)} < 1$, and
- since $u \to f_\pm$ uniformly, for a sufficiently large $R > 0$, λ_1 can be chosen so that there are no jumps on its restriction to $\mathbb{R} - [-R, R]$.

Defining λ_2 is a somewhat more delicate problem. We would like to exclude the solutions which lie in the unstable manifold of f_+, since they cannot lie in the space of heteroclines from $f_- \to f_+$. We do this by separating the eigenvalues corresponding to the intersection of the unstable manifolds of f_- and f_+ from those which lie in the stable manifold of f_+. However, there is an obstruction to this technique. In particular, the eigenvalues of $H(t) = \frac{\partial^2}{\partial x^2} - 2u(t)$ vary with time, and can bifurcate. To avoid this issue, we need some kind of regularity for the eigenvalues to prevent them from bifurcating. We follow Floer [7] in the following way:

Conjecture 5.1. There is a generic subset (a Baire subset) of choices for ϕ in (1.1) so that if u is a heteroclinic orbit, all of the eigenvalues of $H(t)$ are simple.

Numerical evidence, as exhibited in [19] and Section 6 suggests that the above Conjecture is true. When we assume that all of the eigenvalues of $H(t)$ are simple,

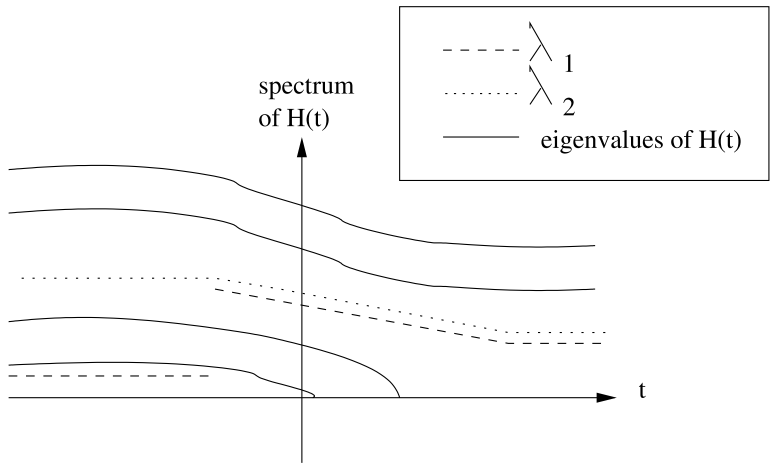

FIGURE 2. Definition of λ_1 and λ_2

and therefore do not undergo any bifurcations other than passing through zero, we shall say u is a heterocline contained in U_{reg}.

Let λ_2 be in $C^1(\mathbb{R})$ such that

- $\lambda_2 = \lambda_1$ on $[R, \infty)$, and
- $\lambda_2(t)$ is not an eigenvalue of $H(t)$ for any t.

We can do this when $u \in U_{reg}$. See Figure 2.

Definition 5.2. Define the Banach algebra $Y_{\lambda_i}(X)$ (for $i = 1, 2$) to be the set of u in $C^1(X, C^{0,\alpha}(\mathbb{R}))$ such that the norm

$$\left\| e^{-\int_0^t \lambda_i(\tau) d\tau} \|u(t)\|_{C^{0,\alpha}} \right\|_{C^1} < \infty,$$

where X is an interval containing zero. Likewise, we can define the spaces $Z_{\lambda_i}(X) \subset C^0(X, C^{0,\alpha}(\mathbb{R}))$ in a similar way. That these are Banach spaces follows from the boundedness of the λ_i. It is also elementary to see that these are Banach algebras.

We then consider N_i, L_i as $Y_{\lambda_i}(\mathbb{R}) \to Z_{\lambda_i}(\mathbb{R})$, where L_i is the linearization of N_i about u for $i = 1, 2$. (Again, since Y_{λ_i} and Z_{λ_i} are Banach algebras, L_i is the Frechét derivative of N_i.) For a $i \in \{1, 2\}$, consider the restriction L_i^- of L_i to a map $Y_{\lambda_i}((-\infty, 0]) \to Z_{\lambda_i}((-\infty, 0])$. We rewrite

$$L_i^- = \Big(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + 2f_-\Big) + (2f_- - 2u). \tag{5.1}$$

Likewise, we can define $L_i^+ : Y_{\lambda_i}([0, \infty)) \to Z_{\lambda_i}([0, \infty))$.

We define the positive eigenspaces V^+ for the equilibria as well

$$V^+(f_\pm) = \operatorname{span}\big\{v \in C^{0,\alpha}(\mathbb{R}) : \text{ there is a } \lambda > 0 \text{ with } \big(\frac{\partial^2}{\partial x^2} - 2f_\pm\big)v = \lambda v\big\}. \tag{5.2}$$

Note in particular that $\dim V^+(f_\pm) < \infty$.

Lemma 5.3. *If $u \in U_{reg}$ is a heterocline that converges to $f_\pm$ as $t \to \pm\infty$, then the operator L_i has a finite dimensional kernel for $i \in \{1, 2\}$, and in particular*

$$\lim_{t \to -\infty} \dim V^+(u(t)) - \lim_{t \to +\infty} \dim V^+(u(t)) \le \dim \ker L_i \le \dim \ker L_i^- < \infty.$$

(The condition $u \in U_{reg}$ is only necessary for the $i = 2$ case.)

Proof. Notice that the first term of (5.1) has finite dimensional kernel by Lemma 4.4 and closed image by Theorem 4.5. The second term of (5.1) is a compact operator since $u \to f_-$ uniformly. Thus L_i^- has a finite dimensional kernel. Let $\operatorname{span}\{v_m\}_{m=1}^M = \ker L_i^-$ and consider the set of Cauchy problems

$$\begin{gathered} \frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} - 2uh \quad \text{for } t > 0 \\ h(0,x) = v_m(0,x). \end{gathered} \tag{5.3}$$

Standard parabolic theory gives uniqueness of solutions to (5.3), and that a solution h lies in the kernel of L_i^+, the restriction of L_i to $[0,\infty)\times\mathbb{R}$. Therefore $\dim\ker L_i \leq \dim\ker L_i^- < \infty$.

For the other inequality, modify u outside of $[-R,R]\times\mathbb{R}$ to get a $\bar{u}$ so that the linearization $\overline{L_i}$ of N about $\bar{u}$ satisfies

- $\ker\overline{L_i}$ is isomorphic to $\ker L_i$ as vector spaces,
- $\bar{u}|_{(-\infty,-R)\times\mathbb{R}} = f_-$, and
- $\bar{u}|_{(R,\infty)\times\mathbb{R}} = f_+$.

We can do this for a sufficiently large R, since u tends uniformly to equilibria. Then the flow of

$$\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} + 2\bar{u}h$$

defines an injective linear map from the timeslice at $-R$ to the timeslice at R. (That is, it gives an injective map from $C^{0,\alpha}(\mathbb{R})$ to itself – injectivity being an expression of the uniqueness of solutions.) Each element v of the kernel of $\overline{L_i}$ evidently must have $v(-R) \in V^+(f_-)$ and $v(R) \notin V^+(f_+)$. Therefore, the injectivity ensures that the intersection of the image under the flow of $V^+(f_-)$ with the complement of $V^-(f_+)$ has at least dimension $\dim V^+(f_-) - \dim V^+(f_+)$. □

Remark 5.4. Multiplication by u, $C^1(\mathbb{R}^2, C^{0,\alpha}(\mathbb{R})) \to C^0(\mathbb{R}^2)$ is not a compact operator, in particular note that $\dim\ker L_i^+ = \infty$.

Theorem 5.5. *Let u be a heterocline of* (1.1) *which connects equilibria $f_\pm$. There exists a union $\cup M_u$ of finite dimensional submanifolds M_u of $C^1(\mathbb{R}, C^{0,\alpha}(\mathbb{R}))$ which*

- *contains u and*
- *consists of heteroclines connecting f_- to f_+.*

If $u \in U_{reg}$, then M_u has dimension $\lim_{t\to-\infty}\dim V^+(u(t)) - \lim_{t\to\infty}\dim V^+(u(t))$, *and this is maximal among such submanifolds M_u.*

Proof. Observe that L_1 is surjective, since it is easy to show that the formula

$$\Gamma_1(w)(t) = \int_t^0 e^{-\int_0^{T-t} H(\tau)d\tau} w(T,x)dT$$

is a well defined right inverse of L_1. This involves showing that

$$e^{-\int_0^t H(\tau)d\tau} = \frac{1}{2\pi i}\int_{C(t)} e^{st}(H(t)+s)^{-1}ds$$

converges, where we note that the contour changes with time. As it happens, the computation in [10] goes through with the only change that at $t = 0$, we deflect the contour to the right, rather than the left (as in Figure 1). Since Lemma 5.3 shows that L_1 has finite dimensional kernel, then it follows that $M_u = N_1^{-1}(0)$ is a union of finite dimensional manifolds, with a finite maximal dimension. It is obvious that M_u consists entirely of heteroclinic orbits and contains u.

It remains to show that the dimension of M_u is as advertised and maximal. Observe that L_2 is a compact perturbation of an operator $L_2' : Y_{\lambda_2}(\mathbb{R}) \to Z_{\lambda_2}(\mathbb{R})$ which is time-translation invariant. This follows from the precise choice of λ_2 being continous and not intersecting the eigenvalues of H. L_2 and L_2' are both surjective by exactly the same reasoning as for L_1. L_2' is injective by using separation of variables as in Lemma 4.4 (noting that all nontrivial solutions blow up in the Y_{λ_2} norm). Therefore the Fredholm index of L_2', hence L_2 is zero. However, this implies that L_2 is injective.

Since L_2 is bijective, any solution to $L_2 u = 0$ which decays faster than $e^{\int \lambda_2(t)dt}$ as $t \to -\infty$ ends up growing faster than $e^{\int \lambda_2(t)dt}$ as $t \to +\infty$, and in particular does not tend to zero. As a result, such a solution cannot be in $\ker L_1$. This implies that $\dim \ker L_1 \leq \lim_{t\to-\infty} \dim V^+(u(t)) - \lim_{t\to\infty} \dim V^+(u(t))$, which with the estimate in Lemma 5.3 completes the proof. □

Remark 5.6. Even if $u \notin U_{reg}$ (when there exist nonsimple eigenvalues of $H(t)$), the function λ_1 can still be constructed. As a result, we *always* get that the connecting manifold M_u is finite-dimensional.

Corollary 5.7. *The space of heteroclinic orbits has the structure of a cell complex with finite dimensional cells. This cell complex structure is evidently finite dimensional if there exist only finitely many equilibria for* (1.1).

6. An extended example

Consider the following special case of (1.1)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u^2 + (x^2 - c)e^{-x^2/2}, \tag{6.1}$$

where the choice of ϕ in (1.1) has been fixed. The bifurcation diagram for the equilibria of (6.1) can be found in Figure 3. The bifurcation diagram is parametrized by three variables: c, $f(0)$, $f'(0)$. (Since the equilibrium equation is a second-order ODE, it suffices to specify each solution by its value and first derivative at 0.) Based on the Theorem 4.5, the number of positive eigenvalues shown in Figure 3 corresponds exactly to the dimension of the unstable manifold of each equilibrium.

6.1. **Frontier of the stable manifold.** According to Figure 3, when $c = -1.2$, there is only one equilibrium, f_0. It has empty unstable manifold, though of course it is asymptotically unstable (as is shown in [17]). On the other hand, f_0 has an infinite dimensional stable manifold, which is not all of $C^{0,\alpha}(\mathbb{R})$, as a consequence of the asymptotic instability. As a result, its stable manifold has a frontier in $C^{0,\alpha}(\mathbb{R})$ (which may not be a boundary in the sense of a manifold with boundary). We are interested in the qualitative behavior of solutions near and along this frontier. We know by Lemma 6 of [16] that if they tend to f_0 uniformly on compact subsets, then they do so uniformly. It is enlightening to use a numerical procedure to this end. We start solutions at the following family of initial conditions

$$u_A(x) = f_0(x) + Ae^{-x^2/10}. \tag{6.2}$$

Using the Fujita technique (exactly as shown in [17]), we can show that for sufficiently negative A, the solution started at u_A will not be eternal. As a result, the

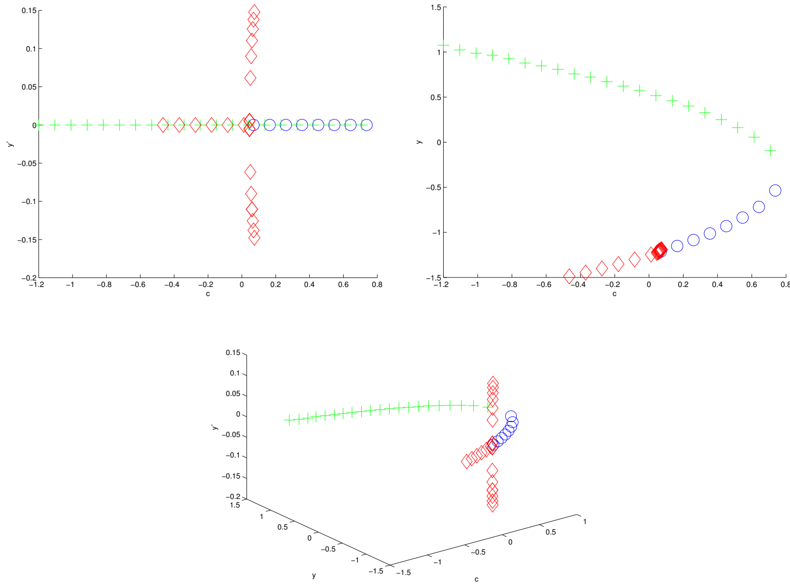

FIGURE 3. Bifurcation diagram, coded by spectrum of $\frac{d^2}{dx^2} - 2f$: green = nonpositive spectrum, blue = one positive eigenvalue, red = two positive eigenvalues

family of initial conditions u_A intersects the frontier of the stable manifold of f_0. An approximation to the value of A which corresponds to the frontier can be easily found using a binary search. Some typical such solutions are shown in Figure 4, and the approximate value of A corresponding to the frontier is $A \approx -2.15$

The qualitative behavior shown in Figure 4 indicates that there is some kind of traveling disturbance in the frontier solutions, which seems like a traveling wave. However, such a solution also appears to tend uniformly on compact subsets to f_0, so in fact it converges uniformly. (The uniform convergence is not obvious from the figure, due to the numerical solution being truncated at a finite time.) The leading edge of this disturbance collapses to $-\infty$ in finite time for solutions just outside the stable manifold of f_0.

6.2. **Flow near equilibria with two-dimensional unstable manifolds.** Also of interest is the structure of the flow in the unstable manifold of the "fork arms" which occur at $c = 0.0740$, as they approach the pitchfork bifurcation at $c = 0.0501$. Figure 5 shows a schematic of the flow based on numerical evidence. Of particular interest is the behavior near the boundary marked A. Solutions to the right of the boundary are not eternal solutions – they fail to exist for all t. Solutions to the left of A are heteroclinic orbits connecting the equilibrium with an unstable manifold of dimension 2 to the equilibrium with an unstable manifold of dimension zero. A typical such solution is shown in Figure 6.

To examine solutions near the boundary A, we center our attention on the case $c = 0$, which has two equilibria, one of which (call it f_1) has a 2-dimensional unstable manifold. (This corresponds to the right pane of Figure 5.) If we linearize about f_1, the operator $H = \frac{\partial^2}{\partial x^2} - 2f_1 : C^{0,\alpha}(\mathbb{R}) \to C^{0,\alpha}(\mathbb{R})$ has a pair of simple

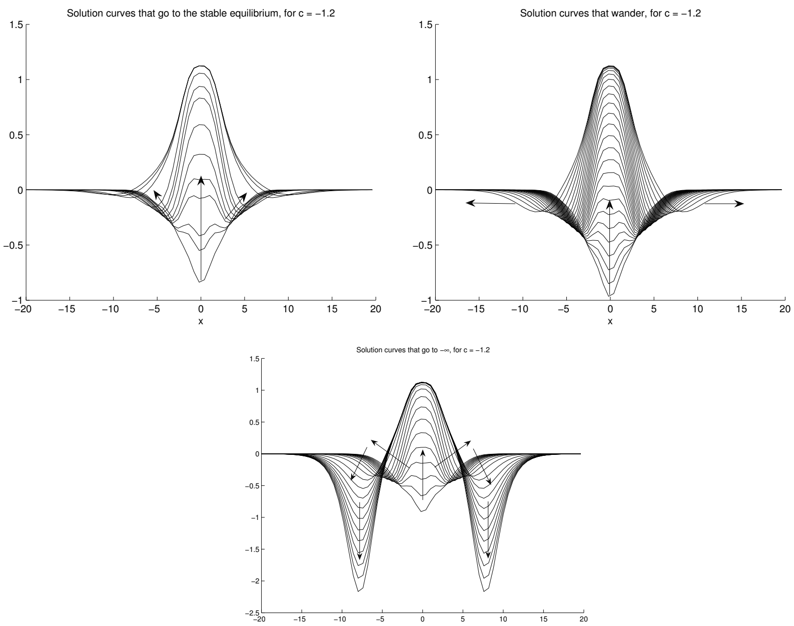

FIGURE 4. Behavior of solutions near the frontier of the stable manifold of f_0 (horizontal axis is x)

eigenvalues, as is easily seen in the right pane of Figure 6 at $t = 0$. One of these eigenvalues is smaller, to which is associated the eigenfunction e_1 in Figure 7. The eigenfunction e_2 is associated to the larger eigenvalue. In Figure 5, e_1 corresponds to the horizontal direction, and e_2 corresponds to the vertical direction. ¿From the proof of Lemma 4.4, it is clear that $\{e_1, e_2\}$ spans the tangent space of the unstable manifold at f_1. Therefore, we specify initial conditions $u_{A,\theta}(x)$ for a numerical solver using

$$u_{A,\theta}(x) = f_1(x) + A\left(e_1(x)\cos\theta + e_2(x)\sin\theta\right). \tag{6.3}$$

(Taking A small allows us to approximate solutions which tend to f_1 in backwards time.) Since the perturbations along e_1, e_2 are quite small, and indeed the eigenvalue associated to e_1 is much smaller than that associated to e_2, examining the numerical results of evolving $u_{A,\theta}$ is quite difficult. The behavior along the boundary occurs at a much smaller scale than f_1, yet is crucial in determining the long-time behavior of the solution. To remedy this, the boundary behavior is better emphasized by plotting $u_{A,\theta}(t,x) - f_1(x)$ instead. Figure 8 shows the results of evolving initial conditions (6.3) for $A = 0.1$ and various values of θ.

Solutions in Figure 8 show a similar kind of behavior as in the case of the frontier of f_0. There is a traveling front, which moves very slowly in the negative x-direction. However, the behavior is quite a bit more delicate. The determining factor in locating the frontier of f_0 is the perturbation in a direction roughly like e_2, which has a large eigenvalue. On the other hand, for f_1, Figure 5 indicates that such a direction is not parallel to the boundary of the connecting manifold. (The boundary direction is some linear combination of e_1 and e_2, with a numerical value for the angle θ being roughly 1.114975 radians.) The eigenvalue associated to e_1 is

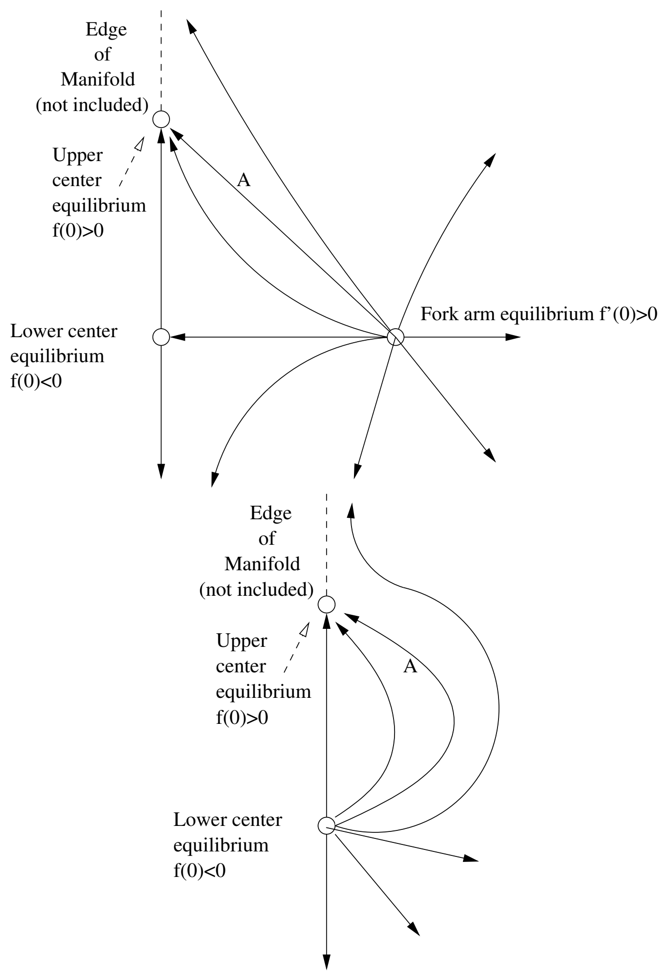

FIGURE 5. Flow in the unstable manifold of a "fork arm." $c = 0.0600$ (left); $c = 0.0501$ (right)

roughly ten times smaller, and therefore perturbations in that direction are much more sensitive. Additionally, the action of the flow is therefore primarily in the direction of e_1, which tends to mask effects in other directions. For this reason, it was visually necessary to postprocess the numerical solutions by subtracting f_1 from them. Otherwise the presence of the traveling front was unclear.

Conclusions. We have shown that the tangent space at an equilibrium splits into a finite dimensional unstable subspace, and infinite dimensional center and stable subspaces. However, it is quite clear by [17] that the center subspace is nonempty and large. Indeed, considering the work of [20], the center and stable subspaces are not closed complements of each other. Additionally, we have given conditions for the space of heteroclinic orbits to have a finite dimensional cell complex structure.

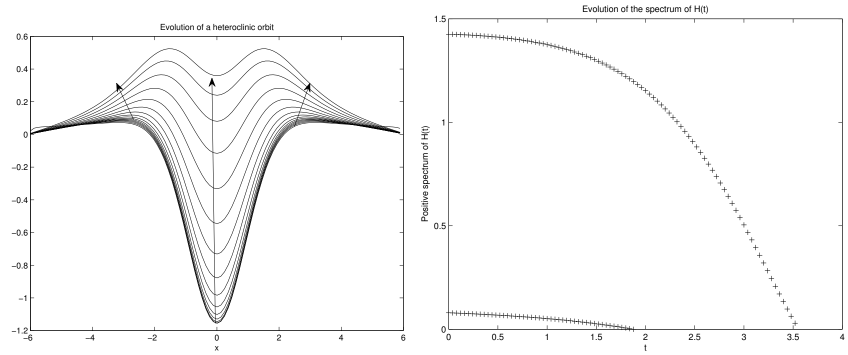

Figure 6. A typical heteroclinic orbit to the left of boundary A, with the spectrum of $H(t)$ as a function of t.

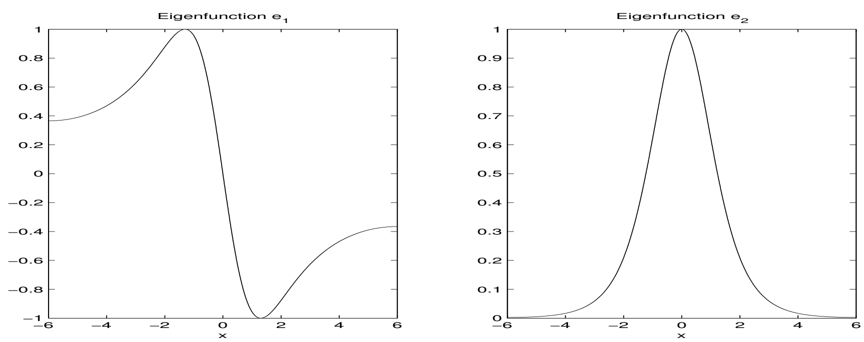

Figure 7. Eigenfunctions describing unstable directions at f_1

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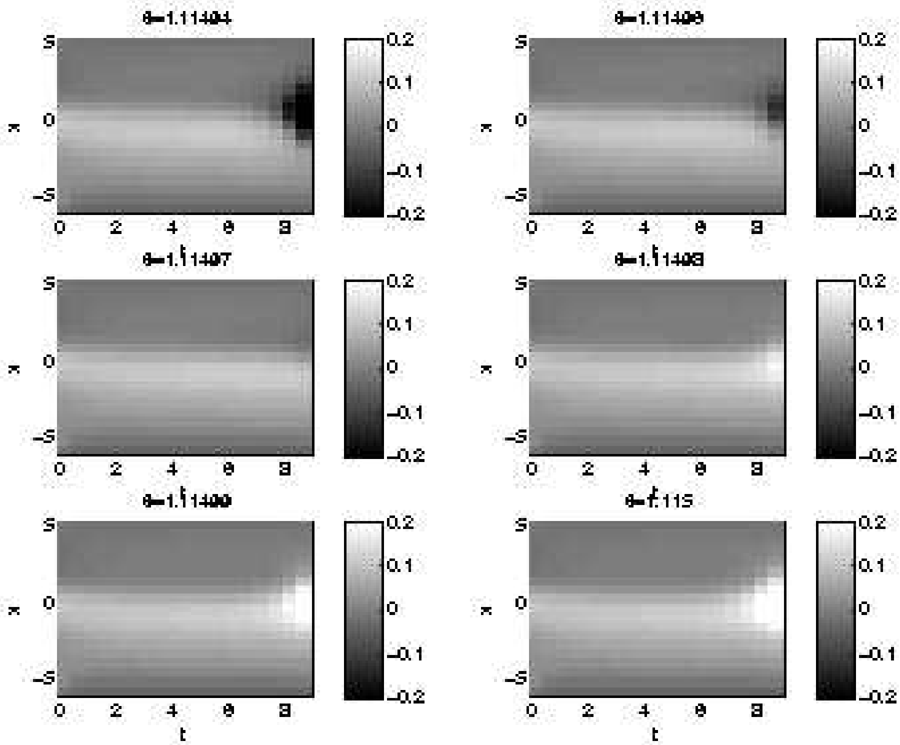

FIGURE 8. Difference between equilibrium f_1 and the numerical solution started at $u_{A,\theta}$, where black indicates a value of -0.2, and white indicates 0.2. The horizontal axis represents t, and the vertical axis represents x. $A = 0.1$ in all figures. Starting from the upper left, $\theta = 1.11494, 1.11496, 1.11497, 1.11498, 1.11499, 1.115$.

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