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A NONLINEAR WAVE EQUATION WITH A NONLINEAR INTEGRAL EQUATION INVOLVING THE BOUNDARY VALUE

THANH LONG NGUYEN, TIEN DUNG BUI

Abstract. We consider the initial-boundary value problem for the nonlinear wave equation

$$\begin{gathered}u_{tt}-u_{xx}+f(u,u_t)=0,\quad x\in\Omega=(0,1),\ 0<t<T,\\ u_x(0,t)=P(t),\quad u(1,t)=0,\\ u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\end{gathered}$$

where u_0, u_1, f are given functions, the unknown function $u(x,t)$ and the unknown boundary value $P(t)$ satisfy the nonlinear integral equation

$$P(t)=g(t)+H(u(0,t))-\int_0^t K(t-s,u(0,s))ds,$$

where g, K, H are given functions. We prove the existence and uniqueness of weak solutions to this problem, and discuss the stability of the solution with respect to the functions g, H and K. For the proof, we use the Galerkin method.

1. Introduction

In this paper we consider the problem of finding a pair of functions (u,P) that satisfy

$$u_{tt}-u_{xx}+f(u,u_t)=0,\quad x\in\Omega=(0,1),\ 0<t<T,\tag{1.1}$$

$$u_x(0,t)=P(t)\tag{1.2}$$

$$u(1,t)=0,\tag{1.3}$$

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\tag{1.4}$$

where u_0, u_1, f are given functions satisfying conditions to be specified later and the unknown function $u(x,t)$ and the unknown boundary value $P(t)$ satisfy the nonlinear integral equation

$$P(t)=g(t)+H(u(0,t))-\int_0^t K(t-s,u(0,s))ds,\tag{1.5}$$

where g, H, K are given functions. Ang and Dinh [2] established the existence of a unique global solution for the initial and boundary value problem (1.1)-(1.4)

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with u_0, u_1, P given functions and $f(u, u_t) = |u_t|^\alpha \operatorname{sign}(u_t)$, $(0 < \alpha < 1)$. As a generalization of the results in [2], Long and Dinh [7, 9, 10] have considered problem (1.1), (1.3), (1.4) associated with the following nonhomogeneous boundary condition at $x = 0$,

$$u_x(0,t) = g(t) + H(u(0,t)) - \int_0^t K(t-s, u(0,s))ds. \tag{1.6}$$

We have considered it with $K \equiv 0$, $H(s) = hs$, where $h > 0$ [9]; $K \equiv 0$ [7], $H(s) = hs$, $K(t,u) = k(t)u$, where $h > 0$, $k \in H^1(0,T)$, for all $T > 0$ [10]. In the case of $H(s) = hs$, $K(t,u) = h\omega(\sin\omega t)u$, where $h > 0$, $\omega > 0$ are given constants, the problem (1.1)-(1.5) is formed from the problem (1.1)-(1.4) wherein, the unknown function $u(x,t)$ and the unknown boundary value $P(t)$ satisfy the following Cauchy problem

$$P''(t) + \omega^2 P(t) = hu_{tt}(0,t), \quad 0 < t < T, \tag{1.7}$$

$$P(0) = P_0, \quad P'(0) = P_1, \tag{1.8}$$

where $\omega > 0$, $h \geq 0$, P_0, P_1 are given constants [10]. An and Trieu [1], studied a special case of problem (1.1)-(1.4), (1.7), (1.8) with $u_0 = u_1 = P_0 = 0$ and with $f(u,u_t)$ linear, i.e., $f(u,u_t) = Ku + \lambda u_t$ where K, λ are given constants. In the later case the problem (1.1)-(1.4), (1.7), and (1.8) is a mathematical model describing the shock of a rigid body and a linear visoelastic bar resting on a rigid base [1]. Our problem is thus a nonlinear analogue of the problem considered in [1]. In the case where $f(u,u_t) = |u_t|^\alpha \operatorname{sign}(u_t)$ the problem (1.1)-(1.4), (1.7), and (1.8) describes the shock between a solid body and a linear viscoelastic bar with nonlinear elastic constraints at the side, and constraints associated with a viscous frictional resistance. From (1.7), (1.8) we represent $P(t)$ in terms of P_0, P_1, ω, h, $u_{tt}(0,t)$ and then by integrating by parts, we have

$$P(t) = g(t) + hu(0,t) - \int_0^t k(t-s)u(0,s)ds, \tag{1.9}$$

where

$$g(t) = (P_0 - hu_0(0))\cos\omega t + (P_1 - hu_1(0))\frac{\sin\omega t}{\omega}, \tag{1.10}$$

$$k(t) = h\omega(\sin\omega t). \tag{1.11}$$

By eliminating an unknown function $P(t)$, we replace the boundary condition (1.2) by

$$u_x(0,t) = g(t) + hu(0,t) - \int_0^t k(t-s)u(0,s)ds. \tag{1.12}$$

Then, we reduce problem (1.1)-(1.4), (1.7), (1.8) to (1.1)-(1.4), (1.9)-(1.11) or to (1.1), (1.3), (1.4), (1.10)-(1.12).

In this paper, we consider two main parts. In Part 1, we prove a theorem of global existence and uniqueness of a weak solution of problem (1.1)-(1.5). The proof is based on a Galerkin method associated to a priori estimates, weak-convergence and compactness techniques. We remark that the linearization method in [6, 11, 13] cannot be used for the problems in [2, 4, 5, 7, 9, 10]. In Part 2 we prove that the solution (u,P) of this problem is stable with respect to the functions g, H and K. The results obtained here generalize the ones in [1, 2, 4, 7, 9, 10].

2. The existence and uniqueness theorem

We first set notations $\Omega = (0,1)$, $Q_T = \Omega \times (0,T)$, $T > 0$, $L^p = L^p(\Omega)$, $H^1 = H^1(\Omega)$, $H^2 = H^2(\Omega)$, where H^1, H^2 are the usual Sobolev spaces on Ω.

The norm in L^2 is denoted by $\|\cdot\|$. We also denote by $\langle\cdot,\cdot\rangle$ the scalar product in L^2 or pair of dual scalar product of continuous linear functional with an element of a function space. We denote by $\|\cdot\|_X$ the norm of a Banach space X and by X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \le p \le \infty$ for the Banach space of the real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \Big(\int_0^T \|u(t)\|_X^p dt\Big)^{1/p} \quad \text{for } 1 \le p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X \quad \text{for } p = \infty.$$

We put

$$V = \{v \in H^1 : v(1) = 0\}, \quad a(u,v) = \langle \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\rangle = \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} dx.$$

Here V is a closed subspace of $H^1$and on V, $\|v\|_{H^1}$ and $\|v\|_V = \sqrt{a(v,v)}$ are two equivalent norms.

Lemma 2.1. *The imbedding $V \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \le \|v\|_V \tag{2.1}$$

for all $v \in V$.

The proof is straightforward and we omit it. We make the following assumptions:

(A) $u_0 \in H^1$ and $u_1 \in L^2$

(G) $g \in H^1(0,T)$ for all $T > 0$

(H) $H \in C^1(\mathbb{R})$, $H(0) = 0$ and there exists a constant $h_0 > 0$ such that

$$\widehat{H}(y) = \int_0^y H(s)ds \ge -h_0$$

(K1) K and $\frac{\partial K}{\partial t}$ are in $C^0(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$

(K2) There exist the nonnegative functions $k_1 \in L^2(0,T)$, $k_2 \in L^1(0,T)$, $k_3 \in L^2(0,T)$, and $k_4 \in L^1(0,T)$, such that
  (i) $|K(t,u)| \le k_1(t)|u| + k_2(t)$,
  (ii) $|\frac{\partial K}{\partial t}(t,u)| \le k_3(t)|u| + k_4(t)$.

The function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies $f(0,0) = 0$ and the following conditions:

(F1)

$$(f(u,v) - f(u,\widetilde{v}))(v - \widetilde{v}) \ge 0 \quad \text{for all } u, v, \widetilde{v} \in \mathbb{R}$$

(F2) There is a constant α in $(0,1]$ and a function $B_1 : \mathbb{R}_+ \to \mathbb{R}_+$ continuous and satisfying

$$|f(u,v) - f(u,\widetilde{v})| \le B_1(|u|)|v - \widetilde{v}|^\alpha \quad \text{for all } u, v, \widetilde{v} \in \mathbb{R}$$

(F3) There is a constant β in $(0,1]$ and a function $B_2 : \mathbb{R}_+ \to \mathbb{R}_+$ continuous and satisfying

$$|f(u,v) - f(\widetilde{u},v)| \le B_2(|v|)|u - \widetilde{u}|^\beta \quad \text{for all } u, \widetilde{u}, v \in \mathbb{R}$$

We will use the notation $u' = u_t = \partial u/\partial t$, $u'' = u_{tt} = \partial^2 u/\partial t^2$. Then we have the following theorem.

Theorem 2.2. *Let (A),(G),(H),(K1),(K2), (F1), (F3) hold. Then, for every* $T > 0$*, there exists a weak solution* (u, P) *to problem* (1.1)-(1.5) *such that*

$$u \in L^\infty(0,T;V), \quad u_t \in L^\infty(0,T;L^2), \quad u(0,\cdot) \in H^1(0,T), \tag{2.2}$$

$$P \in H^1(0,T). \tag{2.3}$$

Furthermore, if $\beta = 1$ *in (F3) and the functions* H, K, f *satisfying, in addition*

- (H1) $H \in C^2(\mathbb{R})$, $H'(s) > -1$ *for all* $s \in \mathbb{R}$
- (K3) *For all* M *positive and all* T *positive, there exists* $p_{M,T}$, $q_{M,T}$ *in* $L^2(0,T)$, $p_{M,T}(t) \geq 0$, $q_{M,T}(t) \geq 0$ *such that*
  - (i) $|K(t,u) - K(t,v)| \leq p_{M,T}(t)|u - v|$ *for all* u, v *in* $\mathbb{R}$, $|u|, |v| \leq M$,
  - (ii) $|\frac{\partial K}{\partial t}(t,u) - \frac{\partial K}{\partial t}(t,v)| \leq q_{M,T}(t)|u - v|$ *for all* u, v *in* $\mathbb{R}$, $|u|, |v| \leq M$.
- (F4) $B_2(|v|) \in L^2(Q_T)$ *for all* $v \in L^2(Q_T)$ *for all* $T > 0$.

Then the solution is unique

Remark 2.3. This result is stronger than that in [9]. Indeed, corresponding to the same problem (1.1)-(1.5) with $K(t,u) \equiv 0$ and $H(s) = hs$, $h > 0$ the following assumptions made in [9] are not needed here: $0 < \alpha < 1$, $B_1(|u|) \in L^{2/(1-\alpha)}(Q_T)$ for all $u \in L^\infty(0,T;V)$ and all $T > 0$; B_1, B_2 are nondecreasing functions.

Proof of Theorem 2.2. It is done in several steps.
Step 1. The Galerkin approximation. Consider the orthonormal basis on V consisting of eigenvectors of the Laplacian, $-\partial^2/\partial x^2$,

$$w_j(x) = \sqrt{2/(1+\lambda_j^2)}\cos(\lambda_j x), \quad \lambda_j = (2j-1)\frac{\pi}{2}, \quad j = 1, 2, \dots.$$

Put

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t) w_j,$$

where $c_{mj}(t)$ satisfy the system of nonlinear differential equations

$$\langle u_m''(t), w_j \rangle + a(u_m(t), w_j) + P_m(t) w_j(0) + \langle f(u_m(t), u_m'(t)), w_j \rangle = 0, \tag{2.4}$$

$$P_m(t) = g(t) + H(u_m(0,t)) - \int_0^t K(t-s, u_m(0,s))ds, \tag{2.5}$$

with

$$\begin{gathered} u_m(0) = u_{0m} = \sum_{j=1}^{m} \alpha_{mj} w_j \to u_0 \quad \text{strongly in } H^1, \\ u_m'(0) = u_{1m} = \sum_{j=1}^{m} \beta_{mj} w_j \to u_1 \quad \text{strongly in } L^2, \end{gathered} \tag{2.6}$$

This system of equations is rewritten in form

$$c_{mj}''(t) + \lambda_j^2 c_{mj}(t) = \frac{-1}{\|w_j\|^2}(P_m(t) w_j(0) + \langle f(u_m(t), u_m'(t)), w_j \rangle),$$

$$P_m(t) = g(t) + H(u_m(0,t)) - \int_0^t K(t-s, u_m(0,s))ds,$$

$$c_{mj}(0) = \alpha_{mj}, \quad c_{mj}'(0) = \beta_{mj}, \quad 1 \leq j \leq m.$$

This system is equivalent to the system of integrodifferential equations

$$\begin{aligned} &c_{mj}(t)\\ &= G_{mj}(t) - \frac{1}{\|w_j\|^2}\int_0^t N_j(t-\tau)(H(u_m(0,\tau))w_j(0) + \langle f(u_m(\tau),u_m'(\tau)),w_j\rangle)d\tau\\ &\quad + \frac{w_j(0)}{\|w_j\|^2}\int_0^t N_j(t-\tau)d\tau\int_0^\tau K(\tau-s,u_m(0,s))ds, \quad 1\le j\le m, \end{aligned} \tag{2.7}$$

where $N_j(t) = \sin(\lambda_j t)/\lambda_j$ and

$$G_{mj}(t) = \alpha_{mj}N_j'(t) + \beta_{mj}N_j(t) - \frac{w_j(0)}{\|w_j\|^2}\int_0^t N_j(t-\tau)g(\tau)d\tau. \tag{2.8}$$

We then have the following lemma.

Lemma 2.4. *Let (A), (G), (H), (K1), (K2), (F1),(F3) hold. For fixed $T>0$, the system* (1.10)-(1.11) *has solution $c_m = (c_{m1}, c_{m2}, \dots, c_{mm})$ on an interval $[0,T_m]\subset[0,T)$.*

Proof. Omitting the index m, system (2.7), (2.8) is rewritten in the form

$$c = Uc,$$

where $c = (c_1, c_2, \dots, c_m)$, $Uc = ((Uc)_1, (Uc)_2, \dots, (Uc)_m)$,

$$(Uc)_j(t) = G_j(t) + \int_0^t N_j(t-\tau)(Vc)_j(\tau)d\tau, \tag{2.9}$$

$$(Vc)_j(t) = f_{1j}(c(t), c'(t)) + \int_0^t f_{2j}(t-s, c(s))ds, \tag{2.10}$$

$$G_j(t) = \alpha_{mj}N_j'(t) + \beta_{mj}N_j(t) - \frac{w_j(0)}{\|w_j\|^2}\int_0^t N_j(t-\tau)g(\tau)d\tau, \tag{2.11}$$

the functions $f_{1j}:\mathbb{R}^{2m}\to\mathbb{R}$ $f_{2j}:[0,T_m]\times\mathbb{R}^m\to\mathbb{R}$ satisfy

$$f_{1j}(c,d) = \frac{-1}{\|w_j\|^2}\big[H(\sum_{i=1}^m c_iw_i(0))w_j(0) + \langle f(\sum_{i=1}^m c_iw_i, \sum_{i=1}^m d_iw_i), w_j\rangle\big], \tag{2.12}$$

$$f_{2j}(t,c) = \frac{w_j(0)}{\|w_j\|^2}K(t, \sum_{i=1}^m c_iw_i(0)), \quad 1\le j\le m. \tag{2.13}$$

For every $T_m>0$, $M>0$ we put

$$S = \{c\in C^1([0,T_m];\mathbb{R}^m) : \|c\|_1\le M\}, \quad \|c\|_1 = \|c\|_0 + \|c'\|_0,$$

$$\|c\|_0 = \sup_{0\le t\le T_m}|c(t)|_1, \quad |c(t)|_1 = \sum_{i=1}^m |c_i(t)|.$$

Clearly S is a closed convex and bounded subset of $Y = C^1([0,T_m];\mathbb{R}^m)$. Using the Schauder fixed point theorem we shall show that the operator $U: S\to Y$ defined by (2.9)-(2.13) has a fixed point. This fixed point is the solution of (2.7).

(a) First we show that U maps S into itself. Note that $(Vc)_j\in C^0([0,T_m];\mathbb{R})$ for all $c\in C^1([0,T_m];\mathbb{R}^m)$, hence it follows from (2.9), and the equality

$$(Uc)_j'(t) = G_j'(t) + \int_0^t N_j'(t-\tau)(Vc)_j(\tau)d\tau, \tag{2.14}$$

that $U : Y \to Y$. Let $c \in S$, we deduce from (2.8), (2.13) that

$$|(Uc)(t)|_1 \le |G(t)|_1 + \frac{1}{\lambda_1} T_m \|Vc\|_0, \tag{2.15}$$

$$|(Uc)'(t)|_1 \le |G'(t)|_1 + T_m \|Vc\|_0. \tag{2.16}$$

On the other hand, it follows from (H), (K1), (K2),(F2),(F3), (2.10), (2.12), (2.13) that

$$\|Vc\|_0 \le \sum_{j=1}^{m} [N_1(f_{1j}, M) + TN_2(f_{2j}, M, T)] \equiv \beta(M, T) \quad \text{for all } c \in S, \tag{2.17}$$

where

$$\begin{gathered} N_1(f_{1j}, M) = \sup\{|f_{1j}(y, z)| : \|y\|_{\mathbb{R}^m} \le M, \quad \|z\|_{\mathbb{R}^m} \le M\}, \\ N_2(f_{2j}, M, T) = \sup\{|f_{2j}(t, y)| : 0 \le t \le T, \quad \|y\|_{\mathbb{R}^m} \le M\}. \end{gathered} \tag{2.18}$$

Hence, from (2.15)-(2.18) we obtain

$$\|Uc\|_1 \le \|G\|_{1T} + (1 + \frac{1}{\lambda_1}) T_m \beta(M, T),$$

where

$$\|G\|_{1T} = \|G\|_{0T} + \|G'\|_{0T} = \sup_{0 \le t \le T} |G(t)|_1 + \sup_{0 \le t \le T} |G'(t)|_1.$$

Choosing M and $T_m > 0$ such that

$$M > 2\|G\|_{1T} \quad \text{and} \quad (1 + \frac{1}{\lambda_1}) T_m \beta(M, T) \le M/2.$$

Hence, $\|Uc\|_1 \le M$ for all $c \in S$, that is, the operator U maps S the set into itself.
(b) Now we show that the operator U is continuous on S. Let $c, d \in S$, we have

$$(Uc)_j(t) - (Ud)_j(t) = \int_0^t N_j(t - \tau)[(Vc)_j(\tau) - (Vd)_j(\tau)] d\tau.$$

Hence

$$\|Uc - Ud\|_0 \le \frac{1}{\lambda_1} T_m \|Vc - Vd\|_0. \tag{2.19}$$

Similarly, we obtain from the equality

$$(Uc)_j'(t) - (Ud)_j'(t) = \int_0^t N_j'(t - \tau)((Vc)_j(\tau) - (Vd)_j(\tau)) d\tau,$$

which implies

$$\|(Uc)' - (Ud)'\|_0 \le T_m \|Vc - Vd\|_0. \tag{2.20}$$

By estimates (2.19), (2.20), we only have to prove that the operator $V : Y \to C^0([0, T_m]; \mathbb{R}^m)$ is continuous on S. We have

$$\begin{aligned} (Vc)_j(t) - (Vd)_j(t) &= f_{1j}(c(t), c'(t)) - f_{1j}(d(t), d'(t)) \\ &\quad + \int_0^t (f_{2j}(t - s, c(s)) - f_{2j}(t - s, d(s))) ds. \end{aligned} \tag{2.21}$$

From the assumptions (H),(F2) and (F3), it follows that there exists a constant $K_M > 0$ such that

$$\sup_{0 \le t \le T_m} \sum_{j=1}^{m} |f_{1j}(c(t), c'(t)) - f_{1j}(d(t), d'(t))| \le K_M (\|c - d\|_0 + \|c - d\|_0^{\beta} + \|c' - d'\|_0^{\alpha}), \tag{2.22}$$

for all $c, d \in S$. Then we have the following lemma.

Lemma 2.5. *Let $f_{2j} : [0, T_m] \times \mathbb{R}^m \to R$ be continuous, and let*
$$(W_j c)(t) = \int_0^t f_{2j}(t-s, c(s))ds, c \in C^0([0, T_m]; \mathbb{R}^m). \tag{2.23}$$
Then, the operator $W_j : C^0([0, T_m]; \mathbb{R}^m) \to C^0([0, T_m]; \mathbb{R})$ is continuous on S.

The proof of this lemma follows easily from f_{2j} being uniformly continuous on $[0, T_m] \times [-M, M]^m$. We omit the proof.

From (2.21), (2.22), (2.23), we deduce that
$$\begin{aligned} \|Vc - Vd\|_0 &= \sup_{0 \le \tau \le T_m} \sum_{j=1}^m |(Vc)_j(\tau) - (Vd)_j(\tau)| \\ &\le K_M \big(\|c-d\|_0 + \|c-d\|_0^\beta + \|c' - d'\|_0^\alpha \big) \\ &\quad + \sup_{0 \le t \le T_m} \sum_{j=1}^m |(W_j c)(t) - (W_j d)(t)|, \quad \forall c, d \in S\,. \end{aligned} \tag{2.24}$$
Thus, Lemma 2.5 and inequality (2.24) show that $V : S \to C^0([0, T_m]; \mathbb{R}^m)$ is continuous.

(c) Now, we shall show that the set $\overline{US}$ is a compact subset of Y. Let $c \in S, t, t' \in [0, T_m]$. From (2.9), we rewrite
$$\begin{aligned} &(Uc)_j(t) - (Uc)_j(t') \\ &= G_j(t) - G_j(t') + \int_0^t N_j(t-\tau)(Vc)_j(\tau)d\tau - \int_0^{t'} N_j(t'-\tau)(Vc)_j(\tau)d\tau \\ &= G_j(t) - G_j(t') + \int_0^t (N_j(t-\tau) - N_j(t'-\tau))(Vc)_j(\tau)d\tau \\ &\quad - \int_t^{t'} N_j(t'-\tau)(Vc)_j(\tau)d\tau\,. \end{aligned} \tag{2.25}$$
From the inequality $|N_j(t) - N_j(s)| \le |t-s|$ for all $t, s \in [0, T_m]$ and (2.17), we obtain
$$\begin{aligned} |(Uc)(t) - (Uc)(t')|_1 &= \sum_{j=1}^m |(Uc)_j(t) - (Uc)_j(t')| \\ &\le |G(t) - G(t')|_1 + (T_m + \frac{1}{\lambda_1})|t-t'|\|Vc\|_0 \\ &\le |G(t) - G(t')|_1 + \beta(M,T)(T_m + \frac{1}{\lambda_1})|t-t'|. \end{aligned} \tag{2.26}$$
Similarly, from (2.14) and (2.17), we also obtain
$$|(Uc)'(t) - (Uc)'(t')|_1 \le |G'(t) - G'(t')|_1 + \beta(M,T)(\lambda_m T_m + 1)|t-t'|. \tag{2.27}$$
Since $US \subset S$, from estimates (2.26), (2.27) we deduce that the family of functions $US = \{Uc, c \in S\}$, are bounded and equicontinuous with respect to the norm $\|\cdot\|_1$ of the space Y. Applying Arzela-Ascoli's theorem to the space Y, we deduce that $\overline{US}$ is compact in Y. By the Schauder fixed-point theorem, U has a fixed point $c \in S$, which satisfies (2.7). The proof of Lemma 2.4 is complete. □

Using Lemma 2.4, for $T > 0$, fixed, system (2.4) - (2.6) has solution $(u_m(t), P_m(t))$ on an interval $[0, T_m]$. The following estimates allow one to take $T_m = T$ for all m.

Step 2. A priori estimates. Substituting (2.5) into (2.4), then multiplying the j^{th} equation of (2.4) by $c'_{mj}(t)$ and summing up with respect to j, integrating by parts with respect to the time variable from 0 to t, by (G) and (F1), we have

$$\begin{aligned} S_m(t) \leq & -2\widehat{H}(u_m(0,t)) + 2\widehat{H}(u_{0m}(0)) + S_m(0) + 2g(0)u_{0m}(0) \\ & - 2g(t)u_m(0,t) + 2\int_0^t g'(s)u_m(0,s)ds - 2\int_0^t \langle f(u_m(s),0), u'_m(s)\rangle ds \\ & + 2\int_0^t u'_m(0,s)ds \int_0^s K(s-\tau, u_m(0,\tau))d\tau, \end{aligned} \tag{2.28}$$

where

$$S_m(t) = \|u'_m(t)\|^2 + \|u_m(t)\|_V^2 \,. \tag{2.29}$$

Then, using (2.6), (2.29), (H), and Lemma 2.1, we have

$$\begin{aligned} & - 2\widehat{H}(u_m(0,t)) + 2\widehat{H}(u_{0m}(0)) + S_m(0) + 2|g(0)u_{0m}(0)| \\ & \leq 2h_0 + 2\widehat{H}(u_{0m}(0)) + S_m(0) + 2|g(0)u_{0m}(0)| \\ & \leq \frac{1}{4}C_1, \quad \text{for all } m \text{ and all } t, \end{aligned} \tag{2.30}$$

where C_1 is a constant depending only on u_0, u_1, h_0, H, and g.

Again using Lemma 2.1 and the inequality $2ab \leq 4a^2 + \frac{1}{4}b^2$, we obtain

$$\begin{aligned} & | - 2g(t)u_m(0,t) + 2\int_0^t g'(s)u_m(0,s)ds| \\ & \leq 4g^2(t) + 4\int_0^t |g'(s)|^2 ds + \frac{1}{4}S_m(t) + \frac{1}{4}\int_0^t S_m(s)ds. \end{aligned} \tag{2.31}$$

Using Lemma 2.1, from (F3) it follows that

$$\begin{aligned} \big| - 2\int_0^t \langle f(u_m(s),0), u'_m(s)\rangle ds\big| & \leq 2B_2(0)\int_0^t S_m(s)^{(1+\beta)/2}ds \\ & \leq (1+\beta)B_2(0)\int_0^t S_m(s)ds + (1-\beta)B_2(0)t. \end{aligned}$$

Note that the last integral in (2.28), after integrating by parts, gives

$$\begin{aligned} I &= 2\int_0^t u'_m(0,s)ds \int_0^s K(s-\tau, u_m(0,\tau))d\tau \\ &= 2u_m(0,t)\int_0^t K(t-\tau, u_m(0,\tau))d\tau \\ &\quad - 2\int_0^t u_m(0,s)ds\big[K(0,u_m(0,s)) + \int_0^s \frac{\partial K}{\partial t}(s-\tau, u_m(0,\tau))d\tau\big]. \end{aligned}$$

Hence

$$\begin{aligned}
|I| &\le 2\sqrt{S_m(t)}\int_0^t (k_1(t-\tau)\sqrt{S_m(\tau)}+k_2(t-\tau))d\tau \\
&\quad + 2\int_0^t \sqrt{S_m(s)}ds\big[k_1(0)\sqrt{S_m(s)}+k_2(0) \\
&\quad + \int_0^s (k_3(s-\tau)\sqrt{S_m(\tau)}+k_4(s-\tau))d\tau\big] \\
&= 2\sqrt{S_m(t)}\int_0^t k_1(t-\tau)\sqrt{S_m(\tau)}d\tau + 2\sqrt{S_m(t)}\int_0^t k_2(\tau)d\tau \\
&\quad + 2k_1(0)\int_0^t S_m(s)ds + 2k_2(0)\int_0^t \sqrt{S_m(s)}ds \\
&\quad + 2\int_0^t \sqrt{S_m(s)}ds\int_0^s k_3(s-\tau)\sqrt{S_m(\tau)}d\tau + 2\int_0^t \sqrt{S_m(s)}ds\int_0^s k_4(\tau)d\tau \\
&\equiv I_1 + I_2 + 2k_1(0)\int_0^t S_m(s)ds + I_4 + I_5 + I_6.
\end{aligned} \tag{2.32}$$

By the inequality $2ab \le 4a^2 + \frac{1}{4}b^2$ and the Cauchy- Schwarz inequality we estimate without difficulty the following integrals in the right-hand side of the above expression as follows

$$I_1 = 2\sqrt{S_m(t)}\int_0^t k_1(t-\tau)\sqrt{S_m(\tau)}d\tau \le \frac{1}{4}S_m(t) + 4\int_0^t k_1^2(\tau)d\tau.\int_0^t S_m(\tau)d\tau,$$

$$I_2 = 2\sqrt{S_m(t)}\int_0^t k_2(\tau) \le \frac{1}{4}S_m(t) + 4\Big(\int_0^t k_2(\tau)d\tau\Big)^2,$$

$$I_4 = 2k_2(0)\int_0^t \sqrt{S_m(s)}ds \le 4k_2^2(0) + \frac{1}{4}t\int_0^t S_m(s)ds,$$

$$I_5 = 2\int_0^t \sqrt{S_m(s)}ds\int_0^s k_3(s-\tau)\sqrt{S_m(\tau)}d\tau \le 2\sqrt{t}\Big(\int_0^t k_3^2(\tau)d\tau\Big)^{1/2}\int_0^t S_m(s)ds,$$

$$I_6 = 2\int_0^t \sqrt{S_m(s)}ds\int_0^s k_4(\tau)d\tau \le \frac{1}{4}\int_0^t S_m(s)ds + 4t\Big(\int_0^t k_4(\tau)d\tau\Big)^2.$$

It follows from the estimates for I_1, I_2, I_4, I_5, I_6 that

$$\begin{aligned}
|I| &\le 4\Big(\int_0^t k_2(\tau)d\tau\Big)^2 + 4k_2^2(0) + 4t\Big(\int_0^t k_4(\tau)d\tau\Big)^2 + \frac{1}{2}S_m(t) \\
&\quad + \frac{1}{4}\Big[1 + t + 16\int_0^t k_1^2(\tau)d\tau + 8k_1(0) + 8\sqrt{t}\Big(\int_0^t k_3^2(\tau)d\tau\Big)^{1/2}\Big]\int_0^t S_m(s)ds.
\end{aligned} \tag{2.33}$$

It follows from (2.28)-(2.30), (2.31)-(2.32), and (2.33) that

$$S_m(t) \le D_1(t) + D_2(t)\int_0^t S_m(\tau)d\tau, \tag{2.34}$$

where

$$\begin{aligned}
D_1(t) &= C_1 + 16k_2^2(0) + 4(1-\beta)B_2(0)t + 16g^2(t) \\
&\quad + 16\int_0^t |g'(s)|^2ds + 16\Big(\int_0^t k_2(\tau)d\tau\Big)^2 + 16t\Big(\int_0^t k_4(\tau)d\tau\Big)^2,
\end{aligned} \tag{2.35}$$

$$\begin{aligned} D_2(t) &= 2+4(1+\beta)B_2(0)+8k_1(0)+t+\int_0^t k_1^2(\tau)d\tau+8\sqrt{t}\Big(\int_0^t k_3^2(\tau)d\tau\Big)^{1/2} \\ &\le 2+4(1+\beta)B_2(0)+8k_1(0)+T+\|k_1\|^2_{L^2(0,T)}+8\sqrt{T}\|k_3\|_{L^2(0,T)} \equiv C_T^{(2)}. \end{aligned}$$

Since $H^1(0,T) \hookrightarrow C^0([0,T])$, from the assumptions (G), (K2), we deduce that

$$|D_1(t)| \le C_T^{(1)}, \quad \text{a.e. in } [0,T], \tag{2.36}$$

where $C_T^{(1)}$, is a constant depending only on T. By Gronwall's lemma, from (2.34)-(2.36) we obtain that

$$S_m(t) \le C_T^{(1)} \exp(tC_T^{(2)}) \le C_T \quad \forall t \in [0,T],\ \forall T > 0. \tag{2.37}$$

Now we need an estimate on the integral $\int_0^t |u_m'(0,s)|^2 ds$. Put

$$K_m(t) = \sum_{j=1}^m \frac{\sin(\lambda_j t)}{\lambda_j}, \tag{2.38}$$

$$\begin{aligned} \gamma_m(t) &= \sum_{j=1}^m w_j(0)[\alpha_{mj}\cos(\lambda_j t)+\beta_{mj}\frac{\sin(\lambda_j t)}{\lambda_j}] \\ &\quad - \sqrt{2}\sum_{j=1}^m \int_0^t \frac{\sin[\lambda_j(t-\tau)]}{\lambda_j}\langle f(u_m(\tau),u_m'(\tau)), \frac{w_j}{\|w_j\|}\rangle d\tau. \end{aligned}$$

Then $u_m(0,t)$ can be rewritten as

$$u_m(0,t) = \gamma_m(t) - 2\int_0^t K_m(t-\tau)P_m(\tau)d\tau. \tag{2.39}$$

We shall require the following lemma which proof can be found in [2].

Lemma 2.6. *There exist a constant $C_2 > 0$ and a positive continuous function $D(t)$ independent of m such that*

$$\int_0^t |\gamma_m'(\tau)|^2 d\tau \le C_2 + D(t)\int_0^t \|f(u_m(\tau),u_m'(\tau))\|^2 d\tau \quad \forall t \in [0,T], \forall T > 0.$$

Lemma 2.7. *There exist two positive constants $C_T^{(3)}$ and $C_T^{(4)}$ depending only on T such that*

$$\int_0^t ds|\int_0^s K_m'(s-\tau)P_m(\tau)d\tau|^2 \le C_T^{(3)} + C_T^{(4)}\int_0^t ds\int_0^s |u_m'(0,\tau)|^2 d\tau, \tag{2.40}$$

for all $t \in [0,T]$ and all $T > 0$.

Proof. Integrating by parts, we have

$$\int_0^s K_m'(s-\tau)P_m(\tau)d\tau = K_m(s)P_m(0) + \int_0^t K_m(s-\tau)P_m'(\tau)d\tau,$$

then

$$\begin{aligned}
&\int_0^t ds|\int_0^s K_m'(s-\tau)P_m(\tau)d\tau|^2 \\
&\le 2P_m^2(0)\int_0^t K_m^2(s)ds + 2\int_0^t ds\int_0^s K_m^2(r)dr\int_0^s |P_m'(\tau)|^2 d\tau \\
&\le 2\int_0^t K_m^2(s)ds\big[P_m^2(0) + \int_0^t ds\int_0^s |P_m'(\tau)|^2 d\tau\big].
\end{aligned} \tag{2.41}$$

From (2.5), we have

$$P_m(0) = g(0) + H(u_{0m}(0)), \tag{2.42}$$

$$P_m'(\tau) = g'(\tau) + H'(u_m(0,\tau))u_m'(0,\tau) - K(0,u_m(0,\tau)) - \int_0^\tau \frac{\partial K}{\partial t}(\tau - s, u_m(0,s))ds. \tag{2.43}$$

Using the inequality $(a+b+c+d)^2 \le 4(a^2+b^2+c^2+d^2)$, for all $a,b,c,d \in \mathbb{R}$, we deduce from (2.37), (2.43), and (G),(H),(K2) that

$$\begin{aligned}
&\int_0^s |P_m'(\tau)|^2 d\tau \\
&\le 4\int_0^s |g'(\tau)|^2 d\tau + 4\max_{|s|\le\sqrt{C_T}} |H'(s)|^2 \int_0^s |u_m'(0,\tau)|^2 d\tau \\
&\quad + 4\int_0^s |K(0,u_m(0,\tau))|^2 d\tau + 4\int_0^s d\tau|\int_0^\tau \frac{\partial K}{\partial t}(\tau - s, u_m(0,s))ds|^2 \\
&\le 4\int_0^s |g'(\tau)|^2 d\tau + 4\max_{|s|\le\sqrt{C_T}} |H'(s)|^2 \int_0^s |u_m'(0,\tau)|^2 d\tau \\
&\quad + 8k_1^2(0)\int_0^s |u_m(0,\tau)|^2 d\tau + 8k_2^2(0)s \\
&\quad + 8\int_0^s d\tau\int_0^\tau k_3^2(s)ds\int_0^\tau u_m^2(0,s)ds + 8\int_0^s d\tau(\int_0^\tau k_4(s)ds)^2 \\
&\le 4\int_0^s |g'(\tau)|^2 d\tau + 8[k_1^2(0)C_T + k_2^2(0)]s + 4C_T s^2\int_0^s k_3^2(\tau)d\tau \\
&\quad + 8s(\int_0^s k_4(\tau)d\tau)^2 + 4\max_{|s|\le\sqrt{C_T}} |H'(s)|^2 \int_0^s |u_m'(0,\tau)|^2 d\tau.
\end{aligned} \tag{2.44}$$

Hence

$$\begin{aligned}
\int_0^t ds\int_0^s |P_m'(\tau)|^2 d\tau &\le 4t\int_0^t |g'(\tau)|^2 d\tau + 4[k_1^2(0)C_T + k_2^2(0)]t^2 \\
&\quad + \frac{4}{3}C_T t^3\int_0^t k_3^2(\tau)d\tau + 4t^2\Big(\int_0^t k_4(\tau)d\tau\Big)^2 \\
&\quad + 4\max_{|s|\le\sqrt{C_T}} |H'(s)|^2 \int_0^t ds\int_0^s |u_m'(0,\tau)|^2 d\tau.
\end{aligned}$$

From this inequality, (2.41), and (2.42), it follows that

$$\begin{aligned}
&\int_0^t ds|\int_0^s K_m'(s-\tau)P_m(\tau)d\tau|^2 \\
&\le 2\int_0^t K_m^2(s)ds\Big[(g(0)+H(u_{0m}(0)))^2+4t\int_0^t|g'(\tau)|^2d\tau+4[k_1^2(0)C_T+k_2^2(0)]t^2 \\
&\quad+\frac{4}{3}C_Tt^3\int_0^t k_3^2(\tau)d\tau+4t^2\Big(\int_0^t k_4(\tau)d\tau\Big)^2 \\
&\quad+4\max_{|s|\le\sqrt{C_T}}|H'(s)|^2\int_0^t ds\int_0^s|u_m'(0,\tau)|^2d\tau\Big].
\end{aligned} \tag{2.45}$$

Note that for every $T>0$, $K_m\to\widetilde{K}$, strongly in $L^2(0,T)$ as $m\to+\infty$. Using the assumptions (G), (H),(K2) and the results (2.6) and (2.45), we obtain (2.40). The proof of Lemma 2.7 is complete. □

Lemma 2.8. *There exist two positive constants $C_T^{(5)}$ and $C_T^{(6)}$ depending only on T such that*

$$\int_0^t|u_m'(0,\tau)|^2d\tau\le C_T^{(5)}\quad\forall t\in[0,T],\forall T>0. \tag{2.46}$$

$$\int_0^t|P_m'(\tau)|^2d\tau\le C_T^{(6)}\quad\forall t\in[0,T],\forall T>0. \tag{2.47}$$

Proof. Since (2.47) is a consequence of (2.44) and (2.46), we only have to prove (2.46). From (2.39), using Lemmas 2.6 and 2.7, we obtain

$$\begin{aligned}
\int_0^t|u_m'(0,s)|^2ds&\le 2\int_0^t|\gamma_m'(s)|^2ds+8\int_0^t ds|\int_0^s K_m'(s-\tau)P_m(\tau)d\tau|^2 \\
&\le 2C_2+2D(t)\int_0^t\|f(u_m(\tau),u_m'(\tau))\|d\tau \\
&\quad+8C_T^{(3)}+8C_T^{(4)}\int_0^t ds\int_0^s|u_m'(0,\tau)|^2d\tau.
\end{aligned} \tag{2.48}$$

On the other hand, from the assumptions (F2),(F3), we obtain

$$\|f(u_m(t),u_m'(t))\|^2\le 2(\max_{|s|\le\sqrt{C_T}}B_1^2(s))\|u_m'(t)\|^{2\alpha}+2B_2^2(0)\|u_m(t)\|_V^{2\beta}, \tag{2.49}$$

since $0<\alpha\le1$ we have $\|\cdot\|\le\|\cdot\|_{L^{2\alpha}}$. Hence, using (2.37) and (2.49) we have

$$\|f(u_m(t),u_m'(t))\|\le C_T^{(7)}. \tag{2.50}$$

At last from this inequality and (2.48) we obtain the inequality

$$\int_0^t|u_m'(0,s)|^2ds\le C_T^{(8)}+8C_T^{(4)}\int_0^t ds\int_0^s|u_m'(0,\tau)|^2d\tau,$$

which implies (2.46), by Gronwall's lemma. Therefore, Lemma 2.8 is proved. □

Step 3. Passing to limit. From (2.5), (2.29), (2.37), (2.46), (2.47), and (2.50), we deduce that, there exists a subsequence of sequence $\{(u_m,P_m)\}$, still denoted by

$\{(u_m, P_m)\}$, such that

$$u_m \to u \quad \text{in } L^\infty(0,T;V) \text{ weak}*, \tag{2.51}$$

$$u'_m \to u' \quad \text{in } L^\infty(0,T;L^2) \text{ weak}*, \tag{2.52}$$

$$u_m(0,t) \to u(0,t) \quad \text{in } L^\infty(0,T) \text{ weak}*, \tag{2.53}$$

$$u'_m(0,t) \to u'(0,t) \quad \text{in } L^2(0,T) \text{ weak}, \tag{2.54}$$

$$f(u_m, u'_m) \to \chi \quad \text{in } L^\infty(0,T;L^2) \text{ weak}*, \tag{2.55}$$

$$P_m \to \widehat{P} \quad \text{in } H^1(0,T) \text{ weak}, \tag{2.56}$$

By the compactness lemma of Lions (see [9]), we can deduce from (2.51)-(2.54) that there exists a subsequence still denoted by $\{u_m\}$ such that

$$u_m(0,t) \to u(0,t) \quad \text{strongly in } C^0([0,T]), \tag{2.57}$$

$$u_m \to u \quad \text{strongly in } L^2(Q_T) \text{ and a.e. } (x.t) \in Q_T. \tag{2.58}$$

By (H),(K) and using (2.5), (2.57) we obtain

$$P_m(t) \to g(t) + H(u(0,t)) - \int_0^t K(t-s, u(0,s))ds \equiv P(t) \quad \text{strongly in } C^0([0,T]). \tag{2.59}$$

From (2.56) and (2.59) we have

$$P \equiv \widehat{P} \ \text{ a.e. in } Q_T. \tag{2.60}$$

Passing to the limit in (2.4) by (2.51), (2.52), (2.59), and (2.60) we have

$$\frac{d}{dt}\langle u'(t), v\rangle + a(u(t), v) + P(t)v(0) + \langle \chi, v\rangle = 0 \quad \forall v \in V.$$

As in [9], we can prove that

$$u(0) = u_0, \quad u'(0) = u_1.$$

To prove the existence of solution u, we have to show that $\chi = f(u, u')$. We need the following lemma which proof can be found in [2].

Lemma 2.9. *Let u be the solution of the problem*

$$\begin{gathered} u_{tt} - u_{xx} + \chi = 0, \quad 0 < x < 1, \quad 0 < t < T, \\ u_x(0,t) = P(t), \quad u(1,t) = 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \\ u \in L^\infty(0,T;V), \quad u' \in L^\infty(0,T;L^2) \\ u(0,\cdot) \in H^1(0,T). \end{gathered}$$

Then

$$\frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|u(t)\|_V^2 + \int_0^t P(s)u'(0,s)ds + \int_0^t \langle \chi(s), u'(s)\rangle ds \geq \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|u_0\|_V^2 ,$$

a.e. $t \in [0,T]$. Furthermore, if $u_0 = u_1 = 0$ there is equality in the above expression.

Now, from (2.4)-(2.6) we have

$$\begin{aligned}&\int_0^t \langle f(u_m(s),u_m'(s)),u_m'(s)\rangle ds\\&=\frac{1}{2}\|u_{1m}\|^2+\frac{1}{2}\|u_{0m}\|_V^2-\frac{1}{2}\|u_m'(t)\|^2-\frac{1}{2}\|u_m(t)\|_V^2-\int_0^t P_m(s)u_m'(0,s))ds.\end{aligned} \tag{2.61}$$

By Lemma 2.9, it follows from (2.6), (2.51), (2.52), (2.54), (2.59) and (2.61), that

$$\begin{aligned}&\limsup_{m\to+\infty}\int_0^t \langle f(u_m(s),u_m'(s)),u_m'(s)\rangle ds\\&\le \frac{1}{2}\|u_1\|^2+\frac{1}{2}\|u_0\|_V^2-\frac{1}{2}\|u'(t)\|^2-\frac{1}{2}\|u(t)\|_V^2-\int_0^t P(s)u'(0,s))ds\\&\le \int_0^t\langle\chi(s),u'(s)\rangle ds,\quad \text{a.e. } t\in[0,T].\end{aligned}$$

Using the same arguments as in [9], we can show that $\chi=f(u,u')$ a.e. in Q_T. The existence of the solution is proved.

Step 4. Uniqueness of the solution. Assume now that $\beta=1$ in (F3), and that H, K, f satisfy (H1),(K3), and (F4). Let (u_1,P_1), (u_2,P_2) be two weak solutions of the problem (1.1)-(1.5). Then $u=u_1-u_2$, $P=P_1-P_2$ satisfy the problem

$$\begin{gathered}u''-u_{xx}+\chi=0,\quad 0<x<1,\quad 0<t<T,\\ u_x(0,t)=P(t),\quad u(1,t)=0,\\ u(x,0)=u'(x,0)=0,\\ \chi=f(u_1,u_1')-f(u_2,u_2'),\\ \begin{aligned}P(t)&=P_1(t)-P_2(t)\\&=H(u_1(0,t))-H(u_2(0,t))\\&\quad-\int_0^t(K(t-s,u_1(0,s))-K(t-s,u_2(0,s)))ds,\end{aligned}\\ u_i\in L^\infty(0,T;V),\quad u_i'\in L^\infty(0,T;L^2),\quad u_i(0,\cdot)\in H^1(0,T),\\ P_i\in H^1(0,T),\quad i=1,2.\end{gathered}$$

Using Lemma 2.9 with $u_0=u_1=0$, we obtain

$$\frac{1}{2}\|u'(t)\|^2+\frac{1}{2}\|u(t)\|_V^2+\int_0^t P(s)u'(0,s)ds+\int_0^t\langle\chi(s),u'(s)\rangle ds=0\,, \tag{2.62}$$

a.e. $t\in[0,T]$. Put

$$\begin{gathered}\sigma(t)=\|u'(t)\|^2+\frac{1}{2}\|u(t)\|_V^2,\\ \widetilde{H}_1(t)=H(u_1(0,t))-H(u_2(0,t)),\\ \widetilde{K}_1(t,s)=K(t-s,u_1(0,s))-K(t-s,u_2(0,s)).\end{gathered}$$

Substituting $P(t)$, χ into (2.62) and using that f is nondecreasing with respect to the second variable, we have

$$\begin{aligned}&\sigma(t)+2\int_0^t \widetilde{H}_1(s)u'(0,s)ds\\&\le 2\int_0^t \|f(u_1(s),u_2'(s))-f(u_2(s),u_2'(s))\|\,\|u'(s)\|ds\\&\quad+2\int_0^t u'(0,s)ds\int_0^s \widetilde{K}_1(s,r)dr.\end{aligned}\tag{2.63}$$

Using assumption (F3),

$$\|f(u_1(s),u_2'(s))-f(u_2(s),u_2'(s))\|\le \|B_2(|u_2'(s)|)\|\|u(s)\|_V\,.$$

Using integration by parts in the last integral of (2.63), we get

$$\begin{aligned}J&=2\int_0^t u'(0,s)ds\int_0^s \widetilde{K}_1(s,r)dr\\&=2u(0,t)\int_0^t \widetilde{K}_1(t,r)dr-2\int_0^t u(0,s)ds\big[\widetilde{K}_1(s,s)+\int_0^s\frac{\partial\widetilde{K}_1}{\partial s}(s,r)dr\big].\end{aligned}\tag{2.64}$$

From assumption (K3), we have

$$\begin{gathered}|\widetilde{K}_1(s,r)|\le p_{M,T}(t-r)|u(0,r)|\le p_{M,T}(t-r)\sqrt{\sigma(r)},\\|\widetilde{K}_1(s,s)|\le p_{M,T}(0)|u(0,s)|\le p_{M,T}(0)\sqrt{\sigma(s)},\\|\frac{\partial\widetilde{K}_1}{\partial s}(s,r)|\le q_{M,T}(t-r)|u(0,r)|\le q_{M,T}(t-r)\sqrt{\sigma(r)}\,,\end{gathered}\tag{2.65}$$

where $M=\max_{i=1,2}\|u_i\|_{L^\infty(0,T;V)}$. It follows from (2.64) and (2.65) that

$$\begin{aligned}|J|&\le 2\sqrt{\sigma(t)}\int_0^t p_{M,T}(t-r)\sqrt{\sigma(r)}dr+2p_{M,T}(0)\int_0^\sigma(s)\,ds\\&\quad+2\int_0^t\sqrt{\sigma(s)}ds\int_0^s q_{M,T}(s-r)\sqrt{\sigma(r)}dr\\&\le\beta_1\sigma(t)+\frac{1}{\beta_1}\int_0^t p_{M,T}^2(r)dr\int_0^t\sigma(r)dr\\&\quad+2p_{M,T}(0)\int_0^t\sigma(s)ds2\sqrt{t}\Big(\int_0^t q_{M,T}^2(r)dr\Big)^{1/2}\int_0^t\sigma(s)ds\\&=\beta_1\sigma(t)+\Big[2p_{M,T}(0)+\frac{1}{\beta_1}\int_0^t p_{M,T}^2(r)dr\\&\quad+2\sqrt{t}\Big(\int_0^t q_{M,T}^2(r)dr\Big)^{1/2}\Big]\int_0^t\sigma(s)ds,\end{aligned}\tag{2.66}$$

for all $\beta_1>0$. Put

$$m_1=\min_{|s|\le M}H'(s),\quad m_2=\max_{|s|\le M}\max|H''(s)|.\tag{2.67}$$

From assumption (H1) we have

$$m_1>-1\,.\tag{2.68}$$

On the other hand, using integration by parts and (2.67) it follows that

$$
\begin{aligned}
&2\int_0^t \widetilde{H}_1(s)u'(0,s)ds \\
&= 2\int_0^t \Big[\int_0^1 \frac{d}{d\theta}H(u_2(0,s)+\theta u(0,s))d\theta\Big]u'(0,s)ds \\
&= u^2(0,t)\int_0^1 H'(u_2(0,s)+\theta u(0,s))d\theta \\
&\quad -\int_0^t u^2(0,s)ds\int_0^1 H''(u_2(0,s)+\theta u(0,s))(u_2'(0,s)+\theta u'(0,s))d\theta \\
&\geq m_1u^2(0,t) - m_2\int_0^t u^2(0,s)(|u_1'(0,s)|+|u_2'(0,s)|)ds \\
&\geq m_1u^2(0,t) - m_2\int_0^t \sigma(s)(|u_1'(0,s)|+|u_2'(0,s)|)ds.
\end{aligned}
$$

From the above inequality, (2.63)-(2.64) and (2.66), we obtain

$$
\begin{aligned}
\sigma(t)+m_1u^2(0,t) \leq m_2\int_0^t \sigma(s)(|u_1'(0,s)|+|u_2'(0,s)|)ds \\
+\int_0^t \|B_2(|u_2'(s)|)\|\sigma(s)ds + |J| \equiv \eta(t).
\end{aligned}
\tag{2.69}
$$

From (2.1), (2.68), and (2.69), we have

$$
(1+m_1)u^2(0,t) \leq \sigma(t)+m_1u^2(0,t) \leq \eta(t). \tag{2.70}
$$

It follows from (2.69) and (2.70) that

$$
\begin{aligned}
&\sigma(t)+[m_1+\beta_2(1+m_1)]u^2(0,t) \\
&\leq (1+\beta_2)\eta(t) \\
&\leq (1+\beta_2)\int_0^t [m_2(|u_1'(0,s)|+|u_2'(0,s)|)+\|B_2(|u_2'(s)|)\|]\sigma(s)ds \\
&\quad + (1+\beta_2)\beta_1\sigma(t) + (1+\beta_2)\Big[2p_{M,T}(0)+\frac{1}{\beta_1}\int_0^t p_{M,T}^2(r)dr \\
&\quad + 2\sqrt{t}\Big(\int_0^t q_{M,T}^2(r)dr\Big)^{1/2}\Big]\int_0^t \sigma(s)ds,
\end{aligned}
\tag{2.71}
$$

for all $\beta_1>0$, $\beta_2>0$. Choose $\beta_1>0$, $\beta_2>0$ such that $m_1+\beta_2(1+m_1)\geq 1/2$, $(1+\beta_2)\beta_1 \leq 1/2$ and denote

$$
\begin{aligned}
R_1(t) = 2(1+\beta_2)[m_2(|u_1'(0,s)|+|u_2'(0,s)|)+\|B_2(|u_2'(s)|)\| \\
+\frac{1}{\beta_1}\|p_{M,T}\|_{L^2(0,T)}^2 + 2p_{M,T}(0) + 2\sqrt{T}\|q_{M,T}\|_{L^2(0,T)}].
\end{aligned}
\tag{2.72}
$$

Then from (2.71) and (2.72) we have

$$
\sigma(t)+u^2(0,t) \leq \int_0^t R_1(s)[\sigma(s)+u^2(0,s)]ds; \tag{2.73}
$$

i.e. $\sigma(t)+u^2(0,t)\equiv 0$ by Gronwall's lemma. Then Theorem 2.2 is proved. □

In the special cases

$$H(s) = hs, \quad h > 0;$$
$$K(t,u) = k(t)u, \quad k \in H^1(0,T), \quad \forall T > 0, k(0) = 0,$$

the following theorem is a consequence of Theorem 2.2.

Theorem 2.10. *Let* $(A), (G)$ *and* $(F_1)-(F_3)$ *hold. Then, for every* $T > 0$*, problem* (1.1)- (1.4) *and* (1.9) *has at least a weak solution* (u, P) *satisfying* (2.2), (2.3).

Furthermore, if $\beta = 1$ *in (F3) and* B_2 *satisfies (F4), then this solution is unique.*

We remark that Theorem 2.10 gives the same result as in [10], but we do not need the assumption "B_1 is nondecreasing" used there.

In the special case with $K(t, u) \equiv 0$, the following result is the consequence of Theorem 2.2.

Theorem 2.11. *Let (A), (G), (H), (F1)–(F3) hold. Then, for every* $T > 0$*, the problem* (1.1)-(1.4) *corresponding to* $P = g$ *has at least a weak solution* u *satisfying* (2.2).

Furthermore, if $\beta = 1$ *in (F3) and the functions* H*,* B_2 *satisfy the assumptions (H1), (F4), then this solution is unique.*

We remark that Theorem gives same result in [7] but without using the assumption "B_1 is nondecreasing" used there.

3. Stability of the solutions

In this section, we assume that $\beta = 1$ in (F3) and that the functions H, B_2 satisfying (H), (H1), (F4), respectively. By Theorem 2.2 problem (1.1)-(1.5) admits a unique solution (u, P) depending on g, H, K:

$$u = u(g, H, K), \quad P = P(g, H, K),$$

where g, H, K satisfy the assumptions (G), (H),(H1),(K1)-(K3), and u_0, u_1, f are fixed functions satisfying (A), (F1)-(F4).

Let $h_0 > 0$ be a given constant and $H_0 : \mathbb{R}_+ \to \mathbb{R}_+$ be a given function. We put

$$\begin{aligned}\Im(h_0, H_0) = \big\{&H \in C^2(\mathbb{R}) : H(0) = 0, \int_0^x H(s)ds \geq -h_0, \ \forall x \in \mathbb{R}, \\ &H'(s) > -1, \ \forall s \in \mathbb{R}, \sup_{|s|\leq M} (|H(s)| + |H'(s)|) \leq H_0(M), \ \forall M > 0\big\}.\end{aligned}$$

Given $t \geq 0$, $M > 0$, and $K \in C^0(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$, we put

$$N_h(M, K, t) = \sup_{|u|,|v|\leq M, \ u\neq v} \Big|\frac{K(t,u) - K(t,v)}{u - v}\Big|.$$

Given the family $\{p_{M,T}\}$, $M > 0$, $T > 0$ which consists of nonnegative functions $p_{M,T}(t) = p(M, T, t)$, $M > 0$, $T > 0$ such that $p_{M,T} \in L^2(0,T)$, for all $M, T > 0$.

Let $k_1 \in L^2(0,T)$, $k_2 \in L^1(0,T)$, for all $T > 0$. We put

$$\begin{aligned}&\Gamma(k_1, k_2, \{p_{M,T}\}) \\ &= \big\{K \in C^0(\mathbb{R}_+ \times \mathbb{R}) : \partial K/\partial t \in C^0(\mathbb{R}_+ \times \mathbb{R}), \\ &\quad N_h(M, K, t) + N_h(M, \partial K/\partial t, t) \leq p_{M,T}(t), \ \forall t \in [0,T], \ \forall M, T > 0, \\ &\quad |K(t,u)| + |\partial K/\partial t(t,u)| \leq k_1(t)|u| + k_2(t), \ \forall u \in \mathbb{R}, \ \forall t \in [0,T], \ \forall T > 0\big\}.\end{aligned}$$

Then we have the following theorem.

Theorem 3.1. *Let $\beta = 1$ and (A),(F1)–(F4) hold. Then, for every $T > 0$, the solutions of* (1.1)-(1.5) *are stable with respect to the data g, H, K; i.e., if (g, H, K), $(g_j, H_j, K_j) \in H^1(0,T) \times \Im(h_0, H_0) \times \Gamma(k_1, k_2, \{p_{M,T}\})$, are such that*

$$(g_j, H_j) \to (g, H) \quad \text{in } H^1(0,T) \times C^1([-M, M]) \tag{3.1}$$

strongly, and

$$(K_j, \partial K_j/\partial t) \to (K, \partial K/\partial t) \text{ in } [C^0([0,T] \times [-M, M])]^2 \tag{3.2}$$

strongly, as $j \to +\infty$, for all $M, T > 0$. Then

$$(u_j, u_j', u_j(0,t), P_j) \to (u, u', u(0,t), P)$$

in $L^\infty(0,T;V) \times L^\infty(0,T;L^2) \times C^0([0,T]) \times C^0([0,T])$ strongly, as $j \to +\infty$, for all $M, T > 0$, where $u_j = u(g_j, H_j, K_j)$, $P_j = P(g_j, H_j, K_j)$.

Proof. First, we note that if the data (g, H, K) satisfy

$$\|g\|_{H^1(0,T)} \le G_0, \quad H \in \Im(h_0, H_0), \quad K \in \Gamma(k_1, k_2, \{p_{M,T}\}), \tag{3.3}$$

then, the a priori estimates of the sequences $\{u_m\}$ and $\{P_m\}$ in the proof of the Theorem 2.2 satisfy

$$\|u_m'(t)\|^2 + \|u_m(t)\|_V^2 \le C_T^2 \quad \forall t \in [0,T],\ \forall T > 0, \tag{3.4}$$

$$\int_0^t |u_m'(0,s)|^2 ds \le C_T^2 \quad \forall t \in [0,T],\ \forall T > 0, \tag{3.5}$$

$$\int_0^t |P_m'(s)|^2 ds \le C_T^2 \quad \forall t \in [0,T],\ \forall T > 0, \tag{3.6}$$

where C_T is a constant depending only on T, u_0, u_1, f, G_0, h_0, H_0, k_1, k_2, $\{p_{M,T}\}$ (independent of g, H, K). Hence, the limit (u, P) in suitable function spaces of the sequence $\{(u_m, P_m)\}$ is defined by (2.4)-(2.6), which is a solution of (1.1)-(1.5) satisfying the a priori estimates (3.4)-(3.6).

Now, by (3.1), (3.2) we can assume that there exists constant $G_0 > 0$ such that the data (g_j, H_j, K_j) satisfy (3.3) with $(g, H, K) = (g_j, H_j, K_j)$. Then, by the above remark, we have that the solutions (u_j, P_j) of problem (1.1)-(1.5) corresponding to $(g, H, K) = (g_j, H_j, K_j)$ satisfy

$$\|u_j'(t)\|^2 + \|u_j(t)\|_V^2 \le C_T^2 \quad \forall t \in [0,T],\ \forall T > 0, \tag{3.7}$$

$$\int_0^t |u_j'(0,s)|^2 ds \le C_T^2 \quad \forall t \in [0,T],\ \forall T > 0, \tag{3.8}$$

$$\int_0^t |P_j'(s)|^2 ds \le C_T^2 \quad \forall t \in [0,T], \quad \forall T > 0, \tag{3.9}$$

Put $\widetilde{g}_j = g_j - g$, $\widetilde{H}_j = H_j - H$, $\widetilde{K}_j = K_j - K$. Then, $v_j = u_j - u$ and $Q_j = P_j - P$ satisfy the problem

$$\begin{gathered} v_j'' - v_{jxx} + \chi_j = 0, \quad 0 < x < 1,\ 0 < t < T, \\ v_{jx}(0,t) = Q_j(t), \quad v_j(1,t) = 0, \\ v_j(x,0) = v_j'(x,0) = 0, \end{gathered}$$

where

$$\begin{aligned}\chi_j &= f(u_j, u_j') - f(u, u'),\\ Q_j(t) &= \widehat{g}_j(t) + H(u_j(0,t)) - H(u(0,t))\\ &\quad - \int_0^t [K(t-s, u_j(0,s)) - K(t-s, u(0,s))]ds,\end{aligned} \tag{3.10}$$

$$\widehat{g}_j(t) = \widetilde{g}_j(t) + \widetilde{H}_j(u_j(0,t)) - \int_0^t \widetilde{K}_j(t-s, u_j(0,s))ds. \tag{3.11}$$

Applying Lemma 2.9 with $u_0 = u_1 = 0$, $\chi = \chi_j$, $P = Q_j$, we have

$$\|v_j'(t)\|^2 + \|v_j(t)\|_V^2 + 2\int_0^t Q_j(s)v_j'(0,s))ds + 2\int_0^t \langle \chi_j(s), v_j'(s)\rangle ds = 0.$$

Let

$$\begin{gathered}S_j(t) = \|v_j'(t)\|^2 + \|v_j(t)\|_V^2 + v_j^2(0,t),\\ M = C_T, \quad m_1 = \min_{|s|\le M} H'(s) > -1, \quad m_2 = \max_{|s|\le M} |H''(s)|.\end{gathered}$$

Then, we can prove the following inequality in a similar manner

$$\begin{aligned}&\|v_j'(t)\|^2 + \|v_j(t)\|_V^2 + m_1 v_j^2(0,t)\\ &\le \int_0^t \|B_2(|u'(s)|)\| S_j(s)ds + m_2\int_0^t (|u'(0,s)| + |u_j'(0,s)|)S_j(s)ds\\ &\quad + 2\varepsilon S_j(t) + \varepsilon\int_0^t S_j(s)ds + \frac{1}{\varepsilon}(\widehat{g}_j^2(t) + \int_0^t |\widehat{g}_j'(s)|^2 ds)\\ &\quad + (\frac{1}{\varepsilon}\|p_{M,T}\|_{L^2(0,T)}^2 + 2\sqrt{T}\|p_{M,T}\|_{L^2(0,T)})\int_0^t S_j(s)ds\\ &= 2\varepsilon S_j(t) + \frac{1}{\varepsilon}(\widehat{g}_j^2(t) + \int_0^t |\widehat{g}_j'(s)|^2 ds)\\ &\quad + \int_0^t [\|B_2(|u'(s)|)\| + m_2(|u'(0,s)| + |u_j'(0,s)|)]S_j(s)ds\\ &\quad + (\varepsilon + \frac{1}{\varepsilon}\|p_{M,T}\|_{L^2(0,T)}^2 + 2\sqrt{T}\|p_{M,T}\|_{L^2(0,T)})\int_0^t S_j(s)ds \equiv y_j(t),\end{aligned} \tag{3.12}$$

for all $\varepsilon > 0$ and $t \in [0,T]$.

We remark that $v_j^2(0,t) \le \|v_j(t)\|_V^2$, consequently

$$(1+m_1)v_j^2(0,t) \le \|v_j'(t)\|^2 + \|v_j(t)\|_V^2 + m_1 v_j^2(0,t) \le y_j(t). \tag{3.13}$$

Multiplying the two members of (3.13) by a number $\beta_1 > 0$ and adding to (3.12), we have

$$\begin{aligned}&\|v_j'(t)\|^2 + \|v_j(t)\|_V^2 + [(1+m_1)\beta_1 + m_1]v_j^2(0,t)\\ &\le (1+\beta_1)y_j(t)\\ &\le (1+\beta_1)[2\varepsilon S_j(t) + \frac{1}{\varepsilon}(\widehat{g}_j^2(t) + \int_0^t |\widehat{g}_j'(s)|^2 ds)]\\ &\quad + \int_0^t \widetilde{R}_j(\varepsilon, T, s)S_j(s)ds, \quad \forall \varepsilon > 0,\ \beta_1 > 0,\ t \in [0,T].\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}\widetilde{R}_j(\varepsilon,T,s) = (1+\beta_1)\Big[\varepsilon + \frac{1}{\varepsilon}\|p_{M,T}\|^2_{L^2(0,T)} + 2\sqrt{T}\|p_{M,T}\|_{L^2(0,T)} \\ + \|B_2(|u'(s)|)\| + m_2(|u'(0,s)| + |u'_j(0,s)|)\Big].\end{aligned} \tag{3.15}$$

Choose $\beta_1 > 0$ and $\varepsilon > 0$ such that $(1+m_1)\beta_1 + m_1 \geq 1$, $2\varepsilon(1+\beta_1) \leq 1/2$. From $H^1(0,T) \hookrightarrow C^0([0,T])$, and (3.14) we have

$$S_j(t) \leq 2(1+\beta_1)\frac{1}{\varepsilon}C_T^{(9)}\|\widehat{g}_j\|^2_{H^1(0,T)} + 2\int_0^t \widetilde{R}_j(\varepsilon,T,s)S_j(s)ds, \tag{3.16}$$

where $C_T^{(9)}$ is a constant depending only on T. By Gronwall's lemma, we obtain from (3.16) that

$$S_j(t) \leq 2(1+\beta_1)\frac{1}{\varepsilon}C_T^{(9)}\|\widehat{g}_j\|^2_{H^1(0,T)} \exp\big(2\int_0^T \widetilde{R}_j(\varepsilon,T,s)S_j(s)ds\big), \tag{3.17}$$

for all $t \in [0,T]$. On the other hand, we from (3.4), (3.10), (3.11), (3.15), and (3.17) obtain

$$S_j(t) \leq C_T^{(10)}\|\widehat{g}_j\|^2_{H^1(0,T)} \quad \forall t \in [0,T], \tag{3.18}$$

$$|Q_j(t)| \leq |\widehat{g}_j(t)| + \max_{|s|\leq M}|H'(s)|\sqrt{S_j(t)} + \|p_{M,T}\|_{L^2(0,T)}\Big(\int_0^t S_j(s)ds\Big)^{1/2}. \tag{3.19}$$

We again use the embedding $H^1(0,T) \hookrightarrow C^0([0,T])$. Then, it follows from (3.18) and (3.19) that

$$\|Q_j\|_{C^0([0,T])} \leq C_T^{(11)}\|\widehat{g}_j\|^2_{H^1(0,T)}.$$

As a final step, we prove

$$\lim_{j\to+\infty}\|\widehat{g}_j\|^2_{H^1(0,T)} = 0.$$

Indeed, from (3.11) combined with (3.8), we deduce the following inequality

$$\begin{aligned}\|\widehat{g}_j\|_{H^1(0,T)} &\leq \|\widetilde{g}_j\|_{H^1(0,T)} + \sqrt{T+M^2}\|\widetilde{H}_j\|_{C^1([-M,M])} \\ &\quad + \sqrt{2T(1+T^2)}(\|\widetilde{K}_j\|_{C^0([0,T]\times[-M,M])} + \|\partial\widetilde{K}_j/\partial t\|_{C^0([0,T]\times[-M,M])}).\end{aligned}$$

Then the proof is complete.

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Thanh Long Nguyen
Department of Mathematics and Computer Science, University of Natural Science, Vietnam National University HoChiMinh City, 227 Nguyen Van Cu Str., Dist.5, HoChiMinh City, Vietnam
E-mail address: `longnt@hcmc.netnam.vn`

Tien Dung Bui
Department of Mathematics, University of Architecture of HoChiMinh City, 196 Pasteur Str., Dist. 3, HoChiMinh City, Vietnam