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LARGE-TIME DYNAMICS OF DISCRETE-TIME NEURAL NETWORKS WITH MCCULLOCH-PITTS NONLINEARITY

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ABSTRACT. We consider a discrete-time network system of two neurons with McCulloch-Pitts nonlinearity. We show that if a parameter is sufficiently small, then network system has a stable periodic solution with minimal period 4k, and if the parameter is large enough, then the solutions of system converge to single equilibrium.

1. INTRODUCTION

We consider the following discrete-time neural network system

$$\begin{aligned} x(n) &= \lambda x(n-1) + (1-\lambda) f(y(n-k)), \\ y(n) &= \lambda y(n-1) - (1-\lambda) f(x(n-k), \end{aligned} \tag{1.1}$$

where the signal function f is given by the following McCulloch-Pitts nonlinearity

$$f(\zeta) = \begin{cases} -1, & \zeta > \sigma, \\ 1, & \zeta \le \sigma. \end{cases} \tag{1.2}$$

in which $\lambda \in (0,1)$ represents the internal decay rate, the positive integer k is the synaptic transmission delay,and σ is the threshold. System (1.1) can be regarded as the discrete analog of the following artificial neural network of two neurons with delayed feedback and McCulloch-Pitts nonlinearity signal function

$$\begin{aligned} \frac{dx}{dt} &= -x(t) + f(y(t-\tau)), \\ \frac{dy}{dt} &= -y(t) - f(x(t-\tau)). \end{aligned} \tag{1.3}$$

where $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are replaced by the backward difference $x(n) - x(n-1)$ and $y(n) - y(n-1)$ respectively.

Model (1.3) has interesting applications in, for example, image processing of moving objects, and has been extensively studied in the literature (see [1-3] and reference herein). But, to the best of our knowledge, the dynamics of the discrete

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model (1.1) are less studied (see [4,5]).For other discrete neural networks, we refer to [6,7].

For the sake of convenience, let Z denote the set of all integers. For any $a, b \in Z$, $a \le b$ define $N(a) = \{a, a+1, \cdots\}$, $N(a,b) = \{a, a+1, \cdots, b\}$,and $N = N(0)$. Also, let $X = \{\phi | \phi = (\varphi, \psi) : N(-k,-1) \to R^2\}$. For the given $\sigma \in R$, let

$$R_\sigma^+ = \{\varphi \mid \varphi : N(-k,-1) \to R \text{ and } \varphi(i) - \sigma > 0, \text{ for } i \in N(-k,-1)\},$$
$$R_\sigma^- = \{\varphi \mid \varphi : N(-k,-1) \to R \text{ and } \varphi(i) - \sigma \le 0, \text{ for } i \in N(-k,-1)\},$$
$$X_\sigma^{\pm,\pm} = \{\phi \in X \mid \phi = (\varphi, \psi), \varphi \in R_\sigma^\pm \text{ and } \psi \in R_\sigma^\pm\},$$
$$X_\sigma = X_\sigma^{+,+} \cup X_\sigma^{+,-} \cup X_\sigma^{-,+} \cup X_\sigma^{-,-}.$$

By a solution of (1.1), we mean a sequence $\{(x(n), y(n))\}$ of points in R^2 that is defined for all $n \in N(-k)$ and satisfies (1.1) for $n \in N$ .Clearly, for any $\phi = (\varphi, \psi) \in X_\sigma$,system (1.1) has an unique solution $(x^\phi(n), y^\phi(n))$ satisfying the initial conditions

$$x^\phi(i) = \varphi(i), \quad y^\phi(i) = \psi(i), \quad \text{for } i \in N(-k,-1).$$

Our goal is to determine the large time behaviors of $(x^\phi(n), y^\phi(n))$ for every $\phi \in X_\sigma$.Our analysis shows that for all $\phi = (\varphi, \psi) \in X_\sigma$, the behaviors of $(x^\phi(n), y^\phi(n))$ as $n \to \infty$ are completely determined by the value $(\varphi(-1), \psi(-1))$ and the size of σ.

The main results of this paper as follows.

Theorem 1.1. *Let* $|\sigma| \le \frac{1+\lambda^{2k+1}-2\lambda}{1-\lambda^{2k+1}}$, $\phi = (\varphi, \psi) \in X_\sigma$ *satisfy:*

(1) $\varphi(-1) \le \frac{\sigma+1}{\lambda} - 1$, $\psi(-1) \le \frac{\sigma+1-2\lambda}{\lambda^{k+1}} + 1$ *for* $\phi \in X_\sigma^{+,+}$;
(2) $\varphi(-1) > \frac{\sigma-1}{\lambda^{k+1}} + 1$, $\psi(-1) \le \frac{\sigma+1}{\lambda} - 1$ *for* $\phi \in X_\sigma^{-,+}$;
(3) $\varphi(-1) > \frac{\sigma-1}{\lambda} + 1$, $\psi(-1) > \frac{\sigma-1+2\lambda}{\lambda^{k+1}} - 1$ *for* $\phi \in X_\sigma^{-,-}$;
(4) $\varphi(-1) \le \frac{\sigma+1}{\lambda^{k+1}} - 1$, $\psi(-1) > \frac{\sigma-1}{\lambda} + 1$ *for* $\phi \in X_\sigma^{+,-}$.

Then there exists $\phi_0 = (\varphi_0, \psi_0) \in X_\sigma$ *such that the solution* $\{x^{\phi_0}(n), y^{\phi_0}(n)\}$ *of (1.1) with initial value* $\phi_0 = (\varphi_0, \psi_0)$ *is* $4k$ *periodic. Moreover, for any solutions* $\{(x^\phi(n), y^\phi(n))\}$ *of (1.1) with initial value* $\phi \in X_\sigma$, *we have*

$$\lim_{n\to\infty}[x^\phi(n) - x_0^\phi(n)] = 0 \qquad \lim_{n\to\infty}[y^\phi(n) - y_0^\phi(n)] = 0.$$

Theorem 1.2. *Let* $|\sigma| > 1$ *and* $\phi = (\varphi, \psi) \in X_\sigma$. *Then* $\lim_{n\to\infty}(x^\phi(n), y^\phi(n)) = (1,-1)$, *if* $\sigma > 1$; *and* $\lim_{n\to\infty}(x^\phi(n), y^\phi(n)) = (-1,1)$, *if* $\sigma < -1$.

Theorem 1.3. *Let* $\sigma = 1$, *Then* $\lim_{n\to\infty}(x^\phi(n), y^\phi(n)) = (1,-1)$, *if* $\phi \in X_\sigma^{+,+} \cup X_\sigma^{-,+} \cup X_\sigma^{-,-}$; *and* $\lim_{n\to\infty}(x^\phi(n), y^\phi(n)) = (1,1)$, *if* $\phi \in X_\sigma^{+,-}$.

Theorem 1.4. *Let* $\sigma = -1$, *Then* $\lim_{n\to\infty}(x^\phi(n), y^\phi(n)) = (-1,1)$, *if* $\phi \in X_\sigma^{+,+} \cup X_\sigma^{+,-} \cup X_\sigma^{-,-}$; *and* $\lim_{n\to\infty}(x^\phi(n), y^\phi(n)) = (-1,-1)$. *if* $\phi \in X_\sigma^{-,+}$.

For the sake of simplicity, in the remaining part of this paper, for a given $n \in N$ and a sequence $z(n)$ defined on $N(-k)$, we define $z_n : N(-k,-1) \to R$ by $z_n(m) = z(n+m)$ for all $m \in N(-k,-1)$.

2. Preliminary Lemmas

In this section, we establish several technical lemmas, important in the proofs of our main results. Assume $n_0 \in N$, we first note the difference equation

$$x(n) = \lambda x(n-1) - 1 + \lambda, \quad n \in N(n_0) \tag{2.1}$$

with initial condition $x(n_0-1)=a$ is given by

$$x(n)=(a+1)\lambda^{n-n_0+1}-1, \quad n\in N(n_0). \tag{2.2}$$

And that the solution of the difference equation

$$x(n)=\lambda x(n-1)+1-\lambda, \quad n\in N(n_0) \tag{2.3}$$

with initial condition $x(n_0-1)=a$ is given by

$$x(n)=(a-1)\lambda^{n-n_0+1}+1, \quad n\in N(n_0). \tag{2.4}$$

Let $(x(n),y(n))$ be a solution of (1.1) with a given initial value $\phi=(\varphi,\psi)\in X_\sigma$.Then we have the following:

Lemma 2.1. *Let $-1<\sigma\le 1$. If there exists $n_0\in N$ such that $(x_{n_0},y_{n_0})\in X_\sigma^{+,+}$, then there exists $n_1\in N(n_0)$ such that $(x_{n_1+k},y_{n_1+k})\in X_\sigma^{-,+}$. Moreover, if $x(n_0-1)\le \frac{\sigma+1}{\lambda}-1$,then $(x_{n_0+k},y_{n_0+k})\in X_\sigma^{-,+}$.*

Proof. Since $(x_{n_0},y_{n_0})\in X_\sigma^{+,+}$, for $n\in N(n_0,n_0+k-1)$ we have

$$\begin{aligned} x(n)&=\lambda x(n-1)-1+\lambda,\\ y(n)&=\lambda y(n-1)+1-\lambda, \end{aligned} \tag{2.5}$$

By (2.2) and (2.4), for $n\in N(n_0,n_0+k-1)$, we get

$$\begin{aligned} x(n)&=[x(n_0-1)+1]\lambda^{n-n_0+1}-1,\\ y(n)&=[y(n_0-1)-1]\lambda^{n-n_0+1}+1. \end{aligned} \tag{2.6}$$

We claim that there exists a $n_1\in N(n_0)$ such that $x(n)>\sigma$ for $n\in N(n_0-k,n_1-1)$ and $x(n_1)\le\sigma$. Assume, for the sake of contradiction, that $x(n)>\sigma$ for all $n\in N(n_0-k)$. From (1.1) and (1.2), we have

$$y(n)=\lambda y(n-1)+1-\lambda, \quad n\in N(n_0),$$

which yield that

$$y(n)=[y(n_0-1)-1]\lambda^{n-n_0+1}+1>(\sigma-1)\lambda^{n-n_0+1}+1>\sigma, \quad n\in N(n_0).$$

Therefore, for all $n\in N(n_0-k)$, we have $y(n)>\sigma$. By(1.1), then

$$x(n)=\lambda x(n-1)-1+\lambda, \quad n\in N(N_0),$$

which implies that

$$x(n)=[x(n_0-1)+1]\lambda^{n-n_0+1}-1, \quad n\in N(N_0).$$

Therefore, $\lim_{n\to\infty}x(n)=-1$, which contradicts $\lim_{n\to\infty}x(n)\ge\sigma>-1$. This proofs our claim. ¿From (1.1) and (1.2), we have

$$y(n)=\lambda y(n-1)+1-\lambda, \quad n\in N(n_0,n_1+k-1),$$

which implies that

$$y(n)=[y(n_0-1)-1]\lambda^{n-n_0+1}+1, \quad n\in N(n_0,n_1+k-1).$$

Note that $y_{n_0}\in R_\sigma^+$ and $\sigma<1$ implies

$$y(n)>\sigma, \quad n\in N(n_0-k,n_1+k-1), \tag{2.7}$$

that is $y_{n_1+k}\in R_\sigma^+$. This, together with (2.1) and (2.2), implies that $x(n)\le\sigma$ for $n\in N(n_1,n_1+2k-1)$, that is $x_{n_1+k}\in R_\sigma^-$. So $(x_{n_1+k},y_{n_1+k})\in X_\sigma^{-,+}$. In addition, if $x(n_0-1)\le\frac{\sigma+1}{\lambda}-1$, then from (2.6) we get $y_{n_0+k}\in R_\sigma^+$ and

$x(n_0) = (x(n_0-1)+1)\lambda - 1 \le \sigma$, Note that $x(n_0-1)+1 > \sigma+1 > 0$,(2.6) implies that

$$x(n_0+k-1) \le x(n_0+k-2) \le \cdots \le x(n_0) \le \sigma,$$

that is $x_{n_0+k} \in R_\sigma^-$. So $(x_{n_0+k}, y_{n_0+k}) \in X_\sigma^{-,+}$. This completes the proof. □

Lemma 2.2. *Let $\sigma > -1$. If there exists $n_0 \in N$ such that $(x_{n_0}, y_{n_0}) \in X_\sigma^{-,+}$,then there exists $n_1 \in N(n_0)$,such that $(x_{n_1+k}, y_{n_1+k}) \in X_\sigma^{-,-}$. Moreover, if $y(n_0-1) \le \frac{\sigma+1}{\lambda} - 1$, then $(x_{n_0+k}, y_{n_0+k}) \in X_\sigma^{-,-}$.*

Proof. Since $(x_{n_0}, y_{n_0}) \in X_\sigma^{-,+}$, from (1.1) and (1.2), it follows that for $n \in N(n_0, n_0+k-1)$,

$$\begin{aligned} x(n) &= \lambda x(n-1) - 1 + \lambda, \\ y(n) &= \lambda y(n-1) - 1 + \lambda. \end{aligned} \tag{2.8}$$

So

$$\begin{aligned} x(n) &= [x(n_0-1)+1]\lambda^{n-n_0+1} - 1, \\ y(n) &= [y(n_0-1)+1]\lambda^{n-n_0+1} - 1. \end{aligned} \tag{2.9}$$

Note that $(x_{n_0}, y_{n_0}) \in X_\sigma^{-,+}$ implies $x(n_0-1) \le \sigma$, $y(n_0-1) > \sigma$. Similar to the proof of Lemma 2.1, we know that there exists $n_1 \in N(n_0)$ such that $y(n) > \sigma$ for $n \in N(n_0-k, n_1-1)$ and $y(n_1) \le \sigma$. Then (2.8) and (2.9) hold for $n \in N(n_0, n_1+k-1)$. So $(x_{n_1+k}, y_{n_1+k}) \in X_\sigma^{-,-}$.

Moreover, if $y(n_0-1) \le \frac{\sigma+1}{\lambda} - 1$, then $x(n) \le \sigma$ for $n \in N(n_0, n_0+k-1)$, that is $x_{n_0+k} \in R_\sigma^-$, and

$$y(n_0) = (y(n_0-1)+1)\lambda - 1 \le \sigma.$$

By (2.9) we get

$$y(n_0+k-1) \le y(n_0+k-2) \le \cdots \le y(n_0) \le \sigma,$$

which implies $y_{n_0+k} \in R_\sigma^-$. So $(x_{n_0+k}, y_{n_0+k}) \in X_\sigma^{-,-}$. □

By a similar argument as that in the proofs of Lemmas 2.1 and 2.2, we obtain the following result.

Lemma 2.3. *Let $-1 \le \sigma < 1$, if there exists $n_0 \in N$ such that $(x_{n_0}, y_{n_0}) \in X_\sigma^{-,-}$, then there exists $n_1 \in N(n_0)$, such that $(x_{n_1+k}, y_{n_1+k}) \in X_\sigma^{+,-}$. Moreover, if $x(n_0-1) > \frac{\sigma-1}{\lambda} + 1$, then $(x_{n_0+k}, y_{n_0+k}) \in X_\sigma^{+,-}$.*

Lemma 2.4. *Let $\sigma < 1$,if there exists $n_0 \in N$ such that $(x_{n_0}, y_{n_0}) \in X_\sigma^{+,-}$, then there exists $n_1 \in N(n_0)$, such that $(x_{n_1+k}, y_{n_1+k}) \in X_\sigma^{+,+}$. Moreover, if $y(n_0-1) > \frac{\sigma-1}{\lambda} + 1$, then $(x_{n_0+k}, y_{n_0+k}) \in X_\sigma^{+,+}$.*

3. Proofs of Main Results

Proof of Theorem 1.1. In view of Lemmas 1-4, it suffices to consider the solution $\{(x(n), y(n))\}$ of (1.1) with initial value $\phi = (\varphi, \psi) \in X_\sigma^{+,+}$. ¿From Lemma1, we obtain $(x_k, y_k) \in X_\sigma^{-,+}$, which implies that for $n \in N(0, k-1)$,

$$\begin{aligned} x(n) &= [\varphi(-1)+1]\lambda^{n+1} - 1, \\ y(n) &= [\psi(-1)-1]\lambda^{n+1} + 1\,. \end{aligned} \tag{3.1}$$

It follows that

$$\begin{aligned} x(k-1) &= [\varphi(-1)+1]\lambda^k - 1, \\ y(k-1) &= [\psi(-1)-1]\lambda^k + 1. \end{aligned}$$

Using $\psi(-1) \leq \frac{\sigma+1-2\lambda}{\lambda^{k+1}}$, then $y(k-1) \leq \frac{\sigma+1}{\lambda} - 1$.

Again by Lemma 2.2, we get $(x_{2k}, y_{2k}) \in X_\sigma^{-,-}$, which implies that for $n \in N(k, 2k-1)$,

$$\begin{aligned} x(n) &= [x(k-1)+1]\lambda^{n-k+1} - 1, \\ y(n) &= [y(k-1)+1]\lambda^{n-k+1} - 1\,. \end{aligned} \tag{3.2}$$

It follows that

$$\begin{aligned} x(2k-1) &= [x(k-1)+1]\lambda^{k} - 1, \\ y(2k-1) &= [y(k-1)+1]\lambda^{k} - 1. \end{aligned}$$

Note that $x(k-1) > (\sigma+1)\lambda^k - 1$ and $\sigma \leq \frac{1+\lambda^{2k+1}-2\lambda}{1-\lambda^{2k+1}}$ yield

$$x(2k-1) > (\sigma+1)\lambda^{2k} - 1 \geq \frac{\sigma-1}{\lambda} + 1.$$

By Lemma 2.3, we obtain $(x_{3k}, y_{3k}) \in X_\sigma^{+,-}$, which implies that for $n \in N(2k, 3k-1)$,

$$\begin{aligned} x(n) &= [x(2k-1)-1]\lambda^{n-2k+1} + 1, \\ y(n) &= [y(2k-1)+1]\lambda^{n-2k+1} - 1\,. \end{aligned} \tag{3.3}$$

It follows that

$$\begin{aligned} x(3k-1) &= [x(2k-1)-1]\lambda^{k} + 1, \\ y(3k-1) &= [y(2k-1)+1]\lambda^{k} - 1. \end{aligned}$$

Note that $y(2k-1) > (\sigma+1)\lambda^k - 1$ and $\sigma \leq \frac{1+\lambda^{2k+1}-2\lambda}{1-\lambda^{2k+1}}$, we have

$$y(3k-1) > (\sigma+1)\lambda^{2k} - 1 \geq \frac{\sigma-1}{\lambda} + 1.$$

By Lemma 2.4, we obtain $(x_{4k}, y_{4k}) \in X_\sigma^{+,+}$, which implies that for $n \in N(3k, 4k-1)$,

$$\begin{aligned} x(n) &= [x(3k-1)-1]\lambda^{n-3k+1} + 1, \\ y(n) &= [y(3k-1)-1]\lambda^{n-3k+1} + 1\,. \end{aligned} \tag{3.4}$$

It follows that

$$\begin{aligned} x(4k-1) &= [x(3k-1)-1]\lambda^{k} + 1, \\ y(4k-1) &= [y(3k-1)-1]\lambda^{k} + 1. \end{aligned}$$

Note that $x(3k-1) \leq (\sigma-1)\lambda^k + 1$ and $\sigma \geq -\frac{1+\lambda^{2k+1}-2\lambda}{1-\lambda^{2k+1}}$, we have

$$x(4k-1) \leq (\sigma-1)\lambda^{2k} + 1 \leq \frac{\sigma+1}{\lambda} - 1.$$

Again by Lemma1, we obtain $(x_{5k}, y_{5k}) \in X_\sigma^{-,+}$, which implies that for $n \in N(4k, 5k-1)$,

$$\begin{aligned} x(n) &= [x(4k-1)+1]\lambda^{n-4k+1} - 1, \\ y(n) &= [y(4k-1)-1]\lambda^{n-4k+1} + 1\,. \end{aligned} \tag{3.5}$$

It follows that

$$\begin{aligned} x(5k-1) &= [x(4k-1)+1]\lambda^{k} - 1, \\ y(5k-1) &= [y(4k-1)-1]\lambda^{k} + 1. \end{aligned}$$

In general, for $i \in N(1)$, we can get:

$$\begin{aligned} x(n) &= [\varphi(-1)+1]\lambda^{n+1} + 2\lambda^{n+1}\frac{\lambda^{-4(i-1)k}-1}{\lambda^{2k}+1} - 1, \\ y(n) &= [\psi(-1)-1]\lambda^{n+1} + 2\lambda^{n+k+1}\frac{\lambda^{-(4i-2)k}+1}{\lambda^{2k}+1} - 1 \end{aligned}$$

for $n \in N((4i-3)k, (4i-2)k-1)$;

$$\begin{aligned} x(n) &= [\varphi(-1)+1]\lambda^{n+1} - 2\lambda^{n+1}\frac{\lambda^{-(4i-2)k}+1}{\lambda^{2k}+1} + 1, \\ y(n) &= [\psi(-1)-1]\lambda^{n+1} + 2\lambda^{n+k+1}\frac{\lambda^{-(4i-2)k}+1}{\lambda^{2k}+1} - 1 \end{aligned}$$

for $n \in N((4i-2)k, (4i-1)k-1)$;

$$\begin{aligned} x(n) &= [\varphi(-1)+1]\lambda^{n+1} - 2\lambda^{n+1}\frac{\lambda^{-(4i-2)k}+1}{\lambda^{2k}+1} + 1, \\ y(n) &= [\psi(-1)-1]\lambda^{n+1} - 2\lambda^{n+k+1}\frac{\lambda^{-4ik}-1}{\lambda^{2k}+1} + 1, \end{aligned}$$

for $n \in N((4i-1)k, 4ik-1)$;

$$\begin{aligned} x(n) &= [\varphi(-1)+1]\lambda^{n+1} + 2\lambda^{n+1}\frac{\lambda^{-4ik}-1}{\lambda^{2k}+1} - 1, \\ y(n) &= [\psi(-1)-1]\lambda^{n+1} - 2\lambda^{n+k+1}\frac{\lambda^{-4ik}-1}{\lambda^{2k}+1} + 1, \end{aligned}$$

for $n \in N(4ik, (4i+1)k-1)$.

Let $\phi_0 = (\varphi_0, \psi_0) \in X_\sigma^{+,+}$, with

$$\varphi_0(-1) = \frac{1-\lambda^{2k}}{1+\lambda^{2k}}, \psi_0(-1) = \frac{1+\lambda^{2k}-2\lambda^k}{1+\lambda^{2k}}.$$

Then

$$\begin{aligned} x^{\phi_0}(n) &= \frac{2}{1+\lambda^{2k}}\lambda^{n-4(i-1)k+1} - 1, \\ y^{\phi_0}(n) &= \frac{2}{1+\lambda^{2k}}\lambda^{n-(4i-3k)+1} - 1 \end{aligned}$$

for $n \in N((4i-3)k, (4i-2)k-1)$;

$$\begin{aligned} x^{\phi_0}(n) &= -\frac{2}{1+\lambda^{2k}}\lambda^{n-(4i-2)k+1} + 1, \\ y^{\phi_0}(n) &= \frac{2}{1+\lambda^{2k}}\lambda^{n-(4i-3k)+1} - 1 \end{aligned}$$

for $n \in N((4i-2)k, (4i-1)k-1)$;

$$\begin{aligned} x^{\phi_0}(n) &= -\frac{2}{1+\lambda^{2k}}\lambda^{n-(4i-2)k+1} + 1, \\ y^{\phi_0}(n) &= -\frac{2}{1+\lambda^{2k}}\lambda^{n-(4i-1)k+1} + 1 \end{aligned}$$

for $n \in N((4i-1)k, 4ik-1)$;

$$x^{\phi_0}(n) = \frac{2}{1+\lambda^{2k}}\lambda^{n-4ik+1} - 1,$$
$$y^{\phi_0}(n) = -\frac{2}{1+\lambda^{2k}}\lambda^{n-(4i-1)k+1} + 1,$$

for $n \in N(4ik, (4i+1)k-1)$.

Clearly, $\{(x^{\phi_0}(n), y^{\phi_0}(n))\}$ is periodic with minimal period $4k$, and as $n \to \infty$,

$$x^{\phi}(n) - x^{\phi_0}(n) = [\varphi(-1)+1]\lambda^{n+1} - \frac{2\lambda^{n+1}}{1+\lambda^{2k}} \to 0,$$
$$y^{\phi}(n) - y^{\phi_0}(n) = [\psi(-1)-1]\lambda^{n+1} + \frac{2\lambda^{n+k+1}}{1+\lambda^{2k}} \to 0.$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We prove only the case where $\sigma > 1$, the case where $\sigma < -1$ is similar. We distinguish several cases.

Case 1 $\phi = (\varphi, \psi) \in X_\sigma^{-,-}$. In view of (1.1), for $n \in N(0, k-1)$ we have

$$\begin{aligned} x(n) &= \lambda x(n-1) + 1 - \lambda, \\ y(n) &= \lambda y(n-1) - 1 + \lambda\,. \end{aligned} \tag{3.6}$$

which yields that for $n \in N(0, k-1)$,

$$\begin{aligned} x(n) &= [\varphi(-1)-1]\lambda^{n+1} + 1, \\ y(n) &= [\psi(-1)+1]\lambda^{n+1} - 1\,. \end{aligned} \tag{3.7}$$

This implies that $x_k(m) \le \sigma, y_k(m) \le \sigma$ for $m \in N(-k,-1)$, therefore $(x_k, y_k) \in X_\sigma^{-,-}$. Repeating the above argument on $N(0, k-1), N(k, 2k-1), \cdots$,consecutively, we can obtain that $(x_n, y_n) \in X_\sigma^{-,-}$ for all $n \in N$. Therefore, (3.7) holds for all $n \in N$, and hence

$$\lim_{n\to\infty}(x(n), y(n)) = (1, -1).$$

Case 2 $\phi = (\varphi, \psi) \in X_\sigma^{-,+} \cup X_\sigma^{+,-} \cup X_\sigma^{+,+}$. By (1.1), for $n \in N$, we have

$$x(n) \le \lambda x(n-1) + 1 - \lambda,$$
$$y(n) \le \lambda y(n-1) + 1 - \lambda\,.$$

By induction, this implies

$$\begin{aligned} x(n) &\le [\varphi(-1)-1]\lambda^{n+1} + 1, \\ y(n) &\le [\psi(-1)-1]\lambda^{n+1} + 1\,. \end{aligned} \tag{3.8}$$

Since

$$\lim_{n\to\infty}[(\varphi(-1)-1)\lambda^{n+1} + 1] = 1 < \sigma,$$
$$\lim_{n\to\infty}[(\psi(-1)-1)\lambda^{n+1} + 1] = 1 < \sigma,$$

then there exists $m \in N(1)$, such that $x(n) < \sigma, y(n) < \sigma$ for $n \in N(m)$. This implies that $(x_{n+k}, y_{n+k}) \in X_\sigma^{-,-}$ for all $n \in N(m)$. Thus, by case 1, we have

$$\lim_{n\to\infty}(x(n), y(n)) = (1, -1).$$

This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. We distinguish several cases.
Case 1 $\phi = (\varphi, \psi) \in X_\sigma^{-,-}$. Using a similar argument to that in Case 1 for the proof of Theorem 1.2, we can show the conclusion is true.
Case 2 $\phi = (\varphi, \psi) \in X_\sigma^{-,+}$. By lemma 2, there exists $n_0 \in N$ such that $(x_{n_0+k}, y_{n_0+k}) \in X_\sigma^{-,-}$. Thus, it follows from Case 1 that conclusion is true.
Case 3 $\phi = (\varphi, \psi) \in X_\sigma^{+,+}$. By Lemma 2.1, there exists $n_0 \in N$, such that $(x_{n_0+k}, y_{n_0+k}) \in X_\sigma^{-,+}$. Thus, it follows from Case 2 that the conclusion is true.
Case 4 $\phi = (\varphi, \psi) \in X_\sigma^{+,-}$. By (1.1) and (1.2) we have that for $n \in N(0, k-1)$,

$$\begin{aligned} x(n) &= \lambda x(n-1) + 1 - \lambda, \\ y(n) &= \lambda y(n-1) + 1 - \lambda \end{aligned}$$

which implies that for $i \in N(-k, -1)$,

$$\begin{aligned} x_k(i) &= [\varphi(-1) - 1]\lambda^{i+k+1} + 1, \\ y_k(i) &= [\psi(-1) - 1]\lambda^{i+k+1} + 1\,. \end{aligned} \tag{3.9}$$

Since $\varphi(-1) > \sigma = 1, \psi(-1) \le \sigma = 1$, then (3.9) implies that $x_k(i) > 1$, $y_k(i) \le 1$ for $i \in N(-k, -1)$, and so $(x_k, y_k) \in X_\sigma^{+,-}$. Repeating the above argument on $N(k, 2k-1), N(2k, 3k-1), \dots$, consecutively, we can get, for all $n \in N$,

$$\begin{aligned} x(n) &= [\varphi(-1) - 1]\lambda^{n+1} + 1, \\ y(n) &= [\psi(-1) - 1]\lambda^{n+1} + 1\,. \end{aligned}$$

Therefore, $\lim_{n\to\infty}(x(n), y(n)) = (1, 1)$. This completes the proof of Theorem 1.3. □

The proof of Theorem 1.4 is similar to that of Theorem 1.3 and we omit it.

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