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THE CONTRACTION MAPPING PRINCIPLE AND SOME APPLICATIONS

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ABSTRACT. These notes contain various versions of the contraction mapping principle. Several applications to existence theorems in the theories of differential and integral equations and variational inequalities are given. Also discussed are Hilbert's projective metric and iterated function systems

CONTENTS

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Part 1. Abstract results

1. Introduction

1.1. **Theme and overview.** The contraction mapping principle is one of the most useful tools in the study of nonlinear equations, be they algebraic equations, integral or differential equations. The principle is a fixed point theorem which guarantees that a contraction mapping of a complete metric space to itself has a unique fixed point which may be obtained as the limit of an iteration scheme defined by repeated images under the mapping of an arbitrary starting point in the space. As such, it is a constructive fixed point theorem and, hence, may be implemented for the numerical computation of the fixed point.

Iteration schemes have been used since the antiquity of mathematics (viz., the ancient schemes for computing square roots of numbers) and became particularly useful in Newton's method for solving polynomial or systems of algebraic equations and also in the Picard iteration process for solving initial value and boundary value problems for nonlinear ordinary differential equations (see, e.g. [58], [59]).

The principle was first stated and proved by Banach [5] for contraction mappings in complete normed linear spaces (for the many consequences of Banach's work see [60]). At about the same time the concept of an abstract metric space was introduced by Hausdorff, which then provided the general framework for the principle for contraction mappings in a complete metric space, as was done by Caccioppoli [17] (see also [75]). It appears in the various texts on real analysis (an early one being, [56]).

In these notes we shall develop the contraction mapping principle in several forms and present a host of useful applications which appear in various places in the mathematical literature. Our purpose is to introduce the reader to several different areas of analysis where the principle has been found useful. We shall discuss among others: the convergence of Newton's method; iterated function systems and how certain fractals are fixed points of set-valued contractions; the Perron-Frobenius theorem for positive matrices using Hilbert's metric, and the extension of this theorem to infinite dimensional spaces (the theorem of Krein-Rutman); the basic existence and uniqueness theorem of the theory of ordinary differential equations (the Picard-Lindelöf theorem) and various related results; applications to the theory of integral equations of Abel-Liouville type; the implicit function theorem; the basic existence and uniqueness theorem of variational inequalities and hence a Lax-Milgram type result for not necessarily symmetric quadratic forms; the basic existence theorem of Cauchy-Kowalevsky for partial differential equations with analytic terms.

These notes have been collected over several years and have, most recently, been used as a basis for an REU seminar which has been part of the VIGRE program of our department. We want to thank here those undergraduate students who participated in the seminar and gave us their valuable feedback.

2. Complete metric spaces

In this section we review briefly some very basic concepts which are part of most undergraduate mathematics curricula. We shall assume these as requisite knowledge and refer to any basic text, e.g., [15], [32], [62].

2.1. **Metric spaces.** Given a set $\mathbb{M}$, a *metric* on $\mathbb{M}$ is a function (also called a *distance*)

$$\mathrm{d} : \mathbb{M} \times \mathbb{M} \to \mathbb{R}_+ = [0, \infty),$$

that satisfies

$$\begin{gathered} \mathrm{d}(x, y) = \mathrm{d}(y, x), \quad \forall x, y \in \mathbb{M} \\ \mathrm{d}(x, y) = 0, \quad \text{if, and only if,} \quad x = y \\ \mathrm{d}(x, y) \le \mathrm{d}(x, z) + \mathrm{d}(y, z), \quad \forall x, y, z \in \mathbb{M}, \end{gathered} \tag{2.1}$$

(the last requirement is called the *triangle inequality*). We call the pair $(\mathbb{M}, \mathrm{d})$ a metric space (frequently we use $\mathbb{M}$ to represent the pair).

A sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{M}$ is said to converge to $x \in \mathbb{M}$ provided that

$$\lim_{n\to\infty} \mathrm{d}(x_n, x) = 0.$$

This we also write as

$$\lim_n x_n = x, \text{ or } x_n \to x \text{ as } n \to \infty.$$

We call a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{M}$ a *Cauchy sequence* provided that for all $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$, such that

$$\mathrm{d}(x_n, x_m) \le \epsilon, \quad \forall n, m \ge n_0.$$

A metric space $\mathbb{M}$ is said to be *complete* if, and only if, every Cauchy sequence in $\mathbb{M}$ converges to a point in $\mathbb{M}$.

Metric spaces form a useful-in-analysis subfamily of the family of topological spaces. We need to discuss some of the concepts met in studying these spaces. We do so, however, in the context of metric spaces rather than in the more general setting. The following concepts are normally met in an advanced calculus or foundations of analysis course. We shall simply list these concepts here and refer the reader to appropriate texts (e.g. [32] or [72]) for the formal definitions. We consider a fixed metric space $(\mathbb{M}, \mathrm{d})$.

- Open balls $B(x, \epsilon) := \{y \in \mathbb{M} : \mathrm{d}(x, y) < \epsilon\}$ and closed balls $B[x, \epsilon] := \{y \in \mathbb{M} : \mathrm{d}(x, y) \le \epsilon\}$;
- open and closed subsets of $\mathbb{M}$;
- bounded and totally bounded sets in $\mathbb{M}$;
- limit point (accumulation point) of a subset of $\mathbb{M}$;
- the closure of a subset of $\mathbb{M}$ (note that the closure of an open ball is not necessarily the closed ball);
- the diameter of a set;
- the notion of one set's being dense in another;
- the distance between a point and a set (and between two sets).

Suppose $(\mathbb{M}, \mathrm{d})$ is a metric space and $\mathbb{M}_1 \subset \mathbb{M}$. If we restrict d to $\mathbb{M}_1 \times \mathbb{M}_1$, then $\mathbb{M}_1$ will be a metric space having the "same" metric as $\mathbb{M}$. We note the important fact that if $\mathbb{M}$ is complete and $\mathbb{M}_1$ is a closed subset of $\mathbb{M}$, then $\mathbb{M}_1$ is also a complete metric space (any Cauchy sequence in $\mathbb{M}_1$ will be a Cauchy sequence in $\mathbb{M}$; hence it will converge to some point in $\mathbb{M}$; since $\mathbb{M}_1$ is closed in $\mathbb{M}$ that limit must be in $\mathbb{M}_1$).

The notion of *compactness* is a crucial one. A metric space $\mathbb{M}$ is said to be *compact* provided that given any family $\{G_\alpha : \alpha \in A\}$ of open sets whose union is $\mathbb{M}$, there is a finite subset $A_0 \subset A$ such that the union of $\{G_\alpha : \alpha \in A_0\}$ is $\mathbb{M}$. (To describe this situation one usually says that every open cover of $\mathbb{M}$ has a finite subcover.) We may describe compactness more "analytically" as follows. Given any sequence $\{x_n\}$ in $\mathbb{M}$ and a point $y \in \mathbb{M}$; we say that y is a *cluster point* of the sequence $\{x_n\}$ provided that for any $\epsilon > 0$ and any positive integer k, there exists $n \geq k$ such that $x_n \in B(y, \epsilon)$. Thus in any open ball, centered at y, infinitely many terms of the sequence $\{x_n\}$ are to be found. We then have that $\mathbb{M}$ is compact, provided that every sequence in $\mathbb{M}$ has a cluster point (in $\mathbb{M}$).

In the remaining sections of this chapter we briefly list and describe some useful examples of metric spaces.

2.2. **Normed vector spaces.** Let $\mathbb{M}$ be a vector space over the real or complex numbers (the scalars). A mapping $\|\cdot\| : \mathbb{M} \to \mathbb{R}_+$ is called a *norm* provided that the following conditions hold:

$$\begin{gathered} \|x\| = 0, \quad \text{if, and only if, } x = 0 \ (\in \mathbb{M}) \\ \|\alpha x\| = |\alpha| \|x\|, \quad \forall \text{ scalar } \alpha, \ \forall x \in \mathbb{M} \\ \|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in \mathbb{M}. \end{gathered} \tag{2.2}$$

If $\mathbb{M}$ is a vector space and $\|\cdot\|$ is a norm on $\mathbb{M}$, then the pair $(\mathbb{M}, \|\cdot\|)$ is called a *normed vector space*. Should no ambiguity arise we simply abbreviate this by saying that $\mathbb{M}$ is a normed vector space. If $\mathbb{M}$ is a vector space and $\|\cdot\|$ is a norm on $\mathbb{M}$, then $\mathbb{M}$ becomes a metric space if we define the metric d by

$$\mathrm{d}(x, y) := \|x - y\|, \ \forall x, y \in \mathbb{M}.$$

A normed vector space which is a complete metric space, with respect to the metric d defined above, is called a *Banach space*. Thus, a closed subset of a Banach space may always be regarded as a complete metric space; hence, a closed subspace of a Banach space is also a Banach space.

We pause briefly in our general discussion to put together, for future reference, a small catalogue of Banach spaces. We shall consider only real Banach spaces, the complex analogues being defined similarly.

In all cases the verification that these spaces are normed linear spaces is straightforward, the verification of completeness, on the other hand usually is more difficult. Many of the examples that will be discussed later will have their setting in complete metric spaces which are subsets or subspaces of Banach spaces.

Examples of Banach spaces.

Example 2.1. $(\mathbb{R}, |\cdot|)$ is a simple example of a Banach space.

Example 2.2. We fix $N \in \mathbb{N}$ (the natural numbers) and denote by $\mathbb{R}^N$ the set

$$\mathbb{R}^N := \{x : x = (\xi_1, \dots, \xi_N), \ \xi_i \in \mathbb{R}, \ i = 1, \dots, N\}.$$

There are many useful norms with which we can equip $\mathbb{R}^N$.

(1) For $1 \leq p < \infty$ define $\|\cdot\|_p : \mathbb{R}^N \to \mathbb{R}_+$ by

$$\|x\|_p := \Big(\sum_{i=1}^N |\xi_i|^p \Big)^{1/p}, \quad x \in \mathbb{R}^N.$$

These spaces are finite dimensional l^p- spaces. Frequently used norms are $\|\cdot\|_1$, and $\|\cdot\|_2$.

(2) We define $\|\cdot\|_\infty : \mathbb{R}^N \to \mathbb{R}_+$ by
$$\|x\|_\infty := \max\{|\xi_i| : 1 \le i \le N\}.$$
This norm is called the *sup norm* on $\mathbb{R}^N$.

The next example extends the example just considered to the infinite dimensional setting.

Example 2.3. We let
$$\mathbb{R}^\infty := \{x : x = \{\xi_i\}_{i=1}^\infty,\ \xi_i \in \mathbb{R},\ i = 1, 2, \dots\}.$$
Then $\mathbb{R}^\infty$, with coordinate-wise addition and scalar multiplication, is a vector space, certain subspaces of which can be equipped with norms, with respect to which they are complete.

(1) For $1 \le p < \infty$ define
$$l^p := \{x = \{\xi_i\} \in \mathbb{R}^\infty : \sum_i |\xi_i|^p < \infty\}.$$
Then l^p is a subspace of $\mathbb{R}^\infty$ and
$$\|x\|_p := \Big(\sum_{i=1}^\infty |\xi_i|^p\Big)^{1/p}$$
defines a norm with respect to which l^p is complete.

(2) We define
$$l^\infty := \{x = \{\xi_i\} \in \mathbb{R}^\infty : \sup_i |\xi_i| < \infty\}.$$
and
$$\|x\|_\infty := \sup_i\{|\xi_i|,\ x \in l^\infty\}.$$
With respect to this (*sup norm*) l^∞ is complete.

Example 2.4. Let H be a complex (or real) vector space. An inner product on H is a mapping $(x, y) \mapsto \langle x, y\rangle$ $(H \times H \to \mathbb{C})$ which satisfies:

(1) for each $z \in H$ $\langle \cdot, z\rangle : H \to \mathbb{C}$ is a linear mapping,
(2) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for $x, y \in H$ ($\langle x, y\rangle = \langle y, x\rangle$ if H is a real vector space and the inner product is a real valued function),
(3) $\langle x, x\rangle \ge 0$, $x \in H$, and equality holds if, and only if, $x = 0$. If one defines
$$\|x\| := \sqrt{\langle x, x\rangle},$$
then $(H, \|\cdot\|)$ will be a normed vector space.

If it is complete, we refer to H as a *Hilbert space*. We note that $(\mathbb{R}^N, \|\cdot\|_2)$ and l^2 are (real) Hilbert spaces.

Spaces of continuous functions are further examples of important spaces in analysis. The following is a brief discussion of such spaces.

Example 2.5. We fix $I = [a, b],\ a, b \in \mathbb{R},\ a < b$, and $k \in \mathbb{N} \cup \{0\}$. Let $\mathbb{K} = \mathbb{R}$, or $\mathbb{C}$ (the reader's choice). Define
$$C^k(I) := \{f : I \to \mathbb{K} : f, f', \dots, f^{(k)},\ \text{exist and are continuous on } I\}.$$
We note that
$$C^0(I) := C(I) = \{f : I \to \mathbb{K} : f \text{ is continuous on } I\}.$$

For $f \in C(I)$, we define

$$\|f\|_\infty := \max_{x\in[a,b]} |f(t)|$$

(the sup-on-I norm). And for $f \in C^k(I)$

$$\|f\| := \sum_{i=0}^{k} \|f^{(i)}\|_\infty.$$

With the usual pointwise definitions of $f + g$ and αf ($\alpha \in \mathbb{K}$) and with the norm defined as above, it follows that $C^k(I)$ is a normed vector space. That the space is also complete follows from the completeness of $C(I)$ with respect to the sup-on-I norm (see, e.g., [15]).

Another useful norm is

$$\|f\|^* := \sup_{x\in I} \sum_{i=0}^{k} |f^{(i)}(x)|.$$

which is equivalent to the norm defined above; this follows from the inequalities

$$\|f\|^* \le \|f\| \le (k+1)\|f\|^*, \quad \forall f \in C^k(I).$$

Equivalent norms give us the same idea of closeness and one may, in a given application, use, of equivalent norms, that which makes calculations or verifications easier or gives us more transparent conclusions.

Example 2.6. Let Ω be an open subset of $\mathbb{R}^N$, and let $\mathbb{K}$ be as above; define

$$C(\Omega) := C^0(\Omega) := \{f : \Omega \to \mathbb{K} \text{ such that } f \text{ is continuous on } \Omega\}.$$

Let

$$\|f\|_\infty := \sup_{x\in\Omega} |f(x)|.$$

Since the uniform limit of a sequence of continuous functions is again continuous, it follows that the space

$$E := \{f \in C(\Omega) : \|f\|_\infty < \infty\}$$

is a Banach space.

If Ω is as above and Ω' is an open set with $\bar{\Omega} \subset \Omega'$, we let

$$C(\bar{\Omega}) := \{\text{the restriction to } \bar{\Omega} \text{ of } f \in C(\Omega')\}.$$

If Ω is bounded and $f \in C(\bar{\Omega})$, then $\|f\|_\infty < +\infty$. Hence $C(\bar{\Omega})$ is a Banach space.

Example 2.7. Let Ω be an open subset of $\mathbb{R}^N$. Let $I = (i_1, \dots, i_N)$ be a multiindex, i.e. $i_k \in \mathbb{N} \cup \{0\}$ (the nonnegative integers), $1 \le k \le N$. We let $|I| = \sum_{k=1}^{N} i_k$. Let $f : \Omega \to \mathbb{K}$. Then the partial derivative of f of order I, $D^I f(x)$, is given by

$$D^I f(x) := \frac{\partial^{|I|} f(x)}{\partial^{i_1} x_1 \dots \partial^{i_N} x_N},$$

where $x = (x_1, \dots, x_N)$. Define

$$C^j(\Omega) := \{f : \Omega \to \mathbb{K} \text{ such that } D^I f \in C(\Omega),\ |I| \le j\}.$$

Let

$$\|f\|_j := \sum_{k=0}^{j} \max_{|I|\le k} \|D^I f\|_\infty.$$

Then, using further convergence results for families of differentiable functions it follows that the space

$$E := \{f \in C^j(\Omega) : \|f\|_j < +\infty\}$$

is a Banach space.

The space $C^j(\bar{\Omega})$ is defined in a manner similar to the space $C(\bar{\Omega})$ and if Ω is bounded $C^j(\bar{\Omega})$ is a Banach space.

2.3. **Completions.** In this section we shall briefly discuss the concept of the completion of a metric space and its application to completing normed vector spaces.

Theorem 2.8. *If* $(\mathbb{M}, \mathrm{d})$ *is a metric space, then there exists a complete metric space* $(\mathbb{M}^*, \mathrm{d}^*)$ *and a mapping* $h : \mathbb{M} \to \mathbb{M}^*$ *such that*

1. *h is an isometry* $\mathrm{d}^*(h(x), h(y)) = \mathrm{d}(x, y),\ x, y \in \mathbb{M})$
2. *$h(\mathbb{M})$ is dense in $\mathbb{M}^*$.*

We give a short sketch of the proof. We let C be the set of all Cauchy sequences in $\mathbb{M}$. We observe that if $\{x_n\}$ and $\{y_n\}$ are elements of $\mathbb{M}$, then $\{\mathrm{d}(x_n, y_n)\}$ is a Cauchy sequence (hence, convergent) sequence in $\mathbb{R}$, as follows from the triangle inequality. We define $\mathrm{d}_C : C \times C \to \mathbb{R}$, by

$$\mathrm{d}_C(\{x_n\}, \{y_n\}) := \lim_{n\to\infty} \mathrm{d}(x_n, y_n).$$

The mapping d_C is a pseudo-metric on C (lacking only the condition

$$\mathrm{d}_C(\{x_n\}, \{y_n\}) = 0 \ \Rightarrow\ \{x_n\} = \{y_n\}$$

from the definition of a metric). The relation R defined on C by

$$\{x_n\}R\{y_n\},\ \text{if, and only if,}\ \lim_{n\to\infty} \mathrm{d}(x_n, y_n) = 0,$$

or equivalently,

$$\mathrm{d}_C(\{x_n\}, \{y_n\}) = 0,$$

is an equivalence relation on C. The space C/R, the set of all equivalence classes in C, shall be denoted by $\mathbb{M}^*$. If we denote by $R\{x_n\}$, the class of all $\{z_n\}$ which are $R-$ equivalent to $\{x_n\}$, we may define

$$\mathrm{d}^*(R\{x_n\}, R\{y_n\}) := \mathrm{d}_C(\{x_n\}, \{y_n\}).$$

This defines a metric on $\mathbb{M}^*$.

We next connect this to $\mathbb{M}$. There is a natural mapping of $\mathbb{M}$ to C given by

$$x \mapsto \{x\}$$

(the sequence, all of whose entries are the same element x). We clearly have

$$\mathrm{d}_C(\{x\}, \{y\}) = \mathrm{d}(x, y)$$

and thus the mapping $x \mapsto R\{x\}$ (which we now call h) is an isometry of $\mathbb{M}$ to $\mathbb{M}^*$. That the image $h(\mathbb{M})$ is dense in $\mathbb{M}^*$, follows easily from the above construction.

Remark 2.9. We observe that $(\mathbb{M}^*, \mathrm{d}^*)$ is "essentially unique". For, if $(\mathbb{M}_1, \mathrm{d}_1)$ and $(\mathbb{M}_2, \mathrm{d}_2)$ are completions of $\mathbb{M}$ with mappings h_1, respectively h_2, then there exists a mapping $g : \mathbb{M}_1 \to \mathbb{M}_2$ such that

1. g is an isometry of $\mathbb{M}_1$ onto $\mathbb{M}_2$,
2. $h_2 = g \circ h_1$.

The above is summarized in the following theorem, whose proof is similar to the proof of the previous result (Theorem 2.8) recalling that a norm defines a metric and where we define (using the notation of that theorem)

$$\|\{x_n\}\|_C := \lim_{n\to\infty} \|x_n\|,$$

for a given Cauchy sequence, and

$$\|R\{x_n\}\|^* := \|\{x_n\}\|_C.$$

It is also clear that the set X^* becomes a vector space by defining addition and scalar multiplication in a natural way.

Theorem 2.10. *If $(X, \|\cdot\|)$ is a normed vector space, then there exists a (essentially unique) complete normed vector space (a Banach space) $(X^*, \|\cdot\|^*)$ and an isomorphism (which is an isometry) $h : X \to X^*$ such that $h(X)$ is dense in X^*.*

2.4. **Lebesgue spaces.** In this section we shall discuss briefly Lebesgue spaces generated by spaces of continuous functions whose domain is $\mathbb{R}^N$, $N \in \mathbb{N}$. If

$$f : \mathbb{R}^N \to \mathbb{K}, \quad \mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}$$

we define the support of f to be the closed set

$$\operatorname{supp}(f) := \overline{\{x : f(x) \neq 0\}}.$$

We say that f has compact support whenever $\operatorname{supp}(f)$ is a compact (i.e., closed and bounded) set and denote by $C_0(\mathbb{R}^N)$ the set of all continuous $\mathbb{K}-$ valued functions defined on $\mathbb{R}^N$ having compact support. (More generally, if Ω is an open set in $\mathbb{R}^N$, one denotes by $C_0^j(\Omega)$ the set of all C^j- functions having compact support in Ω.) This is a vector subspace of $C(\mathbb{R}^N)$, the space of all continuous $\mathbb{K}-$ valued functions defined on $\mathbb{R}^N$.

We first need to define the Riemann integral on $C_0(\mathbb{R}^N)$. To do this, without getting into too many details of this procedure, we assume that the reader is familiar with this concept for the integral defined on closed rectangular boxes

$$B := \{x = (\xi_1, \dots, \xi_N) : \alpha_i \le \xi_i \le \beta_i,\ 1 \le i \le N\},$$

($B = \prod_{i=1}^N [\alpha_i, \beta_i]$), where the numbers $\alpha_i,\ \beta_i,\ 1 \le i \le N$, are fixed real numbers (for each box). We observe that if $f \in C_0(\mathbb{R}^N)$ and if B_1 and B_2 are such boxes, each of which contains $\operatorname{supp}(f)$, then

$$\int_{B_1} f = \int_{B_1 \cap B_2} f = \int_{B_2} f,$$

$B_1 \cap B_2$ also being a box containing $\operatorname{supp}(f)$. This allows us to define the Riemann integral of f over $\mathbb{R}^N$ by

$$\int f \Big(= \int_{\mathbb{R}^N} f \Big) := \int_B f,$$

where B is any closed box containing $\operatorname{supp}(f)$.

The mapping $f \mapsto \int f$ is a linear mapping (linear functional) from $C_0(\mathbb{R}^N)$ to $\mathbb{K}$, which, in addition, satisfies

- if f is non-negative on $\mathbb{R}^N$, then $\int f \ge 0$,

- If $\{f_n\}$ is a sequence of non-negative functions in $C_0(\mathbb{R}^N)$ which is monotonically decreasing (pointwise) to zero, i.e.,
$$f_n(x) \geq f_{n+1}(x),\ n = 1, \dots,\ \lim_{n\to\infty} f_n(x) = 0,\ x \in \mathbb{R}^N,$$
then $\int f_n \to 0$.

Definition 2.11. For $f \in C_0(\mathbb{R}^N)$ we define
$$\|f\|_1 := \int |f|.$$

It is easily verified that $\|\cdot\|_1$ is a norm - called the L^1-norm- on $C_0(\mathbb{R}^N)$. We now sketch the process for completing the normed vector space $\big(C_0(\mathbb{R}^N), \|\cdot\|_1\big)$ in such a way that we may regard the vectors in the completion as *functions* on $\mathbb{R}^N$.

Definition 2.12. A subset $S \subset \mathbb{R}^N$ is called a set of measure zero provided that for any $\epsilon > 0$ there exists a sequence of boxes $\{B_n\}_{n=1}^{\infty}$ such that
$$S \subset \cup_{n=1}^{\infty} B_n,$$
$$\sum_{n=1}^{\infty} \operatorname{vol}(B_n) < \epsilon,$$
where $\operatorname{vol}(B) = \prod_{i=1}^{N}(\beta_i - \alpha_i)$ for the box $B = \prod_{i=1}^{N}[\alpha_i, \beta_i]$.

We say that a property holds "almost everywhere" ("a.e.") if the set of points at which it fails to hold has measure zero.

The proofs of the following theorems may be found in a very complete discussion of $C_0(\mathbb{R}^N)$ and its L^1- completion in [37, Chapter 7].

Definition 2.13. A sequence $\{x_n\}$ in a normed vector space $(X, \|\cdot\|)$ is said to be a fast Cauchy sequence if
$$\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|$$
converges.

Theorem 2.14. *If $\{f_n\}$ is a fast Cauchy sequence in $(C_0(\mathbb{R}^N), \|\cdot\|_1)$, then $\{f_n\}$ converges pointwise a.e. in $\mathbb{R}^N$.*

Definition 2.15. A Lebesgue integrable function on $\mathbb{R}^N$ is a function f such that:

- f is a $\mathbb{K}$ valued function defined a.e. on $\mathbb{R}^N$,
- there is a fast Cauchy sequence in $(C_0(\mathbb{R}^N), \|\cdot\|_1)$ which converges to f a.e. in $\mathbb{R}^N$.

Theorem 2.16. *If f is a Lebesgue integrable function and if $\{f_n\}$ and $\{g_n\}$ are fast Cauchy sequences in $(C_0(\mathbb{R}^N), \|\cdot\|_1)$ converging a.e. to f, then*
$$\lim_{n\to\infty} \int f_n = \lim_{n\to\infty} \int g_n.$$

In light of this result we may define $\int f$ by
$$\int f := \lim_{n\to\infty} \int f_n,$$

where $\{f_n\}$ is any fast Cauchy sequence in $(C_0(\mathbb{R}^N), \|\cdot\|_1)$, converging a.e. to f on $\mathbb{R}^N$. The resulting map

$$f \mapsto \int f$$

is then defined on the space of all Lebesgue integrable functions $L^1(\mathbb{R}^N)$ and is a linear functional on this space which also satisfies

$$|\int f| \leq \int |f|, \quad \forall f \in L^1(\mathbb{R}^N).$$

Theorem 2.17. *The mapping*

$$f \mapsto \|f\|_1 := \int |f|, \quad f \in L^1(\mathbb{R}^N),$$

is a seminorm on $L^1(\mathbb{R}^N)$, i.e., it satisfies all the conditions of a norm, except that $\|f\|_1 = 0$ need not imply that f is the zero of $L^1(\mathbb{R}^N)$. Further $L^1(\mathbb{R}^N)$ is complete with respect to this seminorm and $C_0(\mathbb{R}^N)$ is a dense subspace of $L^1(\mathbb{R}^N)$.

Usually we identify two elements of $L^1(\mathbb{R}^N)$ which agree a.e.; i.e., we define an equivalence relation on $L^1(\mathbb{R}^N)$

$$f \sim g$$

whenever the set $A \cup B$ has measure zero, where

$$A := \{x : f(x) \text{ or } g(x) \text{ fail to be defined}\},$$
$$B := \{x : f(x), g(x) \text{ are defined, but } f(x) \neq g(x)\}.$$

This equivalence relation respects the operations of addition and scalar multiplication and two equivalent functions have the same seminorm. The vector space of all equivalence classes then becomes a complete normed linear space (Banach space). This space, we again call $L^1(\mathbb{R}^N)$,

Remark 2.18. 1. We again refer the reader to [37] for a complete discussion of this topic and others related to it, e.g., convergence theorems for Lebesgue integrals, etc.

2. The ideas above may equally well be employed to define integrals on open regions $\Omega \subset \mathbb{R}^N$ starting with

$$C_0(\Omega) := \{f \in C(\Omega) : \operatorname{supp}(f) \text{ is a compact subset of } \Omega\}.$$

The resulting space being $L^1(\Omega)$.

3. One also may imitate this procedure to obtain the other Lebesgue spaces $L^p(\mathbb{R}^N)$, $1 \leq p < \infty$, by replacing the original norm in $C_0(\mathbb{R}^N)$ by

$$\|f\|_p := \Big(\int |f|^p\Big)^{1/p}, \quad f \in C_0(\mathbb{R}^N).$$

And, of course, in similar vein, one can define $L^p(\Omega)$, $1 \leq p < \infty$.

4. For given $f \in L^1(\mathbb{R})$ define the functional T_f on $C_0^\infty(\mathbb{R})$ as follows

$$T_f(\phi) := \int f\phi.$$

The functional T_f is called the *distribution* defined by f. More generally, the set of all linear functionals on $C_0^\infty(\mathbb{R})$ is called the set of distributions on $\mathbb{R}$ and if T is such, its *distributional derivative* ∂T is defined by

$$\partial T(\phi) := -T(\phi'), \ \forall \phi \in C_0^\infty(\mathbb{R}),$$

hence for $f \in L^1(\mathbb{R})$ and T_f, the distribution determined by f,

$$\partial T_f(\phi) = \int f\phi', \quad \forall \phi \in C_0^\infty(\mathbb{R}).$$

We henceforth, for given $f \in L^1(\mathbb{R})$, we denote by f the distribution T_f determined by f, as well.

5. The Cartesian product

$$E := \prod_{i=1}^{2} L^1(\mathbb{R})$$

may be viewed as a normed linear space with norm defined as

$$\|(u_1, u_2)\| := \sum_{i=1}^{2} \|u_i\|_1 \quad \forall u_i \in L^1(\Omega),\ i = 1, 2,$$

and the space $C_0^1(\mathbb{R})$ may be viewed as a subspace of E by identifying $f \in C_0^1(\mathbb{R})$ with (f, f'). The completion of the latter space in E is called the *Sobolev space* $W^{1,1}(\mathbb{R})$. Where we think of $W^{1,1}(\mathbb{R})$ as a space of tuples of L^1 functions. On the other hand, if $F = (f, g)$ is such an element, then there exists a sequence $\{f_n\} \subset C_0^1(\mathbb{R})$ such that

$$f_n \to f, \quad f_n' \to g,$$

with respect to the L^1 norm. It follows that

$$\int f_n\phi \to \int f\phi, \quad \forall \phi \in C_0^\infty(\mathbb{R}),$$
$$\int f_n'\phi \to \int g\phi, \quad \forall \phi \in C_0^\infty(\mathbb{R}).$$

On the other hand, using integration by parts,

$$\int f_n'\phi = -\int f_n\phi' \to -\int f\phi'$$

and therefore

$$-\int f\phi' = \int g\phi, \quad \forall \phi \in C_0^\infty(\mathbb{R}).$$

I.e., in the sense of distributions $\partial f = g$. This may be summarized as follows: The space $W^{1,1}(\mathbb{R})$ is the set of all L^1 functions whose distributional derivatives are L^1 functions, as well.

If, instead of the L^1 norm, we use the L^2 norm in the above process, one obtains the space $W^{1,2}(\mathbb{R})$ which is usually denoted by $H^1(\mathbb{R})$. Using L^p as an underlying space, one may define the Sobolev spaces $W^{1,p}(\mathbb{R})$, as well. In the case of functions of N variables and open regions $\Omega \subset \mathbb{R}^N$, analogous procedures are used to define the Sobolev spaces $W^{1,p}(\Omega)$. Of particular interest to us later in these notes will be the space $H_0^1(\Omega)$ which is the closure in $H^1(\Omega)$ of the space $C_0^\infty(\Omega)$. We refer the interested reader to the book by Adams [1] for detailed developments and properties of Sobolev spaces.

2.5. **The Hausdorff metric.** Let $\mathbb{M}$ be a metric space with metric d. For $x \in \mathbb{M}$ and $\delta > 0$, we set, as before,

$$B(x,\delta) := \{y \in \mathbb{M} : \mathrm{d}(x,y) < \delta\},$$
$$B[x,\delta] := \{y \in \mathbb{M} : \mathrm{d}(x,y) \le \delta\},$$

the open and closed balls with center at x and radius δ. As pointed out before, the closed ball is closed, but need not be the closure of the open ball.

Let A be a nonempty closed subset of $\mathbb{M}$. For $\delta > 0$ we define

$$\begin{aligned} A_\delta : &= \cup\{B[y,\delta] : y \in A\} \\ &= \{x \in \mathbb{M} : \mathrm{d}(x,y) \le \delta, \text{ for some } y \in A\}. \end{aligned}$$

We observe that

$$A_\delta \subset \{x \in \mathbb{M} : \mathrm{d}(x,A) \le \delta\},$$

where

$$\mathrm{d}(x,A) := \inf\{\mathrm{d}(x,a) : a \in A\}.$$

If A is compact these sets are equal; if A is not compact, the containment may be proper.

Definition 2.19. We let

$$\mathcal{H} := \mathcal{H}(\mathbb{M}) = \{A \subset \mathbb{M} : A \neq \emptyset, \ A \text{ is closed and bounded }\}.$$

For each pair of sets A, B in $\mathcal{H}$ we define

$$\mathrm{D}_1(A,B) := \sup\{\mathrm{d}(a,B) : a \in A\}, \tag{2.3}$$
$$\mathrm{D}_2(A,B) := \inf\{\epsilon > 0 : A \subset B_\epsilon, \}. \tag{2.4}$$

It is a straightforward exercise to prove the following proposition.

Proposition 2.20. *For $A, B \in \mathcal{H}$, $\mathrm{D}_1(A,B) = \mathrm{D}_2(A,B)$.*

We henceforth denote the common value

$$\mathrm{D}(A,B) := \mathrm{D}_1(A,B) = \mathrm{D}_2(A,B). \tag{2.5}$$

Proposition 2.21. *For $A, B \in \mathcal{H}$ let $\mathrm{h} : \mathcal{H} \times \mathcal{H} \to [0,\infty)$ be defined by*

$$\mathrm{h}(A,B) := \mathrm{D}(A,B) \vee \mathrm{D}(B,A) := \max\{\mathrm{D}(A,B), \mathrm{D}(B,A)\}. \tag{2.6}$$

Then h *is a metric on $\mathcal{H}$ (the Hausdorff metric).*

We briefly sketch the proof.

That h is symmetric with respect to its arguments and that

$$\mathrm{h}(A,B) = 0$$

if, and only if, $A = B$, follow easily.

To verify that the triangle inequality holds, we let $A, B, C \in \mathcal{H}$ and let

$$x \in A, \ y \in B, \ z \in C.$$

Then

$$\mathrm{d}(x,z) \le \mathrm{d}(x,y) + \mathrm{d}(y,z),$$

and hence,

$$\mathrm{d}(x,C) \le \mathrm{d}(x,y) + \mathrm{d}(y,z), \ \forall y \in B, \ \forall z \in C.$$

Therefore,

$$\mathrm{d}(x,C) \le \mathrm{d}(x,B) + \mathrm{D}(B,C),$$

which implies that

$$\mathrm{D}(A,C) \le \mathrm{D}(A,B) + \mathrm{D}(B,C) \le \mathrm{h}(A,B) + \mathrm{h}(B,C),$$

and similarly

$$\mathrm{D}(C,A) \le \mathrm{h}(A,B) + \mathrm{h}(B,C).$$

The following corollary, which is an immediate consequence of the definitions, will be of use later.

Proposition 2.22. *Let $A, B \in \mathcal{H}$, $a \in A$, and $\eta > 0$ be given. Then there exists $b \in B$ such that*

$$\mathrm{d}(a,b) \le \mathrm{h}(A,B) + \eta$$

The following examples will serve to illustrate the computation of the Hausdorff distance between two closed sets.

Example 2.23. Let

$$A := [0,1] \times \{0\}, \quad B := \{0\} \times [1,2] \subset \mathbb{R}^2,$$

then

$$\mathrm{D}(A,B) = \sqrt{2}, \quad \mathrm{D}(B,A) = 2,$$

so $\mathrm{h}(A,B) = 2$.

Example 2.24. Let

$$A := B[a,r], \quad B := B[b,s], \quad a, b \in \mathbb{M}, \quad 0 < r \le s,$$

then

$$\mathrm{h}(A,B) = d + s - r,$$

where $d = \mathrm{d}(a,b)$.

There is a natural mapping associating points of $\mathbb{M}$ with elements of $\mathcal{H}$ given by

$$x \mapsto \{x\}.$$

This mapping, as one easily verifies, is an isometry, i.e.,

$$\mathrm{d}(x,y) = \mathrm{h}(\{x\},\{y\}), \ \forall x, y \in \mathbb{M}.$$

We next establish that $(\mathcal{H}, \mathrm{h})$ is a complete metric space, whenever $(\mathbb{M}, \mathrm{d})$ is a complete metric space (see also [25], which contains many very good exercises concerning the Hausdorff metric and the hierarchy of metric spaces constructed in the above manner).

Let $\{A_n\} \subset \mathcal{H}$ be a sequence of sets such that

$$\mathrm{h}(A_n, A_{n+1}) < 2^{-n}, \quad n = 1, 2, \dots.$$

We call a sequence $\{x_n\} \subset \mathbb{M},\ x_n \in A_n,\ n = 1, 2, \dots$ a *fast convergent* sequence, provided that

$$\mathrm{d}(x_n, x_{n+1}) < 2^{-n},\ n = 1, 2, \dots.$$

We have the following lemma whose proof follows immediately from the definition of the Hausdorff metric.

Lemma 2.25. *Let $(\mathbb{M}, \mathrm{d})$ be a complete metric space and $\{A_n\} \subset \mathcal{H}$ be a sequence of sets such that $\mathrm{h}(A_n, A_{n+1}) < 2^{-n},\ n = 1, 2, \dots$. If j is a given positive integer and $y \in A_j$, then there exists a fast convergent sequence $\{x_n\} \subset \mathbb{M},\ x_n \in A_n,\ n = 1, 2, \dots$ with $x_j = y$.*

To see the above one proceeds as follows: Let $x_i = y$ and induct backwards to x_1 and then induct forward through $x_{i+1}, \dots$.

Theorem 2.26. *If* $(\mathbb{M}, \mathrm{d})$ *is a complete metric space, then* $(\mathcal{H}, \mathrm{h})$ *is a complete metric space.*

Proof. Let $\{A_n\} \subset \mathcal{H}$ be a Cauchy sequence. Then, by passing to a subsequence, we may assume that

$$\mathrm{h}(A_n, A_{n+1}) < 2^{-n}, \quad n = 1, 2, \dots .$$

Let

$$A := \{x \in \mathbb{M} : x = \lim_{i\to\infty} x_i\},$$

where $\{x_i\} \subset \mathbb{M},\ x_i \in A_i,\ i = 1, 2, \dots$, is a fast convergent sequence. We claim that the closure of A, $\overline{A}$, is an element of $\mathcal{H}$ and

$$A_i \to \overline{A}.$$

with respect to the Hausdorff metric. To establish that $\overline{A}$ is an element of $\mathcal{H}$, it suffices to show that A is a bounded set. Thus let $x, y \in A$ and let $\{x_i\} \subset \mathbb{M},\ x_i \in A_i,\ i = 1, 2, \dots$, and $\{y_i\} \subset \mathbb{M},\ y_i \in A_i,\ i = 1, 2, \dots$ be fast convergent sequences with

$$\lim_{i\to\infty} x_i = x, \quad \lim_{i\to\infty} y_i = y.$$

Then

$$\mathrm{d}(x, x_1) = \lim_{n\to\infty} \mathrm{d}(x_n, x_1) \le \lim_{n\to\infty} \sum_{k=1}^{k=n-1} \mathrm{d}(x_k, x_{k+1}) < 1,$$

and similarly $\mathrm{d}(y, y_1) < 1$. Hence

$$\mathrm{d}(x, y) \le \mathrm{d}(x, x_1) + \mathrm{d}(y, y_1) + \mathrm{d}(x_1, y_1),$$

or

$$\mathrm{d}(x, y) \le 2 + \sup\{\mathrm{d}(v, w) : v, w \in A_1\} < \infty.$$

We next note that $\mathrm{h}(A_n, \overline{A}) \to 0$, if, and only if,

$$\mathrm{D}(A_n, \overline{A}) \to 0 \quad \text{and} \quad \mathrm{D}(\overline{A}, A_n) \to 0,$$

which, in turn, is equivalent to

$$\sup_{y\in A_n} \mathrm{d}(y, A) \to 0 \quad \text{and} \quad \sup_{z\in A} \mathrm{d}(A, A_n) \to 0.$$

For given $y \in A_n$, there exists a fast convergent sequence $\{x_i\},\ x_n = y,\ x_i \in A_i$, with, say, $x_i \to x \in A$. Hence,

$$\mathrm{d}(y, A) \le \mathrm{d}(y, x) = \mathrm{d}(x_n, x) \le 2^{-n+1},$$

so $\sup_{y\in A_n} \mathrm{d}(y, A) \to 0$, as $n \to \infty$.

Let $z \in A$. Then there exists a fast convergent sequence $\{x_i\},\ x_i \in A_i$, with, say $x_i \to z$. Thus, for each $n = 1, 2, \dots$,

$$\mathrm{d}(z, A_n) \le \mathrm{d}(z, x_n) \le 2^{-n+1},$$

and consequently $\sup_{z\in A} \mathrm{d}(z, A_n) \to 0$ as $n \to \infty$. □

As a consequence of this result we also obtain the following theorem.

Theorem 2.27. *If* $(\mathbb{M}, \mathrm{d})$ *is a metric space which is compact, then* $(\mathcal{H}, \mathrm{h})$ *is a compact metric space.*

Proof. Since $\mathbb{M}$ is compact it is complete (see, e.g., [62]). It follows from the previous theorem that $\mathcal{H}$ is complete. Thus we need to show that $\mathcal{H}$ is totally bounded.

Fix $\epsilon > 0$, and choose $0 < \delta < \epsilon$. Since $\mathbb{M}$ is compact, there exists a finite subset $S \subset \mathbb{M}$ such that

$$\mathbb{M} = \cup\{B(x, \delta) : x \in S\}.$$

If we denote by $\mathcal{S} := 2^S \backslash \emptyset$, the set of nonempty subsets of S, then $\mathcal{S}$ is a finite set and one can easily show that

$$\mathcal{H} = \cup\{B(A, \epsilon) : A \in \mathcal{S}\},$$

where $B(A, \epsilon)$ is the ball, centered at $A \in \mathcal{H}$ with Hausdorff metric radius ϵ. Hence $\mathcal{H}$ is totally bounded and, since complete, also compact. $\square$

3. Contraction mappings

Let $(\mathbb{M}, \mathrm{d})$ be a complete metric space and let

$$T : \mathbb{M} \to \mathbb{M}$$

be a mapping. We call T a *Lipschitz mapping* with *Lipschitz constant* $k \geq 0$, provided that

$$\mathrm{d}(T(x), T(y)) \leq k\mathrm{d}(x, y), \ \forall x, y \in \mathbb{M}. \tag{3.1}$$

We note that Lipschitz mappings are necessarily continuous mappings and that the product of two Lipschitz mappings (defined by composition of mappings) is again a Lipschitz mapping. Thus for a Lipschitz mapping T, and for all positive integers n, the mapping $T^n = T \circ \cdots \circ T$, the mapping T composed with itself n times, is a Lipschitz mapping, as well. We call a Lipschitz mapping T a *nonexpansive* mapping provided the constant k may be chosen so that $k \leq 1$, and a *contraction mapping* provided the Lipschitz constant k may be chosen so that $0 \leq k < 1$. In this case the Lipschitz constant k is also called the *contraction constant* of T.

3.1. **The contraction mapping principle.**

In this section we shall discuss the *contraction mapping principle* or what is often also called the *Banach fixed point theorem*. We shall also give some extensions and examples.

We have the following theorem.

Theorem 3.1. *Let $(\mathbb{M}, \mathrm{d})$ be a complete metric space and let $T : \mathbb{M} \to \mathbb{M}$ be a contraction mapping with contraction constant k. Then T has a unique fixed point $x \in \mathbb{M}$. Furthermore, if $y \in \mathbb{M}$ is arbitrarily chosen, then the iterates $\{x_n\}_{n=0}^{\infty}$, given by*

$$x_0 = y$$
$$x_n = T(x_{n-1}), \ n \geq 1,$$

have the property that $\lim_{n\to\infty} x_n = x$.

Proof. Let $y \in \mathbb{M}$ be an arbitrary point of $\mathbb{M}$ and consider the sequence $\{x_n\}_{n=0}^{\infty}$, given by

$$x_0 = y$$
$$x_n = T(x_{n-1}), \ n \geq 1.$$

We shall prove that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathbb{M}$. For $m < n$ we use the triangle inequality and note that

$$\mathrm{d}(x_m, x_n) \leq \mathrm{d}(x_m, x_{m+1}) + \mathrm{d}(x_{m+1}, x_{m+2}) + \cdots + \mathrm{d}(x_{n-1}, x_n).$$

Since T is a contraction mapping, we have that

$$\mathrm{d}(x_p, x_{p+1}) = \mathrm{d}(T(x_{p-1}), T(x_p)) \le k\mathrm{d}(x_{p-1}, x_p),$$

for any integer $p \ge 1$. Using this inequality repeatedly, we obtain

$$\mathrm{d}(x_p, x_{p+1}) \le k^p \mathrm{d}(x_0, x_1);$$

hence,

$$\mathrm{d}(x_m, x_n) \le \left(k^m + k^{m+1} + \cdots + k^{n-1}\right) \mathrm{d}(x_0, x_1),$$

i.e.,

$$\mathrm{d}(x_m, x_n) \le \frac{k^m}{1-k} \mathrm{d}(x_0, x_1),$$

whenever $m \le n$. From this we deduce that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathbb{M}$. Since $\mathbb{M}$ is complete, this sequence has a limit, say $x \in \mathbb{M}$. On the other hand, since T is continuous, it follows that

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_{n-1}) = T(\lim_{n\to\infty} x_{n-1}) = T(x),$$

and, thus, x is a fixed point of T.

If x and z are both fixed points of T, we get

$$\mathrm{d}(x, z) = \mathrm{d}(T(x), T(z)) \le k\mathrm{d}(x, z).$$

Since $k < 1$, we must have that $x = z$.

The following is an alternate proof. It follows (by induction) that for any $x \in \mathbb{M}$ and any natural number m

$$\mathrm{d}(T^{m+1}(x), T^m(x)) \le k^m \mathrm{d}(T(x), x).$$

Let

$$\alpha := \inf_{x\in\mathbb{M}} \mathrm{d}(T(x), x).$$

Then, if $\alpha > 0$, there exists $x \in \mathbb{M}$ such that

$$\mathrm{d}(T(x), x) < \frac{3}{2}\alpha$$

and hence for any m

$$\mathrm{d}(T^{m+1}(x), T^m(x)) \le k^m \frac{3}{2}\alpha.$$

On the other hand,

$$\alpha \le \mathrm{d}(T(T^m(x)), T^m(x)) = \mathrm{d}(T^{m+1}(x), T^m(x))$$

and thus, for any $m \ge 1$,

$$\alpha \le k^m \frac{3}{2}\alpha$$

which is impossible, since $k < 1$. Thus $\alpha = 0$.

We choose a sequence $\{x_n\}$ (a minimizing sequence) such that

$$\lim_{n\to\infty} \mathrm{d}(T(x_n), x_n)) = \alpha = 0.$$

For any m, n the triangle inequality implies that

$$\mathrm{d}(x_n, x_m) \le \mathrm{d}(T(x_n), x_n)) + \mathrm{d}(T(x_m), x_m)) + \mathrm{d}(T(x_n), T(x_m)).$$

And hence

$$(1-k)\mathrm{d}(x_n, x_m) \le \mathrm{d}(T(x_n), x_n)) + \mathrm{d}(T(x_m), x_m)).$$

Which implies that $\{x_n\}$ is a Cauchy sequence and hence has a limit x in $\mathbb{M}$. One now concludes that

$$\mathrm{d}(T(x), x) = 0$$

and thus x is a fixed point of T. □

It may be the case that $T : \mathbb{M} \to \mathbb{M}$ is not a contraction on the whole space $\mathbb{M}$, but rather a contraction on some neighborhood of a given point. In this case we have the following result:

Theorem 3.2. *Let* $(\mathbb{M}, \mathrm{d})$ *be a complete metric space and let*

$$B = \{x \in \mathbb{M} : \mathrm{d}(z, x) < \epsilon\},$$

where $z \in \mathbb{M}$ *and* $\epsilon > 0$ *is a positive number and let* $T : B \to \mathbb{M}$ *be a mapping such that*

$$\mathrm{d}(T(y), T(x)) \leq k\mathrm{d}(x, y),\ \forall\ x, y \in B,$$

with contraction constant $k < 1$. *Furthermore assume that*

$$\mathrm{d}(z, T(z)) < \epsilon(1 - k).$$

Then T *has a unique fixed point* $x \in B$.

Proof. While the hypotheses do not assume that T is defined on the closure $\overline{B}$ of B, the uniform continuity of T allows us to extend T to a mapping defined on $\overline{B}$ which is a contraction mapping having the same Lipschitz constant as the original mapping. We also note that for $x \in \overline{B}$,

$$\mathrm{d}(z, T(x)) \leq \mathrm{d}(z, T(z)) + \mathrm{d}(T(z), T(x)) < \epsilon(1 - k) + k\epsilon = \epsilon,$$

and hence $T : \overline{B} \to B$. Hence, by Theorem 3.1, since $\overline{B}$ is a complete metric space, T has a unique fixed point in $\overline{B}$ which, by the above calculations, must, in fact, be in B. □

3.2. **Some extensions.**

Example 3.3. Let us consider the space

$$\mathbb{M} = \{x \in \mathbb{R} : x \geq 1\}$$

with metric

$$\mathrm{d}(x, y) = |x - y|, \quad \forall x, y \in \mathbb{M},$$

and let $T : \mathbb{M} \to \mathbb{M}$ be given by

$$T(x) := x + \frac{1}{x}.$$

Then, an easy computation shows that

$$\mathrm{d}(T(x), T(y)) = \frac{xy - 1}{xy}|x - y| < |x - y| = \mathrm{d}(x, y).$$

On the other hand, there does not exist $0 \leq k < 1$ such that

$$\mathrm{d}(T(x), T(y)) \leq k\mathrm{d}(x, y), \quad \forall x, y \in \mathbb{M},$$

and one may verify that T has no fixed points in $\mathbb{M}$.

This shows that if we replace the assumption of the theorem that T be a contraction mapping by the less restrictive hypothesis that

$$\mathrm{d}(T(x), T(y)) < \mathrm{d}(x, y), \ \forall x, y \in \mathbb{M},$$

then T need not have a fixed point. On the other hand, we have the following result of Edelstein [27] (see also [28]):

Theorem 3.4. *Let* $(\mathbb{M}, \mathrm{d})$ *be a metric space and let* $T : \mathbb{M} \to \mathbb{M}$ *be a mapping such that*

$$\mathrm{d}(T(x), T(y)) < \mathrm{d}(x, y), \quad \forall x, y \in \mathbb{M}, \ x \neq y.$$

Furthermore assume that there exists $z \in \mathbb{M}$ *such that the iterates* $\{x_n\}_{n=0}^{\infty}$*, given by*

$$x_0 = z$$
$$x_n = T(x_{n-1}), \ n \geq 1,$$

have the property that there exists a subsequence $\{x_{n_j}\}_{j=0}^{\infty}$ *of* $\{x_n\}_{n=0}^{\infty}$*, with*

$$\lim_{j\to\infty} x_{n_j} = y \in \mathbb{M}.$$

Then y *is a fixed point of* T *and this fixed point is unique.*

Proof. We note from the definition of the iteration process that we may write

$$x_n = T^n(x_0),$$

where, as before, T^n is the mapping T composed with itself n times. We abbreviate by

$$y_j = T^{n_j}(x_0) = T^{n_j}(z),$$

where the sequence $\{n_j\}$ is given by the theorem. Let us assume T has no fixed points. Then the function $f : \mathbb{M} \to \mathbb{R}$ defined by

$$x \mapsto \frac{\mathrm{d}(T^2(x), T(x))}{\mathrm{d}(T(x), x)}$$

is a continuous function. Since the sequence $\{y_j\}_{j=1}^{\infty}$ converges to y, the set K given by

$$K = \{y_j\}_{j=1}^{\infty} \cup \{y\}$$

is compact and, hence, its image under f is compact.

On the other hand, since,

$$f(x)\mathrm{d}(T(x), x) = \mathrm{d}(T^2(x), T(x)) < \mathrm{d}(T(x), x), \ \forall \ x \in \mathbb{M},$$

it follows that $f(x) < 1, \ \forall \ x \in \mathbb{M}$ and, since K is compact, there exists a positive constant $k < 1$ such that

$$f(x) \leq k, \quad \forall x \in K.$$

We now observe that for any positive integer m we have that

$$\mathrm{d}(T^{m+1}(z), T^m(z)) = \Big(\prod_{i=0}^{m-1} f(T^i(z))\Big)\mathrm{d}(T(z), z).$$

Hence, for $m = n_j$, we have

$$\mathrm{d}(T(T^{n_j}(z)), T^{n_j}(z)) = \Big(\prod_{i=0}^{n_j-1} f(T^i(z))\Big)\mathrm{d}(T(z), z),$$

and, since $f(T^i(z)) \leq k < 1$, we obtain that

$$\mathrm{d}(T(y_j), y_j) \leq k^{j-1}\mathrm{d}(T(z), z).$$

On the other hand, as $j \to \infty$, $y_j \to y$ and by continuity

$$T(y_j) \to T(y),$$

and also

$$k^{j-1} \to 0,$$

we obtain a contradiction to the assumption that $T(y) \neq y$. $\square$

The above result has the following important consequence.

Theorem 3.5. *Let* $(\mathbb{M}, \mathrm{d})$ *be a metric space and let* $T : \mathbb{M} \to \mathbb{M}$ *be a mapping such that*

$$\mathrm{d}(T(x), T(y)) < \mathrm{d}(x, y), \quad \forall x, y \in \mathbb{M},\ x \neq y.$$

Further assume that

$$T : \mathbb{M} \to K,$$

where K *is a compact subset of* $\mathbb{M}$*. Then* T *has a unique fixed point in* $\mathbb{M}$*.*

Proof. Since K is compact, it follows that for every $z \in \mathbb{M}$ the sequence $\{T^n(z)\}$ has a convergent subsequence. Hence Theorem 3.4 may be applied.

A direct way of seeing the above is the following. By hypothesis we have that $T : K \to K$, and the function

$$x \mapsto \mathrm{d}(T(x), x)$$

is a continuous function on K and must assume its minimum, say, at a point $y \in K$. If $T(y) \neq y$, then

$$\mathrm{d}(T^2(y), T(y)) < \mathrm{d}(T(y), y),$$

contradicting that $\mathrm{d}(T(y), y)$ is the minimum value. Thus $T(y) = y$. $\square$

In some applications it is the case that the mapping T is a Lipschitz mapping which is not necessarily a contraction, whereas some power of T is a contraction mapping (see e.g. the result of [75]). In this case we have the following theorem.

Theorem 3.6. *Let* $(\mathbb{M}, \mathrm{d})$ *be a complete metric space and let* $T : \mathbb{M} \to \mathbb{M}$ *be a mapping such that*

$$\mathrm{d}(T^m(x), T^m(y)) \leq k\mathrm{d}(x, y), \quad \forall x, y \in \mathbb{M},$$

for some $m \geq 1$*, where* $0 \leq k < 1$ *is a constant. Then* T *has a unique fixed point in* $\mathbb{M}$*.*

Proof. It follows from Theorem 3.1 that T^m has a unique fixed point $z \in \mathbb{M}$. Thus

$$z = T^m(z)$$

implies that

$$T(z) = TT^m(z) = T^m(T(z)).$$

Thus $T(z)$ is a fixed point of T^m and hence by uniqueness of such fixed points $z = T(z)$. $\square$

Example 3.7. Let the metric space $\mathbb{M}$ be given by

$$\mathbb{M} = C[a,b],$$

the set of continuous real valued functions defined on the compact interval $[a,b]$. This set is a Banach space with respect to the norm

$$\|u\| = \max_{t\in[a,b]} |u(t)|, \quad u \in \mathbb{M}.$$

We define $T : \mathbb{M} \to \mathbb{M}$ by

$$T(u)(t) = \int_a^t u(s)ds.$$

Then

$$\|T(u) - T(v)\| \le (b-a)\|u - v\|.$$

(Note that $b-a$ is the best possible Lipschitz constant for T.) On the other hand, we compute

$$T^2(u)(t) = \int_a^t \int_a^s u(\tau)d\tau ds = \int_a^t (t-s)u(s)ds$$

and, inductively,

$$T^m(u)(t) = \frac{1}{(m-1)!}\int_a^t (t-s)^{m-1}u(s)ds.$$

It hence follows that

$$\|T^m(u) - T^m(v)\| \le \frac{(b-a)^m}{m!}\|u-v\|.$$

It is therefore the case that T^m is a contraction mapping for all values of m for which

$$\frac{(b-a)^m}{m!} < 1.$$

It, of course, follows that T has the unique fixed point $u = 0$.

3.3. **Continuous dependence upon parameters.** It is often the case in applications that a contraction mapping depends upon other variables (parameters) also. If this dependence is continuous, then the fixed point will depend continuously upon the parameters, as well. This is the content of the next result.

Theorem 3.8. *Let (Λ, ρ) be a metric space and $(\mathbb{M}, \mathrm{d})$ a complete metric space and let*

$$T : \Lambda \times \mathbb{M} \to \mathbb{M}$$

be a family of contraction mappings with uniform contraction constant k, i.e.,

$$\mathrm{d}\left(T(\lambda,x), T(\lambda,y)\right) \le k\mathrm{d}(x,y),\ \forall \lambda \in \Lambda,\ \forall x,y \in \mathbb{M}.$$

Further more assume that for each $x \in \mathbb{M}$ the mapping

$$\lambda \mapsto T(\lambda, x)$$

is a continuous mapping from Λ to $\mathbb{M}$. Then for each $\lambda \in \Lambda$, $T(\lambda,\cdot)$ has a unique fixed point $x(\lambda) \in \mathbb{M}$, and the mapping

$$\lambda \mapsto x(\lambda),$$

is a continuous mapping from Λ to $\mathbb{M}$.

Proof. The contraction mapping principle may be applied for each $\lambda \in \Lambda$, therefore the mapping $\lambda \mapsto x(\lambda)$, is well-defined. For $\lambda_1, \lambda_2 \in \Lambda$ we have

$$\begin{aligned}
\mathrm{d}\left(x(\lambda_1), x(\lambda_2)\right) &= \mathrm{d}\left(T(\lambda_1, x(\lambda_1)), T(\lambda_2, x(\lambda_2))\right) \\
&\le \mathrm{d}\left(T(\lambda_1, x(\lambda_1)), T(\lambda_2, x(\lambda_1))\right) + \mathrm{d}\left(T(\lambda_2, x(\lambda_1)), T(\lambda_2, x(\lambda_2))\right) \\
&\le \mathrm{d}\left(T(\lambda_1, x(\lambda_1)), T(\lambda_2, x(\lambda_1))\right) + k\mathrm{d}\left(x(\lambda_1)), x(\lambda_2)\right).
\end{aligned}$$

Therefore

$$(1-k)\mathrm{d}\left(x(\lambda_1), x(\lambda_2)\right) \le \mathrm{d}\left(T(\lambda_1, x(\lambda_1)), T(\lambda_2, x(\lambda_1))\right).$$

The result thus follows from the continuity of T with respect to λ for each fixed x. □

3.4. **Monotone Lipschitz mappings.** In this section we shall assume that $\mathbb{M}$ is a Banach space with norm $\|\cdot\|$, which also a Hilbert space, i.e, that $\mathbb{M}$ is an inner product space (over the field of complex numbers) (see [63], [66]) with inner product $(\cdot, \cdot)$, related to the norm by

$$\|u\|^2 = (u, u), \quad \forall u \in \mathbb{M}.$$

We call a mapping $T : \mathbb{M} \to \mathbb{M}$, a *monotone mapping* provided that

$$\mathrm{Re}\left((T(u) - T(v), u - v)\right) \ge 0, \quad \forall u, v \in \mathbb{M},$$

where Re(c) denotes the real part of a complex number c.

The following theorem, due to Zarantonello (see [64]), gives the existence of unique fixed points of mappings which are perturbations of the identity mapping by monotone Lipschitz mappings, without the assumption that they be contraction mappings.

Theorem 3.9. *Let $\mathbb{M}$ be a Hilbert space and let*

$$T : \mathbb{M} \to \mathbb{M},$$

be a monotone mapping such that for some constant $\beta > 0$

$$\|T(u) - T(v)\| \le \beta\|u - v\|, \quad \forall u, v \in \mathbb{M}.$$

Then for any $w \in \mathbb{M}$, the equation

$$u + T(u) = w \tag{3.2}$$

has a unique solution $u = u(w)$, and the mapping $w \mapsto u(w)$ is continuous.

Proof. If $\beta < 1$, then the mapping

$$u \mapsto w - T(u),$$

is a contraction mapping and the result follows from the contraction mapping principle. Next, consider the case that $\beta \ge 1$. We note that for $\lambda \ne 0$, u is a solution of

$$u = (1-\lambda)u - \lambda T(u) + \lambda w, \tag{3.3}$$

if, and only if, u solves (3.2). Let us denote by

$$T_\lambda(u) = (1-\lambda)u - \lambda T(u) + \lambda w.$$

It follows that

$$T_\lambda(u) - T_\lambda(v) = (1-\lambda)(u - v) - \lambda(T(u) - T(v)).$$

Using properties of the inner product, we obtain

$$\begin{aligned}\|T_\lambda(u)-T_\lambda(v)\|^2 \le \lambda^2\beta^2\|u-v\|^2 &- 2\mathrm{Re}\,(\lambda(1-\lambda)(T(u)-T(v),u-v))\\ &+(1-\lambda)^2\|u-v\|^2.\end{aligned}$$

Therefore, if $0<\lambda<1$, the monotonicity of T implies that

$$\|T_\lambda(u)-T_\lambda(v)\|^2 \le (\lambda^2\beta^2+(1-\lambda)^2)\|u-v\|^2.$$

Choosing

$$\lambda=\frac{1}{\beta^2+1},$$

We obtain that T_λ satisfies a Lipschitz condition with Lipschitz constant k given by

$$k^2=\frac{\beta^2}{\beta^2+1},$$

hence is a contraction mapping. The result thus follows by an application of the contraction mapping principle. On the other hand, if u and v, respectively, solve (3.2) with right hand sides w_1 and w_2, then we may conclude that

$$\|u-v\|^2+2\mathrm{Re}\,((T(u)-T(v),u-v))+\|T(u)-T(v)\|^2=\|w_1-w_2\|^2.$$

The monotonicity of T, therefore implies that

$$\|u-v\|^2+\|T(u)-T(v)\|^2\le\|w_1-w_2\|^2,$$

from which the continuity of the mapping $w\mapsto u(w)$ follows. □

3.5. **Multivalued mappings.** Let $\mathbb{M}$ be a metric space with metric d and let

$$T:\mathbb{M}\to\mathcal{H}(\mathbb{M}), \tag{3.4}$$

which is a contraction with respect to the Hausdorff metric h, i.e.,

$$\mathrm{h}(T(x),T(y))\le k\mathrm{d}(x,y),\quad \forall x,y\in\mathbb{M}, \tag{3.5}$$

where $0\le k<1$ is a constant. Such a mapping is called a *contraction correspondence.*

For such mappings we have the following extension of the contraction mapping principle. It is due to Nadler [55]. We note that the theorem is an existence theorem, but that uniqueness of fixed points cannot be guaranteed (easy examples are provided by constant mappings).

Theorem 3.10. *Let $T:\mathbb{M}\to\mathcal{H}(\mathbb{M})$ with*

$$\mathrm{h}(T(x),T(y))\le k\mathrm{d}(x,y),\quad \forall x,y\in\mathbb{M},$$

be a contraction correspondence. Then there exists $x\in\mathbb{M}$ such that $x\in T(x)$.

Proof. The proof uses the Picard iteration scheme. Choose any point $x_0\in\mathbb{M}$, and $x_1\in T(x_0)$. Then choose $x_2\in T(x_1)$ such that

$$\mathrm{d}(x_2,x_1)\le\mathrm{h}(T(x_1),T(x_0))+k,$$

where k is the contraction constant of T (that this may be done follows from Proposition 2.22 of Chapter 2). We then construct inductively the sequence $\{x_n\}_{n=0}^\infty$ in $\mathbb{M}$ to satisfy

$$x_{n+1}\in T(x_n),\ \mathrm{d}(x_{n+1},x_n)\le\mathrm{h}(T(x_n),T(x_{n-1}))+k^n.$$

We then obtain, for $n \geq 1$,

$$\begin{aligned}
\mathrm{d}(x_{n+1}, x_n) &\leq \mathrm{h}(T(x_n), T(x_{n-1})) + k^n \\
&\leq k\mathrm{d}(x_n, x_{n-1}) + k^n \\
&\leq k\left(\mathrm{h}(T(x_{n-1}, T(x_{n-2})) + k^{n-1}\right) + k^n \\
&\leq k^2\mathrm{d}(x_{n-1}, x_{n-2}) + 2k^n \\
&\dots \\
&\leq k^n\mathrm{d}(x_1, x_0) + nk^n.
\end{aligned}$$

Using the triangle inequality for the metric d, we obtain, using the above computation

$$\begin{aligned}
\mathrm{d}(x_{n+m}, x_n) &\leq \sum_{i=n}^{n+m-1} \mathrm{d}(x_{i+1}, x_i) \\
&\leq \sum_{i=n}^{n+m-1} \left(k^i\mathrm{d}(x_1, x_0) + ik^i\right) \\
&\leq \Big(\sum_{i=n}^{\infty} k^i\Big)\mathrm{d}(x_1, x_0) + \Big(\sum_{i=n}^{\infty} ik^i\Big).
\end{aligned}$$

Since both $\sum_{i=0}^{\infty} k^i$ and $\sum_{i=0}^{\infty} ik^i$ converge, it follows that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathbb{M}$, hence has a limit $x \in \mathbb{M}$. We next recall the definition of the Hausdorff metric (see Chapter 2, Section 2.5) and compute

$$\mathrm{d}(x_{n+1}, T(x)) \leq \mathrm{h}(T(x_n), T(x)) \leq k\mathrm{d}(x_n, x).$$

Since $\{T(x)\}$ is a closed set and $\lim_{n\to\infty} x_n = x$, it follows that

$$\mathrm{d}(x, T(x)) = 0,$$

i.e., $x \in T(x)$. $\square$

3.6. **Converse to the theorem.** In this last section of the chapter we discuss a result of Bessaga [9] which provides a converse to the contraction mapping principle. We follow the treatment given in [41], see also [23] (this last reference is also a very good reference to fixed point theory, in general, and to the topics of these notes, in particular). We shall establish the following theorem.

Theorem 3.11. *Let $\mathbb{M} \neq \emptyset$ be a set, $k \in (0,1)$ and let*

$$F : \mathbb{M} \to \mathbb{M}.$$

Then:

(1) *If F^n has at most one fixed point for every $n = 1, 2, \dots$, there exists a metric* d *such that*
$$\mathrm{d}(F(x), F(y)) \leq k\mathrm{d}(x, y), \quad \forall x, y \in \mathbb{M}.$$
(2) *If, in addition, some F^n has a fixed point, then there is a metric* d *such that*
$$\mathrm{d}(F(x), F(y)) \leq k\mathrm{d}(x, y),\ \forall x, y \in \mathbb{M}$$
and $(\mathbb{M}, \mathrm{d})$ is a complete metric space.

The proof of Theorem 3.11 will make use of the following lemma.

Lemma 3.12. *Let F be a selfmap of $\mathbb{M}$ and $k \in (0,1)$. Then the following statements are equivalent:*

1. *There exists a metric d which makes $\mathbb{M}$ a complete metric space such that*
$$\mathrm{d}(F(x), F(y)) \le k\mathrm{d}(x,y), \quad \forall x, y \in \mathbb{M}.$$
2. *There exists a function $\phi : \mathbb{M} \to [0,\infty)$ such that $\phi^{-1}(\{0\})$ is a singleton and*
$$\phi(F(x)) \le k\phi(x), \quad \forall x \in \mathbb{M}. \tag{3.6}$$

Proof. (1. $\Rightarrow$ 2.) The contraction mapping principle implies that F has a unique fixed point $z \in \mathbb{M}$. Put
$$\phi(x) := \mathrm{d}(x,z), \ \forall x \in \mathbb{M}.$$

(2. $\Rightarrow$ 1.) Define
$$\mathrm{d}(x,y) := \phi(x) + \phi(y), \quad x \ne y$$
$$\mathrm{d}(x,x) := 0.$$
Then, one easily notes that d is a metric on $\mathbb{M}$ and that F is a contraction with contraction constant k. Let $\{x_n\} \subset \mathbb{M}$ be a Cauchy sequence. If this sequence has only finitely many distinct terms, it clearly converges. Hence, we may assume it to contain infinitely many distinct terms. Then there exists a subsequence $\{x_{n_k}\}$ of distinct elements, and hence, since
$$\mathrm{d}(x_{n_k}, x_{n_m}) = \phi(x_{n_k}) + \phi(x_{n_m}),$$
it follows that
$$\phi(x_{n_k}) \to 0.$$
Since there exists $z \in \mathbb{M}$ such that $\phi(z) = 0$, it follows that
$$\mathrm{d}(x_{n_k}, z) \to 0,$$
and therefore $x_n \to z$. □

To give a proof of Theorem 3.11 it will therefore suffice to produce such a function ϕ. To do this, we will rely on the use of the Hausdorff maximal principle (see [62]).

Let $z \in \mathbb{M}$ be a fixed point of F^n, as guaranteed by part 2. of the theorem. Uniqueness then implies that
$$z = F(z),$$
as well.

For a given function ϕ defined on a subset of $\mathbb{M}$ we denote by D_ϕ its domain of definition and we let
$$\Phi := \{\phi : D_\phi \to [0,\infty) : z \in D_\phi \subset \mathbb{M}, \ \phi^{-1}(\{0\}) = z, \ F(D_\phi) \subset D_\phi\}.$$
We note that, for the given z, if we put
$$D_{\phi^*} := \{z\}, \quad \phi^*(z) := 0,$$
then $\phi^* \in \Phi$. Hence the collection is not empty. One next defines a partial order on the set Φ as follows:
$$\phi_1 :\preceq \phi_2 \iff D_{\phi_1} \subset D_{\phi_2} \text{ and } \phi_2|_{D_{\phi_1}} = \phi_1.$$
If Φ_0 is a chain in $(\Phi, \preceq)$, then the set
$$D := \cup_{\phi \in \Phi_0} D_\phi$$

is a set which is invariant under F, it contains z and if we define

$$\psi : D \to [0,\infty)$$

by

$$\psi(x) := \phi(x),\ x \in D_\phi,$$

then ψ is an upper bound for Φ_0 with domain $D_\psi := D$. Hence, by the Hausdorff maximal principle, there exists a maximal element

$$\phi_0 : D_0 := D_{\phi_0} \to [0,\infty)$$

in $(\Phi, \preceq)$. We next show that $D_0 = \mathbb{M}$, hence completing the proof.

This we proof indirectly. Thus, let $x_0 \in \mathbb{M} \setminus D_0$ and consider the set

$$O := \{F^n(x_0) : n = 0,1,2,\dots\}.$$

If it is the case that

$$O \cap D_0 = \emptyset,$$

then the elements $F^n(x_0) : n = 0,1,2,\dots$ must be distinct; for, otherwise $z \in O$. We define

$$D_\phi := O \cup D_0,\ \phi|_{D_0} := \phi_0,\ \phi(F^n(x_0)) := k^n,\ n = 0,1,2,\dots.$$

Then

$$\phi \in \Phi, \quad \phi \neq \phi_0, \quad \phi_0 \preceq \phi,$$

contradicting the maximality of ϕ_0. Hence

$$O \cap D_0 \neq \emptyset.$$

Let us set

$$m := \min\{n : F^n(x_0) \in D_0\},$$

then $F^{m-1}(x_0) \notin D_0$. Define

$$D_\phi := \{F^{m-1}(x_0)\} \cup D_0.$$

Then

$$F(D_\phi) = \{F^m(x_0)\} \cup F(D_0) \subset D_0 \subset D_\phi.$$

So D_ϕ is F invariant and contains z. Define $\phi : D_\phi \to [0,\infty)$ as follows:

- $\phi|_{D_0} : \phi_0$.
- If $F^m(x_0) = z$, put $\phi(F^{m-1}(x_0)) := 1$.
- If $F^m(x_0) \neq z$, put $\phi(F^{m-1}(x_0)) := \frac{\phi_0(F^m(x_0))}{k}$.

With this definition we obtain again a contradiction to the maximality of ϕ_0 and hence must conclude that $D_0 = \mathbb{M}$.

Part 2. **Applications**

4. Iterated function systems

In this chapter we shall discuss an application of the contraction mapping principle to the study of iterated function systems. The presentation follows the work of Hutchinson [40] who established that a finite number of contraction mappings on a complete metric space $\mathbb{M}$ define in a natural way a contraction mapping on a subspace of $\mathcal{H}(\mathbb{M})$ with respect to the Hausdorff metric (see Chapter 2, Section 2.5).

4.1. **Set-valued contractions.** Let $\mathbb{M}$ be a complete metric space with metric d and let

$$\mathcal{C}(\mathbb{M}) \subset \mathcal{H}(\mathbb{M})$$

be the metric space of nonempty compact subsets of $\mathbb{M}$ endowed with the Hausdorff metric h. Then if $\{A_n\}$ is a Cauchy sequence in $\mathcal{C}(\mathbb{M})$, its limit A belongs to $\mathcal{H}(\mathbb{M})$ and hence is a closed and thus complete set. On the other hand, since $A_n \to A$, for given $\epsilon > 0$, there exists N, such that for $n \geq N$, $A \subset (A_n)_\epsilon$, and hence A is totally bounded and therefore compact. Thus $\mathcal{C}(\mathbb{M})$ is a closed subspace of $\mathcal{H}(\mathbb{M})$, hence a complete metric space in its own right.

We have the following theorem.

Theorem 4.1. *Let $f_i : \mathbb{M} \to \mathbb{M}$, $i = 1, 2, \dots k$, be k mappings which are Lipschitz continuous with Lipschitz constants $L_1, L_2, \dots, L_k$, i.e.,*

$$\mathrm{d}\left(f_i(x), f_i(y)\right) \leq L_i \mathrm{d}(x, y), \quad i = 1, 2, \dots, k, \ x, y \in \mathbb{M}. \tag{4.1}$$

Define

$$F : \mathcal{C}(\mathbb{M}) \to \mathcal{C}(\mathbb{M}) \tag{4.2}$$

by

$$F(A) := \cup_{i=1}^{k} f_i(A), \quad A \in \mathcal{C}(\mathbb{M}). \tag{4.3}$$

Then F satisfies a Lipschitz condition, with respect to the Hausdorff metric, with Lipschitz constant

$$L := \max_{i=1,2,\dots,k} L_i,$$

i.e.,

$$\mathrm{h}\left(F(A), F(B)\right) \leq L\mathrm{h}(A, B), \quad \forall A, B \in \mathcal{C}(\mathbb{M}). \tag{4.4}$$

In particular, if f_i, $i = 1, 2, \dots, k$, are contraction mappings on $\mathbb{M}$, then F, given by (4.3), is a contraction mapping on $\mathcal{C}(\mathbb{M})$ with respect to the Hausdorff metric, and F has a unique fixed point $A \in \mathcal{C}(\mathbb{M})$.

Proof. We present two arguments based on the two equivalent definitions of the Hausdorff metric. In both cases we establish the result for the case of two mappings. The general case will follow using an induction argument.

We first observe that for any $A \in \mathcal{C}(\mathbb{M})$, because of the compactness of A and the Lipschitz continuity of f_i, $i = 1, 2$ it is the case that $f_i(A) \in \mathcal{C}(\mathbb{M})$, $i = 1, 2$ and hence $F(A) \in \mathcal{C}(\mathbb{M})$.

Let us recall the definition of the Hausdorff metric which may equivalently be stated as

$$\mathrm{h}(A, B) = \sup\{\mathrm{d}(a, B), \mathrm{d}(b, A), \ a \in A, \ b \in B\}.$$

Suppose then that A_1, A_2, B_1, B_2 are compact subsets of $\mathbb{M}$. Letting $A = A_1 \cup A_2$, $B = B_1 \cup B_2$, we claim that

$$\mathrm{h}(A, B) \leq \max_{i=1,2} \mathrm{h}(A_i, B_i) =: m. \tag{4.5}$$

To see this, we let $a \in A, b \in B$ and show that

$$\mathrm{d}(a, B), \mathrm{d}(b, A) \leq m.$$

Now

$$\begin{aligned} \mathrm{d}(a,B) = \mathrm{d}(a,B_1\cup B_2) &= \min\{\mathrm{d}(a,B_i), i=1,2\} \\ &\le \mathrm{d}(a,B_i) \le D(A_i,B_i) \\ &\le \mathrm{h}(A_i,B_i) \le m. \end{aligned}$$

and similarly $\mathrm{d}(b,A) \le m$, which establishes the claim.

Let $A,B \in \mathcal{C}(\mathbb{M})$ and let $\mathrm{h}(A,B) = \epsilon$. Then

$$A \subset B_\epsilon, \quad B \subset A_\epsilon,$$

where A_ϵ, B_ϵ are defined at the beginning of Section 2.5 of Chapter 2. We therefore have

$$\begin{aligned} f_1(B) \subset f_1(A_\epsilon), &\quad f_2(B) \subset f_2(A_\epsilon) \\ f_1(A) \subset f_1(B_\epsilon), &\quad f_2(A) \subset f_2(B_\epsilon). \end{aligned} \tag{4.6}$$

It also follows that

$$\begin{aligned} f_i(A_\epsilon) \subset (f_i(A))_{L_i\epsilon}\,, \ i=1,2 \\ f_i(B_\epsilon) \subset (f_i(B))_{L_i\epsilon}\,, \ i=1,2. \end{aligned} \tag{4.7}$$

Further, we obtain

$$\begin{aligned} f_i(A) \subset (f_1(B)\cup f_2(B))_{L\epsilon}\,, \ i=1,2 \\ f_i(B) \subset (f_1(A)\cup f_2(A))_{L\epsilon}\,, \ i=1,2. \end{aligned} \tag{4.8}$$

Using again the definition of Hausdorff distance, it follows from (4.8) that

$$\mathrm{h}\left(F(A),F(B)\right) \le L\epsilon = L\mathrm{h}(A,B). \tag{4.9}$$

An alternate argument is contained in the following. It follows from the discussion in Chapter 2 (see formula (2.6) there) that $\mathrm{h}\left(F(A),F(B)\right)$ is given by

$$\mathrm{h}\left(F(A),F(B)\right) = \mathrm{D}\left(F(A),F(B)\right) \vee \mathrm{D}(F(B),F(A))\,. \tag{4.10}$$

Formula (4.5), on the other hand implies, since,

$$\mathrm{h}\left(F(A),F(B)\right) = \mathrm{h}\left(f_1(A)\cup f_2(A), f_1(B)\cup f_2(B)\right), \tag{4.11}$$

that

$$\mathrm{h}\left(F(A),F(B)\right) \le \mathrm{h}\left(f_1(A),f_1(B)\right) \vee \mathrm{h}\left(f_2(A),f_2(B)\right). \tag{4.12}$$

Then, using the definition of the Hausdorff metric, we find that

$$\mathrm{h}\left(f_i(A),f_i(B)\right) \le L_i\mathrm{h}(A,B), \quad i=1,2. \tag{4.13}$$

Combining (4.13) with (4.12), we obtain (4.4) for $k=2$. □

Remark 4.2. Given the contraction mappings $f_1,f_2,\dots,f_k$, the mapping F, defined by

$$F(A) := \cup_{i=1}^k f_i(A), \quad A \in \mathcal{C}(\mathbb{M})$$

has become known as the *Hutchinson operator* and the iteration scheme

$$A_{i+1} = F(A_i), \quad i=0,1,\dots \tag{4.14}$$

an *iterated function system*. The iteration scheme (4.14), of course, has, by the contraction mapping principle, a unique limit A, which is independent of the choice of the initial set A_0 and satisfies

$$A = F(A) = f_1(A)\cup f_2(A)\cup\dots\cup f_k(A). \tag{4.15}$$

If it is the case that that $\mathbb{M}$ is a compact subset of $\mathbb{R}^N$ and $f_1, f_2, \dots, f_k$ are similarity transformations, formula (4.15) says that the fixed set A is the union of k similar copies of itself.

4.2. **Examples.** In this section we shall gather some examples which will illustrate the utility of the above theorem in the study and use of fractals.

The Cantor set. Let $\mathbb{M} = [0,1] \subset \mathbb{R}$, with the metric given by the absolute value. We define

$$F : \mathcal{H}(\mathbb{M}) \to \mathcal{H}(\mathbb{M}) = \mathcal{C}(\mathbb{M})$$

by

$$F(A) = f_1(A) \cup f_2(A),$$

where

$$f_1(x) = \frac{1}{3}x, \quad f_2(x) = \frac{1}{3}x + \frac{2}{3}, \quad 0 \le x \le 1.$$

Then f_1 and f_2 are contraction mappings with the same contraction constant $\frac{1}{3}$. Hence, F has the same contraction constant, also.

The unique fixed point of F must satisfy

$$A = F(A) = f_1(A) \cup f_2(A).$$

Considering the nature of the two transformations (similarity transformations), one deduces from the last equation, that the fixed point set A must be the Cantor subset of $[0,1]$.

It is apparent how other types of Cantor subsets of an interval may be constructed using other types of linear contraction mappings.

The Sierpinski triangle. Let $\mathbb{M} = [0,1] \times [0,1] \subset \mathbb{R}$, with metric given by the Euclidean distance. We define

$$F : \mathcal{H}(\mathbb{M}) \to \mathcal{H}(\mathbb{M}) = \mathcal{C}(\mathbb{M})$$

by

$$F(A) = f_1(A) \cup f_2(A) \cup f_3(A),$$

where

$$\begin{aligned} f_1(x,y) &= \frac{1}{2}(x,y), \\ f_2(x,y) &= \frac{1}{2}(x,y) + \big(\frac{1}{2}, 0\big), \\ f_3(x,y) &= \frac{1}{2}(x,y) + \big(\frac{1}{4}, \frac{1}{2}\big). \end{aligned}$$

Here, again, all three contraction constants are equal to $\frac{1}{2}$, hence, the Hutchinson operator is a contraction mapping with the same contraction constant. All three mappings are similarity transformations and, since the fixed point A of F satisfies

$$A = F(A) = f_1(A) \cup f_2(A) \cup f_3(A),$$

A equals a union of three similar copies of itself, i.e., it is a self-similar set, which in this case is, what has become known the *Sierpinski triangle*.

For many detailed examples of the above character, we refer to [6], [57].

5. Newton's method

One of the important numerical methods for computing solutions of nonlinear equations is Newton's method, often also referred to as the Newton-Raphson method. It is an iteration scheme, whose convergence may easily be demonstrated by means of the contraction mapping principle. Many other numerical methods contain Newton's method as one of their subroutines (see, e.g., [2]).

Let G be a domain in $\mathbb{R}^N$ and let

$$F : G \to \mathbb{R}^N$$

be C^2 mapping (i.e., all first and second partial derivatives of all components of F are continuous on G).

Let us assume that the equation

$$F(x) = 0, \tag{5.1}$$

has a solution $x^* \in G$ such that the Jacobian matrix $F'(x^*)$ has full rank (i.e., the matrix $F'(x^*)$ is a nonsingular matrix). It then follows by a simple continuity argument that $F'(x)$ has full rank in a closed neighborhood of x^*, say

$$B_r := \{x : \|x^* - x\| \le r,\ r > 0\},$$

where $\|\cdot\|$, is a given norm in $\mathbb{R}^N$, and that x^* is the unique solution of (5.1) there. The mapping

$$x \mapsto x - \left(F'(x)\right)^{-1} F(x) =: N(x) \tag{5.2}$$

is therefore defined in that neighborhood and we note that x^* is a solution of (5.1) in that neighborhood if, and only if, x^* is a fixed point of N in B_r.

The Newton iteration scheme is then defined by:

$$x_{n+1} = N(x_n), \quad x_1 \in B_r, \quad n = 1, 2, \dots. \tag{5.3}$$

The following theorem holds.

Theorem 5.1. *Assume the above conditions hold. Then, for all $r > 0$, sufficiently small, the Newton iteration scheme, given by (5.3), converges to the solution x^* of (5.1).*

Proof. We use Taylor's theorem to write

$$N(x) = N(x^*) + N'(x^*)(x - x^*) + O(\|x - x^*\|^2).$$

On the other hand, because F is a C^2 mapping and $F'(x^*)$ is nonsingular, we obtain that

$$\begin{aligned}
N(x^* + y) &= N(x^*) + y - \left(F'(x^* + y)\right)^{-1} F(x^* + y) \\
&= N(x^*) + y - \left(F'(x^*)\right)^{-1} F(x^* + y) \\
&\quad + \Big(\left(F'(x^*)\right)^{-1} - \left(F'(x^* + y)\right)^{-1}\Big) F(x^* + y) \\
&= N(x^*) + O(\|y\|^2) + \left(F'(x^* + y)\right)^{-1} \Big(F'(x^* + y) \\
&\quad - F'(x^*)\Big) \left(F'(x^*)\right)^{-1} \left(F'(x^*)y + O(\|y\|^2)\right) \\
&= N(x^*) + O(\|y\|^2).
\end{aligned}$$

From which it follows that

$$N'(x^*) = 0,$$

the zero matrix, and thus, there exists $r > 0$, such that the matrix norm of $N'(x)$ satisfies

$$\|N'(x)\| \le \frac{1}{2}, \text{ for } \|x - x^*\| \le r.$$

Hence, for $x, y \in B_r$

$$\|N(x) - N(y)\| \le \int_0^1 \|N'((1-t)y + tx)\|dt\|y - x\| \le \frac{1}{2}\|x - y\|$$

and for $y = x^*$

$$\|x^* - N(x)\| \le \int_0^1 \|N'((1-t)x^* + tx)\|dt\|x^* - x\| \le r.$$

Hence, $N : B_r \to B_r$ and N is a contraction mapping for such r. The assertion of the theorem then follows from the contraction mapping principle. □

6. Hilbert's metric

This chapter is concerned with a fundamental result of matrix theory, the theorem of Perron-Frobenius about the existence of positive eigenvectors of positive matrices. Upon the introduction of Hilbert's metric, the result may be deduced via the contraction mapping principle. Since the approach also works in infinite dimensions, a version of the celebrated theorem of Krein-Rutman may be established, as well. The approach to establishing these important results using Hilbert's projective metric goes back to Birkhoff [10]. Here we also rely on the work in [16] and [45].

6.1. **Cones.** Let E be a real Banach space with norm $\|\cdot\|$. A closed subset K in E is called a *cone* provided that:

(1) for all $\lambda, \mu \ge 0$, and all $u, v \in K$ $\lambda u + \mu v \in K$,
(2) if $u \in K$, $-u \in K$, then $u = 0 \in K$.

If it is the case that the interior of K, $\operatorname{int} K$, is not empty, the cone is called *solid*. In this chapter we shall always assume that the cone K is solid, even though several of the results presented are valid in the absence of this assumption.

A cone K induces a partial order $\le$ by:

$$u \le v \quad \text{if, and only if,} \quad v - u \in K,$$

and if the cone K is solid, another partial order $<$ by:

$$u < v \quad \text{if, and only if} \quad v - u \in \operatorname{int} K.$$

Since we assume that a cone is closed, it follows that it is also *Archimedean*, i.e.,

$$\text{if } nu \le v, \quad n = 1, 2, \dots, \text{ then } u \le 0.$$

We shall denote by K^+ the set of all nonzero elements of K.

For solid cones we have the following lemma (see [65] for most of the details).

Lemma 6.1. *Let K be a solid cone and let $v \in \operatorname{int} K$, $u \in K$. Then:*

(1) $\{w : w = (1-t)v + tu,\ 0 \le t < 1\} \subset \operatorname{int} K$.
(2) *If $u \in \partial K$, then for $t > 1$,*

$$w = (1-t)v + tu \notin K.$$

(3)

$$K = \overline{\operatorname{int} K}.$$

(4)

$$K + v \subset \operatorname{int} K.$$

Proof. We shall establish the last part of the lemma and leave the remaining parts to the reader for verification.

If $v \in \operatorname{int} K$ and $z \in K$, then $v + z = u \in K$. If it were the case that $u \in \partial K$, then by the first two assertions of the lemma $t = 1$ is the maximal number such that

$$w = (1 - t)v + tu \in K.$$

On the other hand

$$w = (1 - t)v + tu = (1 - t)v + t(v + z) = v + tz \in K, \ \forall t \geq 0,$$

yielding a contradiction. □

6.2. **Hilbert's metric.** We define the mappings

$$m(\cdot, \cdot), \ M(\cdot, \cdot), \ [\cdot, \cdot] : E \times K^+ \to [-\infty, \infty]$$

as follows:

$$m(u, v) := \sup\{\lambda : \lambda v \leq u\} \tag{6.1}$$
$$M(u, v) := \inf\{\lambda : u \leq \lambda v\}, \tag{6.2}$$

with the interpretation that $m(u, v) = -\infty$, if the set $\{\lambda : \lambda v \leq u\}$ is empty, and $M(u, v) = \infty$, if the set $\{\lambda : u \leq \lambda v\}$ is empty, and

$$[u, v] := M(u, v) - m(u, v). \tag{6.3}$$

The last quantity is called the $v-$*oscillation* of u. We remark that the Archimedean property immediately implies that

$$m(u, v) < \infty, \ M(u, v) > -\infty,$$

which makes (6.3) well-defined in the extended real numbers.

In what is to follow many of the statements are to be interpreted in the extended real numbers. If this should be the case, we shall not remark so explicitly; it will be clear from the context. The following lemma is easy to prove and we leave the details to the reader. We remark that (6.6) follows from Lemma 6.1 (above).

Lemma 6.2. *For $u, v, w \in K^+$, the following hold:*

$$m(u,v)v \le u \le M(u,v)v, \quad \text{provided} \quad M(u,v) < \infty, \tag{6.4}$$

$$0 \le m(u,v) \le M(u,v) \le \infty, \tag{6.5}$$

$$m(u,v) > 0, \quad \text{if } u \in \operatorname{int} K, \text{ and } u - m(u,v)v \in \partial K, \tag{6.6}$$

$$m(u,v) = 0, \quad \text{if } v \in \operatorname{int} K, \text{ and } u \in \partial K, \tag{6.7}$$

$$M(u,v) < \infty, \quad \text{if } v \in \operatorname{int} K, \tag{6.8}$$

$$M(u,w) \le M(u,v)M(v,w), \tag{6.9}$$

$$m(u,w) \ge m(u,v)m(v,w), \tag{6.10}$$

$$m(u,v)M(v,u) = 1, \tag{6.11}$$

$$M(\lambda u + \mu v, v) = \lambda M(u,v) + \mu, \quad \forall \lambda, \mu \ge 0, \tag{6.12}$$

$$m(\lambda u + \mu v, v) = \lambda m(u,v) + \mu, \quad \forall \lambda, \mu \ge 0, \tag{6.13}$$

$$[\lambda u + \mu v, v] = \lambda [u,v], \quad \forall \lambda, \mu \ge 0. \tag{6.14}$$

$$[u,v] = 0, \text{ implies } u = \lambda v, \quad \text{for some } \lambda \ge 0. \tag{6.15}$$

Using the properties in the previous lemma, one may establish the following result.

Lemma 6.3. *For $u, v \in K^+$, the following hold:*

$$\begin{aligned} M(u, u+v) &= \frac{1}{m(u+v,u)} \\ &= \frac{1}{1 + m(v,u)} \\ &= \frac{M(u,v)}{1 + M(u,v)} \le 1, \end{aligned} \tag{6.16}$$

and

$$\begin{aligned} m(u, u+v) &= \frac{1}{M(u+v,u)} \\ &= \frac{1}{1 + M(v,u)} \\ &= \frac{m(u,v)}{1 + m(u,v)} \le 1. \end{aligned} \tag{6.17}$$

Hence

$$M(u, u+v) + m(v, u+v) = 1. \tag{6.18}$$

We now define *Hilbert's projective metric*

$$\mathrm{d} : \operatorname{int} K \times \operatorname{int} K \to [0, \infty)$$

as follows:

$$\mathrm{d}(u,v) := \log \frac{M(u,v)}{m(u,v)}. \tag{6.19}$$

We have the following theorem.

Theorem 6.4. *The function* d *defined by (6.19) has the following properties: If* $u, v, w \in \operatorname{int} K$*, then*

$$\mathrm{d}(u,v) = \mathrm{d}(v,u), \ \mathrm{d}(\lambda u, \mu v) = \mathrm{d}(u,v), \quad \forall \lambda > 0, \mu > 0, \tag{6.20}$$

$$\mathrm{d}(u,v) = 0, \quad \textit{if, and only if,} \quad u = \lambda v, \quad \textit{for some } \lambda > 0, \tag{6.21}$$

$$\mathrm{d}(u,v) \le \mathrm{d}(u,w) + \mathrm{d}(w,v). \tag{6.22}$$

Let

$$\mathbb{M} := \{u \in \operatorname{int} K : \|u\| = 1\}, \tag{6.23}$$

then $(\mathbb{M}, \mathrm{d})$ *is a metric space.*

Proof. The symmetry property (6.20) and the triangle inequality (6.22) follow immediately from Lemma 6.3. That (6.21) holds follows from the fact that $\mathrm{d}(u,v) = 0$, if, and only if, $m(u,v) = M(u,v)$, which is the case, if, and only if, $u = M(u,v)v$. The properties together imply that d is a metric on $\mathbb{M}$. $\square$

The following examples will serve to illustrate these concepts. In all examples we shall assume that $u, v \in \operatorname{int} K$.

Example 6.5. Let

$$E := \mathbb{R}^N, \quad K := \{(u_1, u_2, \dots, u_N) : u_i \ge 0, \ i = 1, 2, \dots, N\}.$$

Then

$$\operatorname{int} K = \{(u_1, u_2, \dots, u_N) : u_i > 0, \ i = 1, 2, \dots, N\},$$
$$m(u,v) = \min_i \frac{u_i}{v_i}, \quad M(u,v) = \max_i \frac{u_i}{v_i},$$
$$\mathrm{d}(u,v) = \log \max_{i,j} \frac{u_i v_j}{u_j v_i}.$$

Example 6.6. Let

$$E := \mathbb{R}^N, \quad K := \{(u_1, u_2, \dots, u_N) : 0 \le u_1 \le u_2 \le \dots \le u_N\}.$$

Then

$$\operatorname{int} K = \{(u_1, u_2, \dots, u_N) : 0 < u_1 < u_2 < \dots < u_N\},$$
$$m(u,v) = \min_{i<j} \frac{u_j - u_i}{v_j - v_i}, \quad M(u,v) = \max_{i<j} \frac{u_j - u_i}{v_j - v_i},$$
$$\mathrm{d}(u,v) = \log \max_{i<j, k<l} \frac{(u_j - u_i)(v_l - v_k)}{(u_l - u_k)(v_j - v_i)}.$$

Example 6.7. Let

$$E := C[0,1], \quad K := \{u \in E : u(x) \ge 0, \ 0 \le x \le 1\}.$$

Then

$$\operatorname{int} K = \{u \in E : u(x) > 0, \ 0 \le x \le 1\},$$

$$m(u,v) = \min_{x \in [0,1]} \frac{u(x)}{v(x)}, \quad M(u,v) = \max_{x \in [0,1]} \frac{u(x)}{v(x)},$$
$$\mathrm{d}(u,v) = \log \max_{(x,y) \in [0,1]^2} \frac{u(x)v(y)}{u(y)v(x)}.$$

It is an instructive exercise to compute the various quantities $m(u,v), M(u,v)$, etc., also in the cases that u, v are not necessarily interior elements to the cone K.

6.3. **Positive mappings.** A mapping $T : E \to E$ is called a *positive* mapping (with respect to the cone K) provided that

$$T(K^+) \subset K^+.$$

A positive mapping T is called *homogeneous of degree* p, $p \geq 0$, whenever

$$T(\lambda u) = \lambda^p T(u), \quad \forall \lambda > 0,\ u \in K.$$

A positive mapping is called *monotone* provided that

$$u, v \in K,\ u \leq v, \quad \text{imply} \quad T(u) \leq T(v).$$

In the following we are interested to see under what conditions positive mappings are contractions with respect to Hilbert's projective metric. In order to achieve this, we shall derive some properties of positive mappings with respect to the functions introduced above.

We have the following lemma.

Lemma 6.8. *Let T be a positive monotone mapping which is homogeneous of degree p. Then for any $u, v \in K^+$*

$$m(u,v)^p \leq m(T(u),T(v)) \leq M(T(u),T(v)) \leq M(u,v)^p, \tag{6.24}$$

and if

$$k(T) := \inf\{\lambda : \mathrm{d}(T(u),T(v)) \leq \lambda \mathrm{d}(u,v),\ \mathrm{d}(u,v) < \infty\}, \tag{6.25}$$

where d *is Hilbert's projective metric, then*

$$k(T) \leq p.$$

In particular:

1. *If $p < 1$, then T is a contraction with respect to the projective metric.*
2. *If T is linear, then*

$$\mathrm{d}(T(u),T(v)) \leq \mathrm{d}(u,v), \quad \forall u, v \in K^+.$$

Proof. Since

$$m(u,v)v \leq u \leq M(u,v)v,$$

Inequality (6.24) follows from the monotonicity and homogeneity of T. Using the definition of Hilbert's projective metric and (6.24) we obtain that

$$\mathrm{d}(T(u),T(v)) \leq \log\Big(\frac{M(u,v)}{m(u,v)}\Big)^p = p\mathrm{d}(u,v),$$

from which the result follows. $\square$

Remark 6.9. The constant $k(T)$, above, is called the *contraction ratio* of the mapping T.

We next concentrate on computing the contraction ratio for positive linear mappings T. We define the following constants.

$$\begin{gathered}\Delta(T) := \sup\{\mathrm{d}(T(u),T(v)) : u, v \in K^+\},\\ \Gamma(T) := \frac{e^{\frac{1}{2}\Delta(T)} - 1}{e^{\frac{1}{2}\Delta(T)} + 1}\end{gathered} \tag{6.26}$$

($\Delta(T)$ is called the *projective diameter* of T) and

$$N(T) := \inf\{\lambda : [T(u), T(v)] \le \lambda[u, v],\ u, v \in K^+\}. \tag{6.27}$$

We next establish an extension of a result originally proved by Hopf ([38], [39]) in his studies of integral equations and extended by Bauer [7] to the general setting.

Theorem 6.10. *Let $T : E \to E$ be a linear mapping which is positive with respect to the cone K. Let $u, v \in K^+$ be such that $[u, v] < \infty$. Then*

$$[T(u), T(v)] \le \Gamma(T)[u, v], \tag{6.28}$$

where $\Gamma(T)$ is given by (6.23), i.e. $N(T) \le \Gamma(T)$. Furthermore

$$k(T) \le N(T). \tag{6.29}$$

Proof. If $[u, v] = 0$ (which is the case, if, and only if, u and v are co-linear), then $[T(u), T(v)] = 0$ and the result holds trivially.

In the contrary case, $0 < [u, v] < \infty$, and, since T is a positive operator, we have that for any $u, v \in K^+$, the images of the elements

$$\begin{aligned} p &= u - (m(u, v))v, \\ q &= (M(u, v))v - u, \end{aligned}$$

$T(p)$ and $T(q)$ belong to K^+. Then

$$p + q = [u, v]v,$$

and (see Lemma 6.2)

$$\begin{aligned} m(T(u), T(v)) &= [u, v]m(T(p), T(p) + T(q)) + m(u, v) \\ &= \nu M(u, v) + (1 - \nu)m(u, v), \end{aligned} \tag{6.30}$$

where

$$\nu = m(T(p), T(p) + T(q)).$$

Since T is a positive mapping, it follows from Lemma 6.3 that

$$\nu = \frac{1}{1 + M(T(q), T(p))}.$$

We similarly obtain

$$M(T(u), T(v)) = \mu M(u, v) + (1 - \mu)m(u, v), \tag{6.31}$$

where

$$\mu = M(T(p), T(p) + T(q)) = \frac{1}{1 + m(T(q), T(p))}.$$

Hence

$$[T(u), T(v)] = (\mu - \nu)[u, v], \tag{6.32}$$

where

$$\mu - \nu = \frac{M(T(p), T(q))M(T(q), T(p)) - 1}{(1 + M(T(p), T(q)))(1 + M(T(q), T(p)))} =: \phi(T(p), T(q)). \tag{6.33}$$

We next observe that

$$
\begin{aligned}
\phi(T(p), T(q) &\leq \frac{M(T(p),T(q))M(T(q),T(p)) - 1}{\left(\sqrt{M(T(p),T(q))M(T(q),T(p))} + 1\right)^2} \\
&= \frac{e^{\mathrm{d}(T(p),T(q))} - 1}{\left(\sqrt{e^{\mathrm{d}(T(p),T(q))}} + 1\right)^2} \\
&= \frac{\sqrt{e^{\mathrm{d}(T(p),T(q))}} - 1}{\sqrt{e^{\mathrm{d}(T(p),T(q))}} + 1} \\
&\leq \Gamma(T).
\end{aligned}
\tag{6.34}
$$

proving (6.28).

To verify (6.29), we use the above together with the identities of Lemmas 6.2 and 6.3. Since

$$[T(u), T(v)] \leq N(T)[u, v],$$

we have

$$\frac{1}{m(T(v),T(u))} - \frac{1}{M(T(v),T(u))} \leq N(T)\Big(\frac{1}{m(v,u)} - \frac{1}{M(v,u)}\Big).$$

We now replace v by $cv + u$, $c > 0$ and use Lemma 6.2 to find

$$
\begin{aligned}
&\frac{c[T(v),T(u)]}{(cM(T(v),T(u)) + 1)\,(cm(T(v),T(u)) + 1)} \\
&\leq N(T)\frac{c[v,u]}{(cM(v,u) + 1)\,(cm(v,u) + 1)}.
\end{aligned}
$$

We integrate this inequality with respect to c and obtain

$$\log\frac{cM(T(v),T(u)) + 1}{cm(T(v),T(u)) + 1} \leq N(T)\log\frac{cM(v,u) + 1}{cm(v,u) + 1}.$$

We let $c \to \infty$ and obtain

$$\mathrm{d}(T(v), T(u)) \leq N(T)\mathrm{d}(v, u),$$

or, equivalently

$$\mathrm{d}(T(u), T(v)) \leq N(T)\mathrm{d}(u, v),$$

i.e., (6.29) holds, which completes the proof of the theorem. □

We summarize the above results in the following theorem.

Theorem 6.11. *Let T be a positive monotone mapping which is homogeneous of degree p. Then for any $u, v \in K^+$, with $\mathrm{d}(u, v) < \infty$,*

$$\mathrm{d}(T(u), T(v)) \leq p\mathrm{d}(u, v), \tag{6.35}$$

where d is Hilbert's projective metric.

In particular, if $p < 1$, then T is a contraction with respect to the projective metric. If T is linear, then

$$\mathrm{d}(T(u), T(v)) \leq \Gamma(T)\mathrm{d}(u, v),\ \forall u, v \in K^+. \tag{6.36}$$

Thus, in particular, if $\Delta(T) < \infty$, where $\Delta(T)$ and $\Gamma(T)$ are defined in (6.26), then T is a contraction mapping with respect to Hilbert's projective metric.

6.4. **Completeness criteria.** It follows from Theorem 6.11 that if T is a mapping, satisfying the hypotheses there, it will be a contraction mapping with respect to the projective metric and, hence, if the mapping leaves the unit spheres of the cone K invariant and $\mathbb{M}$ is complete with respect to the topology defined by the metric, then the contraction mapping principle may be applied. We shall now describe situations where completeness prevails.

We shall discuss one such situation, namely, the case that the Banach space E is a Banach space whose norm is monotone with respect to the cone K, i.e.,

$$u, v \in K,\ u \le v,\ \text{then } \|u\| \le \|v\|$$

(e.g. if E is a Banach lattice, see [65], with respect to the partial order induced by the cone K). Each of the cones in the examples discussed earlier generates such a Banach space, as do the cones of nonnegative functions in all L^p- spaces.

We have the following result.

Theorem 6.12. *Let E be a real Banach space whose norm is monotone with respect to a solid cone K. Then*

$$\mathbb{M} := \{u \in \operatorname{int} K : \|u\| = 1\}$$

is complete with respect to Hilbert's projective metric d.

Proof. Assume that $\{u_n\}$ is a Cauchy sequence in $\mathbb{M}$ with respect to the metric d, then for $\epsilon > 0$, given, there exists an integer N, such that

$$n, m \ge N,\ \text{implies that}\ \ 1 \le \frac{M(u_n, u_m)}{m(u_n, u_m)} \le 1 + \epsilon.$$

Furthermore, we have (see the definitions of m and M),

$$m(u_n, u_m)u_m \le u_n \le M(u_n, u_m)u_m \le (1 + \epsilon)m(u_n, u_m)u_m, \tag{6.37}$$

and therefore (using the monotonicity of the norm),

$$\frac{1}{1+\epsilon} \le m(u_n, u_m) \le 1,\ \ n, m \ge N.$$

We next use (6.37) to conclude that

$$0 \le u_n - m(u_n, u_m)u_m \le m(u_n, u_m)\left(e^{\mathrm{d}(u_n,u_m)} - 1\right)u_m,$$

and, therefore

$$\|u_n - m(u_n, u_m)u_m\| \le (e^{\mathrm{d}(u_n,u_m)} - 1).$$

Thus

$$\begin{aligned} \|u_n - u_m\| &\le \|u_n - m(u_n, u_m)u_m\| + \|m(u_n, u_m)u_m - u_m\| \\ &\le (e^{\mathrm{d}(u_n,u_m)} - 1) + (1 - m(u_n, u_m)) \\ &\le \epsilon + \frac{\epsilon}{1+\epsilon}, \end{aligned} \tag{6.38}$$

proving that $\{u_n\}$ is a Cauchy sequence in E. Since E is complete and the unit sphere of E and K are closed, this sequence will have a limit $u \in K$ of norm 1. We next show that $u \in \operatorname{int} K$. This follows from the fact that the boundary of K may be characterized by

$$\partial K = \{v \in K : m(v, w) = 0,\ \forall\ w \in \operatorname{int} K\},$$

(see Lemma 6.2) and that for each $u \in \overline{\mathbb{M}}$, the mapping

$$v \mapsto m(u,v)$$
$$\overline{\mathbb{M}} \to [0,1]$$

is an upper semicontinuous function with respect to the norm. To see this, it suffices to show the sequential upper semicontinuity of m. Thus, let $\{v_n\} \subset \overline{\mathbb{M}}$ be a sequence with

$$v_n \to v,$$

and let $u \in \overline{\mathbb{M}}$. Let

$$\alpha = \limsup_{n\to\infty} m(u, v_n),$$

then $0 \le \alpha \le 1$, and given $\epsilon \in (0,1)$

$$(1-\epsilon)\alpha \le m(u, v_n) \le 1.$$

Hence,

$$(1-\epsilon)\alpha v_n \le u,$$

and consequently,

$$(1-\epsilon)\alpha v \le u,$$

i.e.,

$$(1-\epsilon)\alpha \le m(u,v),$$

showing that $\alpha \le m(u,v)$, proving the upper semicontinuity of m.

Returning to the sequence $\{u_n\}$, above with $u_n \to u$, we see that for fixed m,

$$m(u, u_m) \ge \limsup_{n\to\infty} m(u_n, u_m) \ge \frac{1}{1+\epsilon},$$

and, therefore, $u \in \operatorname{int} K$. One may similarly verify that the mapping M, and hence, d are lower semicontinuous functions, which will further imply that

$$\lim_{n\to\infty} \mathrm{d}(u, u_n) = 0.$$

□

6.5. **Homogeneous operators.** In this section we shall establish an eigenvalue theorem for monotone positive operators which are homogeneous of degree less than one. We have.

Theorem 6.13. *Let E be a real Banach space whose norm is monotone with respect to the cone K. Let*

$$T : K^+ \to K^+$$

be a monotone operator which is homogeneous of degree $p < 1$ and leaves the interior of the cone, $\operatorname{int} K$*, invariant. Then for any positive number μ, there exists $u \in \operatorname{int} K$ such that*

$$T(u) = \mu u. \tag{6.39}$$

Proof. Let

$$f(u) := \frac{T(u)}{\|T(u)\|},$$

then $f : \mathbb{M} \to \mathbb{M}$. We have, by the properties of Hilbert's projective metric that

$$\mathrm{d}(f(u), f(v)) \le p\,\mathrm{d}(u,v), \quad \forall u, v \in \mathbb{M}.$$

Hence f is a contraction mapping. Since $\mathbb{M}$ is complete with respect to this metric, it follows from the contraction mapping theorem that f has a unique fixed point u in $\mathbb{M}$, i.e.

$$u = \frac{T(u)}{\|T(u)\|}$$

or

$$T(u) = \|T(u)\|u.$$

We let $u = \lambda y$ and obtain

$$T(y) = \lambda^{1-p} r y, \quad r = \|T(u)\|,$$

and for given μ choose λ such that $\mu = \lambda^{1-p} r$. $\square$

To provide an example illustrating the above result, we consider the following. Let

$$E := C[0,1],$$

with the usual maximum norm, and let

$$G : [0,1]^2 \to [0,\infty)$$

be a nontrivial continuous function. Let $T : E \to E$ be given by

$$T(u)(t) := \int_0^1 G(t,s)|u(s)|^{p-1} u(s) ds, \tag{6.40}$$

where $0 < p < 1$ is a constant. In E we may consider the solid cone

$$K := \{u \in E : u(t) \geq 0,\ 0 \leq t \leq 1\}.$$

Then T is a monotone operator which is homogeneous of degree p and the norm of E is monotone with respect to K. We hence have the following result.

Example 6.14. Let the above assumptions hold. Then for any $\mu \in (0,\infty)$ there exists a continuous function

$$u : [0,1] \to [0,\infty)$$

with $u : (0,1) \to (0,\infty)$ solving the integral equation

$$\mu u(t) = \int_0^1 G(t,s)|u(s)|^{p-1} u(s) ds. \tag{6.41}$$

6.6. **On positive eigenvectors and eigenvalues.** In this section we shall assume that $T : E \to E$ is a linear operator which is positive with respect to the cone K and satisfies

$$T(K^+) \subset K^+, \quad T(\operatorname{int} K) \subset \operatorname{int} K. \tag{6.42}$$

We shall establish the classical results of Perron-Frobenius and Krein-Rutman (see, for example [46]) about principal eigenvalues of a special class of such operators.

We call a positive linear operator S uniformly positive, provided there exists $u_0 \in \operatorname{int} K$ and a constant $\beta > 1$ such that

$$\lambda(u)u_0 \leq S(u) \leq \beta\lambda(u)u_0, \quad u \in \operatorname{int} K, \tag{6.43}$$

where $\lambda(u)$ is a positive constant depending upon u.

We have the following theorem.

Theorem 6.15. *Let the norm of E be monotone with respect to the cone K and let T be a linear positive operator satisfying (6.42) such that for some integer n, the operator T^n is uniformly positive. Then there exists a unique pair $(\mu, u) \in (0, \infty) \times \mathbb{M}$ such that*

$$T(u) = \mu u. \tag{6.44}$$

Proof. We define the mapping $g : \mathbb{M} \to \mathbb{M}$ by

$$g(u) := \frac{T(u)}{\|T(u)\|},$$

and let

$$f := \underbrace{g \circ \cdots \circ g}_{n},$$

i.e., g composed with itself n times. Then

$$f(u) = \frac{S(u)}{\|S(u)\|},$$

where $S = T^n$. It follows from the properties of Hilbert's projective metric, that

$$\mathrm{d}(f(u), f(v)) = \mathrm{d}(S(u), S(v)), \quad u, v \in \mathbb{M},$$

and that $\mathbb{M}$ is complete, since the norm of E is monotone with respect to the cone K. Thus, f will have a unique fixed point, once we show that f is a contraction mapping, which will follow from Theorem 6.11 once we establish that $\Delta(S) < \infty$.

To compute $\Delta(S)$, we recall the definition of projective diameter (see (6.26)) and find that for any $u, v \in \mathbb{M}$,

$$\mathrm{d}(S(u), S(v)) \le \mathrm{d}(S(u), u_0) + \mathrm{d}(S(v), u_0)$$

and, therefore, by the uniform positivity of T^n,

$$\mathrm{d}(S(u), u_0),\ \mathrm{d}(S(v), u_0) \le \log \beta,$$

implying that

$$\mathrm{d}(S(u), S(v)) \le 2 \log \beta.$$

Thus S, and hence, f, are contraction mappings with respect to the projective metric and therefore, there exists a unique $u \in \mathbb{M}$ such that

$$f(u) = u,$$

i.e. $S(u) = u$, or

$$T^n(u) = \|T^n(u)\| u,$$

and the direction u is unique. Furthermore, since f has a unique fixed point in $\mathbb{M}$, g will have a unique fixed point also, as follows from Theorem 3.6 of Chapter 3. This also implies the uniqueness of the eigenvalue with corresponding unique eigenvector $u \in \operatorname{int} K$, $\|u\| = 1$. □

In the following we provide two examples to illustrate the above theorem. The first example illustrates part of the Perron-Frobenius theorem and the second is an extension of this result to operators on spaces of continuous functions. We remark here that the second result concerns an integral equation which is not given by a compact linear operator (see also [10]).

Example 6.16. Let

$$E = \mathbb{R}^N, \quad K = \{(u_1, u_2, \dots, u_N) : u_i \geq 0,\ i = 1, 2, \dots, N\}.$$

Let $T : K \to K$ be a linear transformation whose $N \times N$ matrix representation is irreducible. Then there exists a unique pair $(\lambda, u) \in (0, \infty) \times \operatorname{int} K$, $\|u\| = 1$, such that

$$Tu = \lambda u.$$

Proof. An $N \times N$ matrix is irreducible (see [48]), provided there does not exist a permutation matrix P such that

$$PTP^T = \begin{pmatrix} B & O \\ C & D \end{pmatrix},$$

where B and D are square submatrices. This is equivalent to saying, that for some positive integer n, the matrix $T^n = (t_{i,j})$ has only positive entries $t_{i,j}, i, j = 1, \dots, N$. Since,

$$\operatorname{int} K = \{(u_1, u_2, \dots, u_N) : u_i > 0,\ i = 1, 2, \dots, N\},$$

if we let

$$m = \min_{i,j} t_{i,j}, \quad M = \max_{i,j} t_{i,j}, \quad u_0 = (1, 1, \dots, 1),$$

then for any $u \in K^+$,

$$m\|u\|_1 u_0 \leq T^n u \leq M\|u\|_1 u_0,$$

where

$$\|u\|_1 = \sum_{i=1}^{N} |u_i|,$$

is the l_1 norm of the vector u. This shows that T^n is a uniformly positive operator. Hence Theorem 6.15 may be applied. $\square$

For many applications of positive matrices (particularly to economics) we refer to [48], [70]. The following example is discussed in [10].

Let again $E := C[0, 1]$, with the usual maximum norm and K the cone of non-negative functions. Let

$$p : [0, 1]^2 \to (0, \infty)$$

be a continuous function. Let

$$0 < I := \inf_{[0,1]^2} p(x, y) \leq \sup_{[0,1]^2} p(x, y) =: \mu I.$$

Suppose

$$g : [0, 1] \to [0, 1]$$

is a continuous function and define $T : E \to E$ by

$$T(u)(x) := \int_0^1 p(x, y)u(y)dy + au(g(x)), \tag{6.45}$$

where a is a positive constant.

We have the following example.

Example 6.17. Let T be defined by (6.45), where p, g, a satisfy the above conditions. Then there exists a unique positive number λ and a continuous function

$$u : [0,1] \to (0,1], \quad \max_{x\in[0,1]} u(x) = 1,$$

such that

$$\lambda u(x) = \int_0^1 p(x,y)u(y)dy + au(g(x)), \quad 0 \le x \le 1. \tag{6.46}$$

To see how the result of Example 6.17 follows from Theorem 6.15 we proceed as follows.

We replace the cone K by the following subcone, which we denote by K_1

$$K_1 := \{u \in K : \max u \le \nu \min u\},$$

where

$$\max u = \max_{x\in[0,1]} u(x), \quad \min u = \min_{x\in[0,1]} u(x),$$

and $\nu > \mu$.

Easy computations show that the norm is monotone with respect to the new cone K_1, and that for any $x \in [0,1]$

$$(I + 1) \min u \le (Tu)(x) \le \nu(\mu I + a) \min u.$$

This inequality shows that the operator T is a uniformly positive operator, as required by the theorem. Furthermore, letting $v = Tu$, we obtain that

$$\max v \le \mu I \int_0^1 udx + a\nu \min u,$$
$$\min v \ge I \int_0^1 udx + a \min u,$$

and therefore

$$\frac{\max v}{\min v} \le \frac{\mu I \int_0^1 udx + a\nu \min u}{I \int_0^1 udx + a \min u} \le \nu,$$

showing that $T : K_1 \to K_1$. We may, hence, apply Theorem 6.15. We remark that the operator T, above, is not a compact operator and hence techniques based on Leray-Schauder degree and fixed point theory may not be applied here.

Let us consider another situation, to which the results derived above apply.

Theorem 6.18. *Let the norm of E be monotone with respect to the solid cone K and let*

$$T : K^+ \to \operatorname{int} K,$$

be a positive, linear, and compact operator. Then T has a unique eigenvector in $\mathbb{M}$.

Proof. Let $v \in \mathbb{M}$ be a fixed element. Then $T(v) \in \operatorname{int} K$. Hence, there exist positive numbers $\epsilon > 0$ and α, depending on v, such that

$$\alpha v \le T(v), \text{ and } \overline{B}(v, \epsilon) \subset \operatorname{int} K.$$

For each integer $n = 1, 2, \dots,$ we define the mapping $S_n : \mathbb{M} \to \mathbb{M}$, by

$$S_n(u) := \frac{T(u + \frac{1}{n}v)}{\|T(u + \frac{1}{n}v)\|}.$$

It follows from earlier considerations, that S_n will be a contraction mapping with respect to Hilbert's metric, once we show that

$$\mathrm{d}(S_n(u), v) \le c,$$

where c is a constant, independent of $u \in \mathbb{M}$. To see this, we observe that

$$S_n(u) \ge \frac{T(\frac{1}{n}v)}{\|T(u + \frac{1}{n}v)\|} \ge \frac{\alpha}{n(1+\frac{1}{n})\|T\|} v,$$

hence,

$$m(S_n(u), v) \ge \frac{\alpha}{n(1+\frac{1}{n})\|T\|}.$$

Furthermore, since $v - \epsilon S_n(u) \in \operatorname{int} K$, it follows that

$$M(S_n(u), v) \le \frac{1}{\epsilon}.$$

Thus,

$$\mathrm{d}(S_n(u), v) \le \log \frac{n(1+\frac{1}{n})\|T\|}{\epsilon\alpha}.$$

Hence, for each $n = 1, 2, \dots$, there exists a unique $u_n \in \mathbb{M}$ such that

$$S_n(u_n) = u_n,$$

i.e.,

$$T(u_n + \frac{1}{n}v) = \lambda_n u_n, \tag{6.47}$$

where

$$\lambda_n = \|T(u_n + \frac{1}{n}v)\|.$$

This implies that $\lambda_n \le 2\|T\|$. Since,

$$m(u_n, v)v \le u_n,$$

and $m(u_n, v) > 0$ is the maximal number λ such that $\lambda v \le u_n$, we obtain that

$$\begin{aligned} u_n &= \frac{1}{\lambda_n} T(u_n + \frac{1}{n}v) \\ &\ge \frac{1}{\lambda_n} T(m(u_n, v)v + \frac{1}{n}v) \\ &\ge \frac{\alpha}{\lambda_n}(m(u_n, v) + \frac{1}{n})v. \end{aligned}$$

Therefore, by the maximality of $m(u_n, v)$, we obtain

$$\frac{\alpha}{\lambda_n}\Big(m(u_n, v) + \frac{1}{n}\Big) \le m(u_n, v),$$

i.e.,

$$\alpha\Big(1 + \frac{1}{nm(u_n, v)}\Big) \le \lambda_n.$$

The sequence $\{\lambda_n\}$ is therefore uniformly bounded away from zero and, as has been shown above, also bounded above. It therefore has a convergent subsequence, $\{\lambda_{n_i}\}$, converging, say, to $\lambda > 0$. We now use equation (6.47) and the compactness of T to obtain that the sequence $\{u_{n_i}\}$ has a convergent subsequence, converging, to, say, u, $\|u\| = 1$, and, since T is continuous,

$$T(u) = \lambda u.$$

Since it must be the case that $u \in K^+$, we see that, in fact, $u \in \operatorname{int} K$, and hence, $u \in \mathbb{M}$.

If $u_1, u_2 \in \mathbb{M}$ are such that

$$T(u_1) = \lambda_1 u_1,\ T(u_2) = \lambda_2 u_2,$$

then

$$u_1 \geq m(u_1, u_2)u_2,\ m(u_1, u_2) > 0,$$

and

$$\lambda_1 u_1 = T(u_1) \geq m(u_1, u_2)T(u_2) = m(u_1, u_2)\lambda_2 u_2,$$

hence,

$$u_1 \geq \Big(m(u_1, u_2)\frac{\lambda_2}{\lambda_1}\Big)u_2.$$

On the other hand

$$m(u_1, u_2) = \sup\{\alpha : \alpha u_2 \leq u_1\},$$

which implies that $\lambda_1 \geq \lambda_2$. Reversing the roles of u_1 and u_2, we obtain $\lambda_1 \leq \lambda_2$, and thus,

$$\lambda_1 = \lambda_2 = \lambda.$$

Since

$$u_1 \geq m(u_1, u_2)u_2,\ m(u_1, u_2) > 0,$$

we have

$$T(u_1 - m(u_1, u_2)u_2) \in \operatorname{int} K,$$

unless

$$u_1 - m(u_1, u_2)u_2 = 0.$$

On the other hand

$$T(u_1 - m(u_1, u_2)u_2) = \lambda(u_1 - m(u_1, u_2)u_2),$$

and thus, if

$$u_1 - m(u_1, u_2)u_2 \in K^+,$$

then

$$u_1 - m(u_1, u_2)u_2 \in \operatorname{int} K,$$

and we obtain a contradiction to the maximality of $m(u_1, u_2)$. Hence,

$$u_1 = m(u_1, u_2)u_2$$

and since, $\|u_1\| = \|u_2\| = 1$, we have that $m(u_1, u_2) = 1$ and we have proved that $u_1 = u_2$. □

7. Integral equations

In this chapter, we shall present the basic existence and uniqueness theorem for solutions of initial value problems for systems of ordinary differential equations. We shall also discuss the existence of mild solutions of integral equations which under additional assumptions provide the existence of solutions of initial value problems for parabolic partial differential equations. We conclude the chapter by presenting some results about functional differential equations and integral equations.

7.1. **Initial value problems.** To this end let D be an open connected subset of $\mathbb{R} \times E$, where E is a Banach space, and let

$$f : D \to E$$

be a continuous and bounded mapping, i.e., it maps bounded sets in D to bounded sets in E.

We consider the differential equation

$$u' = f(t, u), \quad ' = \frac{d}{dt}. \tag{7.1}$$

and seek sufficient conditions for the existence of solutions of (7.1), where $u \in C^1(I, E)$, with I an interval, $I \subset \mathbb{R}$, is called a solution, if $(t, u(t)) \in D$, $t \in I$ and

$$u'(t) = f(t, u(t)), \quad t \in I.$$

By an *initial value problem* we mean the following:

Given a point $(t_0, u_0) \in D$ we seek a solution u of (7.1) defined on some open interval I such that

$$u(t_0) = u_0, \quad t_0 \in I. \tag{7.2}$$

We have the following proposition whose proof is straightforward:

Proposition 7.1. *A function $u \in C^1(I, E)$, with $I \subset \mathbb{R}$, and I an interval containing t_0 is a solution of the initial value problem* (7.1)*, satisfying the initial condition* (7.2) *if, and only if, $(t, u(t)) \in D$, $t \in I$, and*

$$u(t) = u(t_0) + \int_{t_0}^{t} f(s, u(s))ds. \tag{7.3}$$

The integral in (7.3) is a Riemann integral of a continuous function.

We shall now, using Proposition 7.1, establish one of the classical and basic existence and uniqueness theorems.

7.2. **The Picard-Lindelöf theorem.** We say that f satisfies a *local Lipschitz* condition on the domain D, provided for every closed and bounded set $K \subset D$, there exists a constant $L = L(K)$, such that for all $(t, u_1),\ (t, u_2) \in K$

$$\|f(t, u_1) - f(t, u_2)\| \leq L\|u_1 - u_2\|,$$

where $\|\cdot\|$ is the norm in the space E. For such functions, one has the following existence and uniqueness theorem. This result is usually called the Picard-Lindelöf theorem.

Theorem 7.2. *Assume that $f : D \to E$ is a continuous and bounded mapping which satisfies a local Lipschitz condition on the domain D. Then for every (t_0, u_0) in D, equation* (7.1) *has a unique solution on some interval I satisfying the initial condition* (7.2)*.*

We remark that the theorem as stated is a *local* existence and uniqueness theorem, in the sense that the interval I, where the solution exists will depend upon the initial condition.

Proof. Let $(t_0, u_0) \in D$, then, since D is open, there exist positive constants a and b such that

$$Q = \{(t, u) : |t - t_0| \leq a,\ \|u - u_0\| \leq b\} \subset D.$$

Let L be the Lipschitz constant for f associated with the set Q. Further let

$$m := \sup_{(t,u)\in Q} \|f(t,u)\|,$$

$$\alpha := \min\{a, \frac{b}{m}\}.$$

Let $\tilde{L}$ be any constant, $\tilde{L} > L$, and define $I = [t_0 - \alpha, t_0 + \alpha]$, and

$$\mathbb{M} := \{u \in C\,(I, E) : u(I) \subset B_r(u_0)\}.$$

In $C\,(I, E)$ we define a new norm as follows:

$$\|u\|_{\mathbb{M}} := \max_{|t-t_0|\le\alpha} e^{-\tilde{L}|t-t_0|}\|u(t)\|.$$

And we let $\mathrm{d}(u, v) = \|u - v\|_{\mathbb{M}}$, then $(\mathbb{M}, \mathrm{d})$ is a complete metric space. Next define the operator T on $\mathbb{M}$ by:

$$(Tu)(t) := u_0 + \int_{t_0}^{t} f(s, u(s))ds,\ |t - t_0| \le \alpha. \tag{7.4}$$

Then

$$\|(Tu)(t) - u_0\| \le \Big|\int_{t_0}^{t} \|f(s, u(s))\|ds\Big|,$$

and, since $u \in \mathbb{M}$,

$$\|(Tu)(t) - u_0\| \le \alpha m \le b.$$

Hence $T : \mathbb{M} \to \mathbb{M}$. Computing further, we obtain, for $u, v \in \mathbb{M}$ that

$$\begin{aligned}\|(Tu)(t) - (Tv)(t)\| &\le \Big|\int_{t_0}^{t} \|f(s, u(s)) - f(s, v(s))\|ds\Big|\\ &\le L\Big|\int_{t_0}^{t} \|u(s) - v(s)\|ds\Big|,\end{aligned}$$

and hence

$$\begin{aligned}e^{-\tilde{L}|t-t_0|}\|(Tu)(t) - (Tv)(t)\| &\le e^{-\tilde{L}|t-t_0|}L\Big|\int_{t_0}^{t} \|u(s) - v(s)\|ds\Big|\\ &\le \frac{L}{\tilde{L}}\|u - v\|_{\mathbb{M}}.\end{aligned}$$

Therefore,

$$\mathrm{d}(Tu, Tv) \le \frac{L}{\tilde{L}}\mathrm{d}(u, v),$$

proving that T is a contraction mapping. The result therefore follows from the contraction mapping principle. □

We remark that, since T is a contraction mapping, the contraction mapping theorem gives a constructive means for the solution of the initial value problem in Theorem 7.2 and the solution may in fact be obtained via an iteration procedure. This procedure is known as *Picard iteration.*

7.3. **Abel-Liouville integral equations.** In establishing the Picard-Lindelöf theorem we studied an associated integral equation

$$u(t) = u(t_0) + \int_{t_0}^{t} f(s, u(s))ds. \tag{7.5}$$

By translating the time variable appropriately and changing variables, there is no loss in generality in assuming that $t_0 = 0$ in (7.5). In this section we shall consider a generalization of this integral equation, namely an equation of Abel-Liouville type

$$u(t) = v(t) + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} f(t, s, u(s))ds, \quad 0 \le t \le a,\ a > 0, \tag{7.6}$$

where $\mu \in (0, 1]$, and Γ is the Gamma function. Choosing $\mu = 1$ and $v(t) =$ constant, one clearly obtains (7.5) as a special case.

While the result to follow will be valid for Banach space valued functions, we shall restrict ourselves to the case of real-valued functions, remarking that the treatment will be similar in the more general setting. The discussion below follows the paper of Rainermann and Stallbohm [61], see also [42].

We shall introduce the following notations and make the assumptions:

(1) $v : [0, a] \to \mathbb{R}$ is a continuous function.

(2)
$$S := \{z : |v(s) - z| \le b,\ 0 \le s \le a\},$$
where $b > 0$ is a fixed positive constant.

(3)
$$\Delta := \{(t, s) : 0 \le s \le t \le a\}.$$

(4) $f : \Delta \times S \to \mathbb{R}$ is a continuous function and
$$M := \max\{|f(t, s, z)| : (t, s, z) \in \Delta \times S\}.$$

(5) for all $(t, s, z_1), (t, s, z_2) \in \Delta \times S$,
$$s^{\mu}|f(t, s, z_1) - f(t, s, z_2)| \le \Gamma(\mu + 1)|z_1 - z_2|. \tag{7.7}$$

(6)
$$\alpha := \min\Big(a, \Big(\Gamma(\mu+1)\frac{b}{M}\Big)^{1/\mu}\Big).$$

(7)
$$\mathbb{M}_1 := \{w \in C[0, \alpha] : w(0) = v(0),\ \max_{0 \le t \le \alpha} |w(t) - v(t)| \le b\}.$$

(8) We define the operator $T : \mathbb{M}_1 \to C[0, \alpha]$ by

$$T(w)(t) := v(t) + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} f(t, s, u(s))ds, \quad 0 \le t \le \alpha. \tag{7.8}$$

Lemma 7.3. *With the above notation and assumptions, we have*

$$T : \mathbb{M}_1 \to \mathbb{M}_1.$$

Proof. It is clear that $T : \mathbb{M}_1 \to C[0, \alpha]$. Further

$$T(w)(0) = v(0),$$

and for $0 \le t \le \alpha$,

$$
\begin{aligned}
|T(w)(t) - v(t)| &\le M \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} ds \\
&\le M \frac{t^\mu}{\Gamma(\mu+1)} \\
&\le M \frac{\alpha^\mu}{\Gamma(\mu+1)} \\
&\le b.
\end{aligned}
$$

We next let

$$\mathbb{M} := T(\mathbb{M}_1)$$

and find a metric d on $\mathbb{M}$ so that $(\mathbb{M}, \mathrm{d})$ becomes a complete metric space. The above lemma, of course, implies that $T : \mathbb{M} \to \mathbb{M}$.

We next define $\mathrm{d} : \mathbb{M} \times \mathbb{M} \to [0, \infty)$ by

$$\mathrm{d}(w_1, w_2) := \sup_{t \in [0,\alpha]} |w_1(t) - w_2(t)|.$$

Then, it is easily seen that d is a metric on $\mathbb{M}$ and, using the continuity assumptions imposed on f, that for all $u_1, u_2 \in \mathbb{M}$,

$$
\begin{aligned}
|T(u_1)(t) - T(u_2)(t)| &\le \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} |f(t,s,u_1(s)) - f(t,s,u_2(s))| ds \\
&\le t^\mu \max_{0 \le s \le t} |f(t,s,u_1(s)) - f(t,s,u_2(s))|,
\end{aligned}
$$

implying that (recall that $u_1(0) = u_2(0)$)

$$t^{-\mu} |T(u_1)(t) - T(u_2)(t)| \to 0.$$

Also

$$
\begin{aligned}
|T(u_1)(t) - T(u_2)(t)| &\le \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} |f(t,s,u_1(s)) - f(t,s,u_2(s))| ds \\
&\le \int_0^t (t-s)^{\mu-1} s^{-\mu} |u_1(s) - u_2(s)| ds.
\end{aligned}
$$

On the other hand, if $u_1 \ne u_2$, then $\mathrm{d}(u_1, u_2) > 0$. Thus, if $\mathrm{d}(T(u_1), T(u_2)) \ne 0$, we may choose $t_1 \in (0, \alpha]$ so that

$$
\begin{aligned}
\mathrm{d}(T(u_1), T(u_2)) &= t_1^{-\mu} |T(u_1)(t_1) - T(u_2)(t_1)| \\
&\le \frac{\Gamma(\mu+1)}{\Gamma(\mu)} t_1^{-\mu} \int_0^t (t-s)^{\mu-1} s^{-\mu} |u_1(s) - u_2(s))| ds \\
&< \mu t_1^{-\mu} \int_0^t (t-s)^{\mu-1} ds \mathrm{d}(u_1, u_2),
\end{aligned}
$$

where the latter strict inequality follows from the continuity of the functions involved and the fact that there exists $s \in (0, t_1)$ such that

$$s^{-\mu} |u_1(s) - u_2(s))| < \mathrm{d}(u_1, u_2).$$

Using calculations and considerations like the above, it is straightforward to show that the family $\mathbb{M}$ is an equicontinuous family of functions which is uniformly bounded. Hence, for a given $u \in \mathbb{M}$ the sequence of iterates $\{T^n(u)\}$ will have a convergent subsequence, say $\{T^{n_j}(u)\}$, converging to some function $v \in \mathbb{M}_1$. The sequence $\{T^{n_j+1}(u)\}$, will therefore converge to $T(v) \in \mathbb{M}$.

It therefore follows from Edelstein's contraction mapping principle, Theorem III.3.4, that equation (7.6) has a unique solution in the metric space $\mathbb{M}$. $\square$

7.4. **Mild solutions.** Let E be a Banach space and let

$$S : [0, \infty) \to \mathcal{L}(E, E)$$

($\mathcal{L}(E, E)$ are the bounded linear maps from E to E) be a family of bounded linear operators which form a *strongly continuous semigroup* of operators, i.e.,

$$\begin{gathered} S(t+s) = S(t)S(s), \ \forall t, s \geq 0, \\ S(0) = \mathrm{id}, \ \text{the identity mapping} \\ \lim_{t \to t_0} S(t)x = S(t_0)x, \quad \forall t_0 \geq 0, x \in E. \end{gathered}$$

For such semigroups it is the case that there exist constants (see [76]) $\beta \in \mathbb{R}$ and $M > 0$ such that

$$\|S(t)\| \leq M e^{\beta t}, \quad t \geq 0.$$

(In this section, we shall use $\|\cdot\|$ for both the norm in E, and the norm in $\mathcal{L}(E, E)$.)

Let $f : [0, \infty) \to E$ be a continuously differentiable function and let

$$u(t) := S(t)x + \int_0^t S(t-s)f(s)ds. \tag{7.9}$$

The integral in (7.9) is a Riemann integral of a continuous function. Then for $x \in D(A)$, where

$$D(A) := \{x : Ax := \lim_{t \to 0+} \frac{1}{t}\left(S(t)x - x\right) \ \text{exists}\},$$

(the operator A is called the *infinitesimal generator* of the semigroup $\{S(t); t \geq 0\}$) u, given by (7.9) is the solution of the initial value problem (7.10) (see [69], [76])

$$\frac{du}{dt} = Au(t) + f(t), \quad u(0) = x. \tag{7.10}$$

On the other hand, if it is only assumed that f is continuous, then a solution of (7.9) need not necessarily be a solution of (7.10). One calls the function u defined by (7.9) a *mild solution* of (7.10). Thus a mild solution is defined for all $x \in E$, even if $D(A)$ is a proper subset of E.

For example, one may easily verify that, if $A \in \mathcal{L}(E, E)$, then

$$S(t) := e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$$

is a strongly continuous semigroups of operators and if u is defined by (7.9), then u solves (7.10) for any $x \in E$ and any continuous $f : [0, \infty) \to E$.

We have the following theorem for the existence of mild solutions.

Theorem 7.4. *Let $S : [0, \infty) \to \mathcal{L}(E, E)$ be a strongly continuous semigroup of operators and let*

$$f : [0, \infty) \times E \to E$$

be a continuous mapping which satisfies the Lipschitz condition

$$\|f(t, u) - f(t, v)\| \leq L\|u - v\|, \forall t \in [0, \infty), \quad \forall u, v \in E, \tag{7.11}$$

where L is a positive constant. Then the equation

$$u(t) = S(t)x + \int_0^t S(t-s)f(s,u(s))ds \tag{7.12}$$

has a unique continuous solution $u : [0,\infty) \to E$, i.e., the initial value problem

$$\frac{du}{dt} = Au(t) + f(t,u), \quad u(0) = x, \tag{7.13}$$

where A is the generator of $\{S(t); t \geq 0\}$, has a unique mild solution.

Proof. Given any $\tau > 0$, we shall prove the existence of a unique continuous solution of (7.12) defined on the interval $[0,\tau]$. Since τ is chosen arbitrarily the result will follow, observing that if u solves (7.12) on an interval $[0,\tau_1]$ it will solve (7.12) on an interval $[0,\tau_2]$, for any $\tau_2 \leq \tau_1$.

Let us consider the space $\mathcal{E} := C([0,\tau],E)$ with norm

$$\|u\|_\tau := \max_{[0,\tau]} e^{-(\tilde{L}+\beta)t}\|u(t)\|,$$

where $\tilde{L} > 0$ is to be chosen, and define the mapping $T : \mathcal{E} \to \mathcal{E}$ by

$$T(u)(t) := S(t)x + \int_0^t S(t-s)f(s,u(s))ds. \tag{7.14}$$

For $u, v \in \mathcal{E}$ we compute

$$e^{-(\tilde{L}+\beta)t}\|T(u)(t) - T(v)(t)\| \leq MLe^{-\tilde{L}t}\int_0^t e^{\tilde{L}s}e^{-(\tilde{L}+\beta)s}\|u(s)-v(s)\|ds.$$

It follows that

$$\|T(u) - T(v)\|_\tau \leq \frac{ML}{\tilde{L}}\|u-v\|_\tau. \tag{7.15}$$

Therefore T is a contraction mapping provided

$$\frac{ML}{\tilde{L}} < 1. \tag{7.16}$$

We choose $\tilde{L}$ this way and obtain the existence of a unique fixed point u of T in $\mathcal{E}$. $\square$

7.5. **Periodic solutions of linear systems.**

Mild periodic solutions. Let $f : [0,\infty) \to E$ be a continuous function which is periodic of period $\tau > 0$, i.e.

$$f(t+\tau) = f(t), \quad t \geq 0.$$

In this section we show that the integral equation

$$u(t) = S(t)x + \int_0^t S(t-s)f(s)ds, \tag{7.17}$$

where $\{S(t) : t \geq 0\}$ is a strongly continuous semigroup of operators, with infinitesimal generator A, has a unique periodic solution, provided the semigroup is a so-called *asymptotically stable* semigroup. I.e., we shall show that the problem

$$\frac{du}{dt} = Au(t) + f(t), \tag{7.18}$$

has a unique mild solution which is also periodic of period τ.

To this end, we shall employ the contraction mapping principle as given by Theorem 3.6 of Chapter 2.

We call the semigroup $\{S(t) : t \geq 0\}$ asymptotically stable, provided there exist constants $M > 0,\ \beta > 0$ such that

$$\|S(t)\| \leq Me^{-\beta t}, \quad t \geq 0.$$

Well known examples of such asymptotically stable semigroups are given in the finite dimensional case by e^{At}, where A is an $N \times N$ matrix all of whose eigenvalues have negative real parts (see e.g. [36]), or in the infinite dimensional case by certain parabolic partial differential equations (see e.g. [29], [69]).

Let us define the operator $T : E \to E$ by

$$T(x) := S(\tau)x + \int_0^\tau S(\tau - s)f(s)ds. \tag{7.19}$$

It then follows from Theorem 7.4 and the periodicity of the function f, that

$$T^n(x) = S(n\tau)x + \int_0^{n\tau} S(n\tau - s)f(s)ds, \tag{7.20}$$

for any positive integer n. We have

$$\begin{aligned} \|T^n(x) - T^n(y)\| &= \|S(n\tau)x - S(n\tau)(y)\| \\ &\leq \|S(n\tau)\|\|x - y\| \\ &\leq Me^{-\beta n\tau}\|x - y\|, \quad \forall x, y \in E. \end{aligned} \tag{7.21}$$

Since

$$Me^{-\beta n\tau} < 1,$$

for n, large enough, we have that T^n is a contraction mapping for such n. It follows therefore that T^n, hence T, has a unique fixed point $x \in E$. The solution

$$u(t) = S(t)x + \int_0^t S(t - s)f(s)ds, \tag{7.22}$$

is a periodic function of period τ.

Summarizing the above, we have proved the following theorem.

Theorem 7.5. *Let $\{S(t) : t \geq 0\}$ be an asymptotically stable, strongly continuous semigroup with infinitesimal generator A. Then for any continuous $f : [0, \infty) \to E$ which is periodic of period $\tau > 0$, there exists a unique mild periodic solution u, of period τ of equation (7.18).*

7.6. **The finite dimensional case.** We consider next the system of equations

$$\frac{du}{dt} = Au(t) + f(t), \tag{7.23}$$

where A is an $N \times N$ matrix and $f : \mathbb{R} \to \mathbb{R}^N$, is a function of period T.

We shall assume here that A is a matrix all of whose eigenvalues have nonzero real part. If this is the case then there exists a nonsingular matrix P such that (see, e.g. [70])

$$P^{-1}AP = \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix},$$

where A_1 is an $N_1 \times N_1$ and A_2 is an $N_2 \times N_2$ matrix with $N_1 + N_2 = N$, and all eigenvalues of A_1 have negative real parts and those of A_2 have positive real parts, and the matrices O are zero matrices of appropriate dimensions. Hence if we make the transformation $u = Pv$, the the system (7.23 becomes

$$\frac{dv}{dt} = P^{-1}APv(t) + P^{-1}f(t). \tag{7.24}$$

This is a decoupled system which we may rewrite as

$$\begin{aligned} \frac{dv_1}{dt} &= A_1v_1(t) + f_1(t) \\ \frac{dv_2}{dt} &= A_2v_2(t) + f_2(t), \end{aligned} \tag{7.25}$$

where

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad P^{-1}f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

On the other hand v is a solution of (7.25) if and only if

$$w = \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} v_1(t) \\ v_2(-t) \end{pmatrix}$$

is a solution of

$$\begin{aligned} \frac{dw_1}{dt} &= A_1v_1(t) + f_1(t) \\ \frac{dw_2}{dt} &= -A_2w_2(t) - f_2(-t). \end{aligned} \tag{7.26}$$

The fundamental solution $S(t)$ of the system (7.26) is given by

$$S(t) = e^{Bt},$$

where B is the matrix

$$B = \begin{pmatrix} A_1 & O \\ O & -A_2 \end{pmatrix},$$

all of whose eigenvalues have negative real part. Hence there exist constants $M > 0$, $\beta > 0$, such that

$$\|S(t)\| \le Me^{-\beta t}.$$

We may therefore apply Theorem 7.5 to conclude that equation (7.26) and hence (7.23) have unique periodic solutions.

We remark here that the existence of a unique periodic solution of (7.23) also easily follows from the fact that A is assumed not to have any eigenvalues with zero real part, and hence that

$$\mathrm{id} - e^{AT},$$

where id is the $N \times N$ identity matrix, is nonsingular for any $T \neq 0$. Furthermore the reduction made above shows that, without loss in generality we may assume that all the eigenvalues of A have negative real parts, say

$$\mathrm{Re}(\lambda_i) \le -\beta < 0, \tag{7.27}$$

where β is a a positive constant and $\lambda_1, \dots, \lambda_N$ are the eigenvalues of A. Using a further change of basis (using the Jordan canonical form of A) we may assume that A has the form (the, perhaps, unconventional labeling has been chosen for convenience's sake)

$$A = \begin{pmatrix} \lambda_N & a_{2,N}, & \dots, & a_{N,N} \\ 0 & \lambda_{N-1}, & \dots & a_{N-1,N} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \lambda_1 \end{pmatrix}.$$

We thus consider the system

$$\frac{du}{dt} = Au(t) + f(t), \tag{7.28}$$

where

$$u = \begin{pmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_1 \end{pmatrix}, \quad f = \begin{pmatrix} f_N \\ f_{N-1} \\ \vdots \\ f_1 \end{pmatrix}.$$

If then u is a $T-$ periodic solution of (7.28) it must be the case that

$$u_1(t) = e^{\lambda_1 t}\Big(c + \int_0^t e^{-\lambda_1 s} f_1(s) ds\Big).$$

Since u_1 is periodic, it must be bounded on the real line and hence,

$$c + \int_0^t e^{-\lambda_1 s} f_1(s) ds \to 0, \quad \text{as } t \to -\infty;$$

i.e.,

$$c = -\int_0^{-\infty} e^{-\lambda_1 s} f_1(s) ds$$

or

$$u_1(t) = \int_{-\infty}^t e^{\lambda_1 (t-s)} f_1(s) ds.$$

Therefore,

$$|u_1(t)| \le \frac{1}{\beta}\|f_1\|,$$

(see (7.27) for the choice of β) where

$$\|f_1\| = \sup_{t\in\mathbb{R}} |f_1(t)|.$$

We next consider the component u_2 of u. Since u_1 has been found (and estimated), we can employ a similar argument and the estimate on u_1 to find that

$$|u_2(t)| \le \frac{1}{\beta}\|a_{2,N} u_1 + f_2\|,$$

and thus

$$|u_2(t)| \le \frac{c_2}{\beta} \max\{\|f_1\|, \|f_2\|\},$$

where the constant c_2 only depends upon the matrix A. Using an induction argument one obtains for $i = 1, \dots, N$

$$|u_i(t)| \le \frac{c_i}{\beta} \max\{\|f_1\|, \dots, \|f_i\|\}$$

with constants c_i depending on A, only. We therefore have, letting

$$\|u\| = \max_{i=1,\dots,N} \|u_i\|, \quad \|f\| = \max_{i=1,\dots,N} \|f_i\|,$$

that

$$\|u\| \le \frac{c}{\beta}\|f\|,$$

where c is a constant, depending upon A only. We summarize the above in the following theorem.

Theorem 7.6. *Let A be a matrix none of whose eigenvalues has zero real part. Then for any given forcing term f, a periodic function of period T, there exists a unique periodic solution u of (7.28). Further, there exists a constant c which depends upon A only, such that*

$$\|u\| \le \frac{c}{\beta}\|f\|,$$

with the norms defined above.

We note that the conclusion of Theorem 7.6 is equally valid if we replace the requirement that f be periodic with the requirement that f be bounded.

7.7. **Almost periodic differential equations.** In this section we shall return to the equation

$$\frac{du}{dt} = Au(t) + f(t), \tag{7.29}$$

and the more general nonlinear equation

$$\frac{du}{dt} = Au(t) + f(t,u), \tag{7.30}$$

where it is assumed that A is an $N \times N$ matrix and either

$$f : \mathbb{R} \to \mathbb{R}^N \quad \text{or} \quad f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

is a function which is *almost periodic* in the t variable (see definitions below). It will be assumed that all eigenvalues of A have nonzero real part which will imply that for almost periodic f equation (7.29) has a unique almost periodic solution. This fact will be employed for the study of equation (7.30) under the assumption that f satisfies a Lipschitz condition with respect to the dependent variable, in which case (7.30) will be shown to have a unique almost periodic solution, provided the Lipschitz constant of f is small enough. Our presentation relies mainly on the work of Coppel [21], see also [22], [4] and [31], and Theorem 7.6, above.

7.8. **Bounded solutions.** We consider again the system (7.28) under the assumption that none of the eigenvalues of A have zero real part and f a continuous (not necessarily periodic) function. It then follows from the superposition principle that, if (7.29) has a bounded solution, no other bounded solutions may exist (the unperturbed system

$$u' = Au,$$

has the zero solution as the only bounded solution). Furthermore the discussion above, proving Theorem 7.6, may be used to establish the following result.

Theorem 7.7. *Let A be a matrix none of whose eigenvalues has zero real part. Then for any given forcing term f, with f bounded on $\mathbb{R}$, there exists a unique bounded solution u of (7.29). Further, there exists a constant c which depends upon A only, such that*

$$\|u\| \le \frac{c}{\beta}\|f\|,$$

with the norms as defined before.

7.9. **Almost periodic functions.** Let us denote by V the set of continuous functions

$$V := \{f : \mathbb{R} \to \mathbb{R}^N : \exists T > 0 : f(t+T) = f(t),\ t \in \mathbb{R}\};$$

i.e., the set of all continuous functions which are periodic of some period, the period not being fixed. We then define

$$\tilde{V} := \operatorname{span}(V),$$

i.e., the smallest vector space containing V. We note that for $f \in \tilde{V}$,

$$\|f\| := \sup_{t\in\mathbb{R}} \max_{i=1,\dots,N} |f_i(t)| < \infty,$$

and that $\|\cdot\|$ defines a norm in $\tilde{V}$. We then denote by E the completion of $\tilde{V}$ with respect to this norm (the norm of uniform convergence on the real line). This space is the space of *almost periodic* functions which, by definition, includes all periodic and quasiperiodic functions (i.e., finite linear combinations of periodic functions having possibly different periods). It follows that an almost periodic function is the uniform limit (uniform on the real line) of a sequence of quasiperiodic functions.

For detailed studies of almost periodic functions we refer the reader to [4], [12], [30], [22], and [31].

7.10. **Almost periodic systems.** In this section we shall first establish an extension to almost periodic systems of Theorem 7.5 and then use it to establish the existence of a unique almost periodic solution of system (7.30), in case the nonlinear forcing term f satisfies a Lipschitz condition with respect to the dependent variable u. We have the following theorem.

Theorem 7.8. *Let A be a matrix none of whose eigenvalues has zero real part. Then for any given almost periodic forcing term f, there exists a unique almost periodic solution u of (7.29). Further, there exists a constant c which depends upon A only, such that*

$$\|u\| \le \frac{c}{\beta}\|f\|,$$

with the norms defined above.

Proof. Let us consider system (7.29) in case the forcing term f is quasiperiodic. In this case f may be written as a finite linear combination of periodic functions, say

$$f(t) = \sum_{i=1}^{k} f_i(t),$$

where f_i has period T_i, $i = 1, \dots, k$. It follows from Theorem 7.5 that each of the systems

$$u' = Au + f_i(t),$$

has a unique periodic solution $u_i(t)$, of period T_i, $i = 1, \dots, k$. Hence, the superposition principle and Theorem 7.6 imply that (7.30) has the unique quasiperiodic solution

$$u(t) = \sum_{i=1}^{k} u_i(t),$$

and

$$\|u\| \le \frac{c}{\beta}\|f\|.$$

On the other hand, if f is an almost periodic function, there exists a sequence of quasiperiodic functions $\{f_n(t)\}$ such that

$$f(t) = \lim_{n\to\infty} f_n(t),$$

where the limit is uniform on the real line. We let $\{u_n(t)\}$ be the sequence of quasiperiodic solutions of (7.29) with f replaced by f_n, $n = 1, 2, \dots$. Then

$$\|u_n - u_m\| \le \frac{c}{\beta}\|f_n - f_m\|.$$

Hence, $\{u_n(t)\}$ is a Cauchy sequence of quasiperiodic functions which is uniform on the real line and therefore must converge to an almost periodic function u. Using equivalent integral equations, as in Section 1, one shows that u solves (7.29) and, since u is bounded it must satisfy

$$\|u\| \le \frac{c}{\beta}\|f\|.$$

□

We next consider the problem (7.30). We have the following theorem:

Theorem 7.9. *Let A be a matrix none of whose eigenvalues has zero real part. Let*

$$f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

be such that

$$|f(t, u) - f(t, v)| \le L|u - v|, \quad \forall t \in \mathbb{R},\ \forall u, v \in \mathbb{R}^N \tag{7.31}$$

and let $f(\cdot, u)$ be almost periodic for each $u \in \mathbb{R}^N$. Then there exists a constant L_0 such that for each

$$L < L_0,$$

equation (7.30 has a unique almost periodic solution u.

Proof. It follows from the Weierstrass approximation theorem and the definition of almost periodicity (see e.g. [22], [21]) that for each almost periodic function v the composition $f(t, v(t))$ is an almost periodic function. We may thus define an operator $T : E \to E$, $v \mapsto T(v)$ by setting

$$u := T(v),$$

where u is the unique almost periodic solution of

$$u' = Au + f(t, v(t)),$$

whose existence follows from Theorem 7.8. This theorem also implies that

$$|T(v)(t) - T(w)(t)| \le \frac{c}{\beta} \sup_{t\in\mathbb{R}} |f(t, v(t)) - f(t, w(t))|,$$

i.e., using the Lipschitz condition satisfied by f,

$$\|T(v) - T(w)\| \le L\frac{c}{\beta}\|v - w\|.$$

The operator T will therefore be a contraction mapping, provided that

$$L\frac{c}{\beta} < 1.$$

Thus, if we choose

$$L_0 = \frac{\beta}{c}$$

the contraction mapping principle will apply and the existence of a unique almost periodic solution follows. □

8. The implicit function theorem

In this chapter we shall prove the *implicit function theorem* for mappings defined between Banach spaces which are Fréchet differentiable.

8.1. **Fréchet differentiable mappings.** Let us assume we have Banach spaces E, X and let

$$f : U \to X,$$

(where U is open in E) be a continuous mapping. Let $u_0 \in U$, then f is said to be *Fréchet differentiable at* u_0 provided there exists

$$L \in \mathcal{L}(E, X)$$

(the continuous (or bounded) linear mappings from E to X) such that

$$f(u_0 + v) = f(u_0) + L(v) + o(\|v\|).$$

If f is Fréchet differentiable for every $u \in U$, the mapping $U \to \mathcal{L}(E, X)$ given by

$$u \mapsto D_u f(u),$$

where $D_u f(u)$ is the Fréchet derivative of f at u, is then defined. We remark, that the Fréchet derivative is uniquely determined (if it exists) and the above definition provides a Taylor expansion expression. (See [37], [63].)

8.2. **The implicit function theorem.** Let us assume we have Banach spaces E, X, Λ and let

$$f : U \times V \to X,$$

(where U is open in E, V is open in Λ) be a continuous mapping satisfying the following condition:

> For each $\lambda \in V$ the map $f(\cdot, \lambda) : U \to X$ is Fréchet-differentiable on U with Fréchet derivative $D_u f(u, \lambda)$ and the mapping $(u, \lambda) \mapsto D_u f(u, \lambda)$ is a continuous mapping from $U \times V$ to $\mathcal{L}(E, X)$ (the continuous (or bounded) linear mappings from E to X).

Theorem 8.1. *Let f satisfy the above condition and let there exist $(u_0, \lambda_0) \in U \times V$ such that $D_u f(u_0, \lambda_0)$ is a linear homeomorphism of E onto X (i.e. $D_u f(u_0, \lambda_0) \in \mathcal{L}(E, X)$ and $[D_u f(u_0, \lambda_0)]^{-1} \in \mathcal{L}(X, E)$). Then there exist $\delta > 0$, $r > 0$, and unique continuous mapping $u : B_\delta(\lambda_0) = \{\lambda : \|\lambda - \lambda_0\|_\Lambda \le \delta\} \to E$ such that*

$$f(u(\lambda), \lambda) = f(u_0, \lambda_0), \quad u(\lambda_0) = u_0 \tag{8.1}$$

and

$$\|u(\lambda) - u_0\| \le r, \quad \forall \lambda \in B_\delta(\lambda_0).$$

Proof. Let us consider the equation

$$f(u, \lambda) = f(u_0, \lambda_0)$$

which is equivalent to

$$T\left(f(u, \lambda) - f(u_0, \lambda_0)\right) = 0, \tag{8.2}$$

where $T = [D_u f(u_0, \lambda_0)]^{-1}$, or

$$u = u - T\left(f(u, \lambda) - f(u_0, \lambda_0)\right) =: G(u, \lambda). \tag{8.3}$$

The mapping G has the following properties:

- (i) $G(u_0, \lambda_0) = u_0$,
- (ii) G and $D_u G$ are continuous in (u, λ),
- (iii) $D_u G(u_0, \lambda_0) = 0$.

Hence, since

$$G(u_1, \lambda) - G(u_2, \lambda) = \int_0^1 D_u G\left(u_2 + t(u_1 - u_2), \lambda\right)(u_1 - u_2) dt,$$

we obtain

$$\begin{aligned} \|G(u_1, \lambda) - G(u_2, \lambda)\| &\le \Big(\sup_{0 \le t \le 1} \|D_u G(u_1 + t(u_2 - u_1), \lambda)\|_{\mathcal{L}}\Big) \|u_1 - u_2\| \\ &\le \frac{1}{2}\|u_1 - u_2\|, \end{aligned} \tag{8.4}$$

provided $\|u_1 - u_0\| \le r$, $\|u_2 - u_0\| \le r$, $\|\lambda - \lambda_0\|_\Lambda \le \delta$, where $r > 0$ and $\delta > 0$ are small enough. Now

$$\begin{aligned} \|G(u, \lambda) - u_0\| &= \|G(u, \lambda) - G(u_0, \lambda_0)\| \\ &\le \|G(u, \lambda) - G(u_0, \lambda)\| + \|G(u_0, \lambda) - G(u_0, \lambda_0)\| \\ &\le \frac{1}{2}\|u - u_0\| + \|G(u_0, \lambda) - G(u_0, \lambda_0)\| \\ &\le \frac{1}{2} r + \frac{1}{2} r, \end{aligned}$$

provided $\|\lambda - \lambda_0\|_\Lambda \le \delta$, where $\delta > 0$ has been further restricted so that

$$\|G(u_0, \lambda) - G(u_0, \lambda_0)\| \le \frac{1}{2} r.$$

We now think of u as a continuous function

$$u : B_\delta(\lambda_0) \to E$$

and define

$$\begin{aligned} \mathbb{M} := \big\{ &u : B_\delta(\lambda_0) \to E, \text{ such that } u \text{ is continuous}, \\ &u(\lambda_0) = u_0,\ u(B_\delta(\lambda_0)) \subset B_r(u_0) \big\}, \end{aligned}$$

and equip $\mathbb{M}$ with the norm

$$\|u\|_{\mathbb{M}} := \sup_{\lambda \in B_\delta(\lambda_0)} \|u(\lambda)\|.$$

Then $\mathbb{M}$ is a closed subset of the Banach space of bounded continuous functions defined on $B_\delta(\lambda_0)$ with values in E. Since E is a Banach space, it is a complete metric space. Thus, (8.3) defines an equation

$$u(\cdot) = G(u(\cdot), \cdot) \tag{8.5}$$

in $\mathbb{M}$.

Define g by (here we think of u as an element of $\mathbb{M}$)

$$g(u)(\lambda) := G(u(\lambda), \lambda),$$

then $g : \mathbb{M} \to \mathbb{M}$ and it follows from (8.4) that

$$\|g(u) - g(v)\|_{\mathbb{M}} \le \frac{1}{2}\|u - v\|_{\mathbb{M}},$$

hence g has a unique fixed point by the contraction mapping principle (Theorem 3.1 of Chapter 2). □

Remark 8.2. If in the implicit function theorem f is k times continuously differentiable with respect to λ, then the mapping $\lambda \mapsto u(\lambda)$ inherits this property.

Proof. We sketch a proof for the case that f is continuously differentiable with respect to λ. It follows from the above computation that

$$\|D_u G(u(\lambda), \lambda)\|_{\mathcal{L}} \le \frac{1}{2},$$

for $\|\lambda - \lambda_0\|_\Lambda \le \delta$. It follows that

$$[\mathrm{id} - D_u G(u(\lambda), \lambda)]^{-1} D_\lambda G(u(\lambda), \lambda) \in \mathcal{L}(\Lambda, E).$$

Furthermore, a quick calculation shows that

$$u(\lambda + h) = u(\lambda) + [\mathrm{id} - D_u G(u(\lambda), \lambda)]^{-1} D_\lambda G(u(\lambda), \lambda)(h) + o(\|h\|_\Lambda)$$

and thus

$$D_\lambda u(\lambda) = [\mathrm{id} - D_u G(u(\lambda), \lambda)]^{-1} D_\lambda G(u(\lambda), \lambda)$$

is continuous with respect to the parameter λ. □

8.3. **Two examples.**

Example 8.3. Let us consider the nonlinear boundary value problem

$$u'' + \lambda e^u = 0, \quad 0 < x < \pi, \quad u(0) = 0 = u(\pi). \tag{8.6}$$

This is a one space-dimensional mathematical model from the theory of combustion (cf. [8]), where u represents a dimensionless temperature.

The problem may be explicitly solved using quadrature methods and was first posed by Liouville [52]. We shall show, by an application of Theorem 8.1, that for $\lambda \in \mathbb{R}$, in a neighborhood of 0, (8.6) has a unique solution of small norm in $C^2([0, \pi])$. To this end we define

$$E := C_0^2([0, \pi]) := C^2([0, \pi]) \cap \{u : u(0) = 0 = u(\pi)\}$$
$$X := C([0, \pi]), \quad \Lambda := \mathbb{R}.$$

These spaces are Banach spaces when equipped with their usual norms, i.e.,

$$\|u\|_X := \sup_{t \in [0,\pi]} |u(t)|,$$
$$\|u\|_E := \|u\|_X + \|u'\|_X + \|u''\|_X.$$

and $|\cdot|$ represents absolute value.

We let $f : E \times \Lambda \to X$ be given by

$$f(u,\lambda) := u'' + \lambda e^u.$$

Then f is continuous and $f(0,0) = 0$. (When $\lambda = 0$ (no heat generation) the unique solution is $u \equiv 0$.) Furthermore, for $u_0 \in E$, $D_u f(u_0,\lambda)$ is given by (the reader should carry out the verification)

$$D_u f(u_0,\lambda)v = v'' + \lambda e^{u_0} v,$$

and, hence, the mapping

$$(u,\lambda) \mapsto D_u f(u,\lambda)$$

is continuous. Let us consider the linear mapping

$$T := D_u f(0,0) : E \to X.$$

We must show that this mapping is a linear homeomorphism. To see this we note that for every $h \in X$, the unique solution of

$$v'' = h(x), \quad 0 < x < \pi, \quad v(0) = 0 = v(\pi),$$

is given by

$$v(x) = \int_0^\pi G(x,s)h(s)ds, \tag{8.7}$$

where

$$G(x,s) = \begin{cases} -\frac{1}{\pi}(\pi - x)s, & 0 \le s \le x \\ -\frac{1}{\pi}x(\pi - s), & x \le s \le \pi. \end{cases} \tag{8.8}$$

From the representation (8.7) we may conclude that there exists a constant c such that

$$\|v\|_E = \|T^{-1}h\|_E \le c\|h\|_X;$$

i.e. T^{-1} is one to one and continuous. Hence, all conditions of the implicit function theorem are satisfied and we may conclude that for each sufficiently small λ, (8.6) has a unique small solution $u \in C^2([0,\pi].)$, Furthermore, the map $\lambda \mapsto u(\lambda)$ is continuous (in fact, smooth) from a neighborhood of $0 \in \mathbb{R}$ to $C^2([0,\pi])$. We observe that this 'solution branch' $(\lambda, u(\lambda))$ is bounded in the $\lambda-$ direction. To see this, we note that if $\lambda > 0$ is such that (8.6) has a solution, then the corresponding solution u must be positive, $u(x) > 0$, $0 < x < \pi$. Hence

$$0 = u'' + \lambda e^u > u'' + \lambda u. \tag{8.9}$$

Let $v(x) = \sin x$. Then v satisfies

$$v'' + v = 0, \quad 0 < x < \pi, \quad v(0) = 0 = v(\pi). \tag{8.10}$$

From (8.9) and (8.10) we obtain

$$0 > \int_0^\pi (u''v - v''u)dx + (\lambda - 1)\int_0^\pi uvdx,$$

and, hence, integrating by parts,

$$0 > (\lambda - 1)\int_0^\pi uvdx,$$

implying that $\lambda < 1$.

As a second example, we consider the following.

Example 8.4. Given any $l_0 \neq n^2$, $n = 1, 2, \dots$, the forced nonlinear oscillator (periodic boundary value problem)

$$u'' + lu + u^2 = g, \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \tag{8.11}$$

where g is a continuous $2\pi -$ *periodic* function and $l \in \mathbb{R}$, is a parameter, has a unique 2π periodic solution for all g of sufficiently small norm and $|l-l_0|$ sufficiently small.

Let

$$E := C^2([0, 2\pi]) \cap \{u : u(0) = u(2\pi),\ u'(0) = u'(2\pi)\},$$
$$X := C([0, 2\pi]),$$

where both spaces are equipped with the norms as in the previous example. As a parameter space we choose

$$\Lambda := \mathbb{R} \times X.$$

The norm in Λ is given by $\|\lambda = (l, g)\|_\Lambda = |l| + \|g\|_X$. We shall show that, for certain values of l, (8.11) has a unique solution for all forcing terms g of small norm.

To this end let $f : E \times \Lambda \to X$ be given by

$$f(u, \lambda) = f(u, l, g) := u'' + lu + u^2 - g.$$

Then $f(0, 0) = 0$, and $D_u f(u, \lambda)$ is defined by

$$(D_u f(u, \lambda))(v) = v'' + lv + 2uv,$$

and hence the mapping

$$u \mapsto D_u f(u, \lambda)$$

is a continuous mapping of E to $\mathcal{L}(E; X)$, i.e. f is a C^1 mapping. It follows from elementary differential equations theory (see e.g., [11]) that the problem

$$v'' + l_0 v = h,$$

has a unique 2π–periodic solution for every 2π–periodic h as long as $l_0 \neq n^2$, $n = 1, 2, \dots$, and that $\|v\| \leq C\|h\|_X$ for some constant C (only depending upon l_0). Hence, $D_u f(0, l_0, 0)$ is a linear homeomorphism of E onto X whenever $l_0 \neq n^2$, $n = 1, 2, \dots$, and we conclude that for every $g \in X$ of small norm and $|l - l_0|$ sufficiently small, (8.11) has a unique solution $u \in E$ of small norm.

We note that the above example is prototypical for forced nonlinear oscillators. Virtually the same arguments can be applied (the reader might carry out the necessary calculations) to conclude that the forced pendulum equation

$$u'' + l \sin u = g$$

has a unique 2π- periodic response of small norm for every 2π - periodic forcing term g of small norm, as long as $l \neq n^2$, $n = 1, 2, \dots$.

9. Variational inequalities

In this chapter we shall discuss existence results for solutions of variational inequalities which are defined by bilinear forms on a Banach space. The main result proved is a Lax-Milgram type result. The approach follows the basic paper of Lions-Stampacchia [51] and also [44].

9.1. **On symmetric bilinear forms.** Let E be a real reflexive Banach space with norm $\|\cdot\|$ and let

$$a : E \times E \to \mathbb{R}$$

be a continuous, coercive, symmetric, bilinear form, i.e.,

$$|a(u, v)| \le c_1\|u\|\|v\|,\ a(u, u) \ge c_2\|u\|^2,\ a(u, v) = a(v, u),\ \forall u, v \in E,$$

and a is linear in each variable separately, where c_1 and c_2 are positive constants. As is common, we denote by E^* the dual space of E and for $b \in E^*$ we denote by $\langle b, u\rangle$ the value of the continuous linear functional b at the point u, the pairing between E^* and E. The norm in E^* we shall denote by $\|\cdot\|_*$.

Along with the norm topology on E, we shall also have occasion to make use of the weak topology (see below and, e.g., [62], [71], [76]).

For given $b \in E^*$ and a weakly closed set K we consider the functional

$$f(u) = \frac{1}{2}a(u, u) - \langle b, u\rangle. \tag{9.1}$$

An easy computation shows that

$$f(u) \ge \frac{c_2}{2}\|u\|^2 - \|b\|_{E^*}\|u\|$$

and, hence, that

$$f(u) \to \infty,\ \text{as } \|u\| \to \infty$$

(f is *coercive*) and that f is bounded below on E. Hence, it is the case that

$$\alpha := \inf_{v\in K} f(v) > -\infty.$$

Let us choose a sequence $\{u_n\}_{n=1}^\infty$ in K (a *minimizing sequence*) such that

$$f(u_n) \to \alpha.$$

It follows (because f is coercive) that the sequence $\{u_n\}_{n=1}^\infty$ is a bounded sequence and hence has (since E is reflexive, see [62], [71], [76]) a weakly convergent subsequence, converging weakly to, say, u. We denote this subsequence, after appropriate relabeling, again by $\{u_n\}_{n=1}^\infty$ and hence have

$$u_n \rightharpoonup u$$

($\rightharpoonup$ denotes weak convergence), i.e. for any element $h \in E^*$

$$\langle h, u_n\rangle \to \langle h, u\rangle.$$

Since K is weakly closed, we have that $u \in K$. Since a is bilinear and nonnegative (a is coercive), we obtain

$$a(u_n, u_n) \ge a(u_n, u) + a(u, u_n) - a(u, u),$$

and that

$$\liminf_{n\to\infty} a(u_n, u_n) \ge a(u, u)$$

(the form is *weakly sequentially lower semicontinuous*). We may, therefore, conclude that

$$f(u) = \min_{v\in K} f(v). \tag{9.2}$$

Let us now assume that the set K is also convex (hence, it is also closed, since, in reflexive Banach spaces, convex sets are closed if, and only if, they are weakly closed, cf. [71], [76]). Then for any $v \in K$ and $0 \le t \le 1$ we have that

$$f(u) \le f(u + t(v - u)). \tag{9.3}$$

Computing $f(u+t(v-u))-f(u)$, and using the properties of the form a, we obtain

$$0 \le ta(u, v-u) - t\langle b, v-u\rangle + t^2 a(v-u, v-u),\ 0 \le t \le 1,\ v \in K.$$

Hence, upon dividing the latter inequality by $t > 0$, and letting $t \to 0$, we see that there exists $u \in K$ such that

$$f(u) = \min_{v \in K} f(v), \tag{9.4}$$

$$a(u, v-u) \ge \langle b, v-u\rangle, \quad \forall v \in K. \tag{9.5}$$

Hence, if $b_1, b_2 \in E^*$ are given and u_1, u_2 are solutions in K of the corresponding problems (9.4), (9.5), then, denoting by

$$Tb_i = u_i, \quad i = 1, 2,$$

we easily conclude from (9.5) and the coerciveness of a that

$$\|Tb_1 - Tb_2\| \le \frac{1}{c_2}\|b_1 - b_2\|_*. \tag{9.6}$$

Thus, we see that the problems (9.4), (9.5) have a unique solution.

We have the following theorem.

Theorem 9.1. *Let $a : E \times E \to \mathbb{R}$ be a continuous, bilinear, symmetric, and coercive form and let K be a closed convex subset of E. Then for any $b \in E^*$ the variational inequality*

$$a(u, v-u) \ge \langle b, v-u\rangle, \quad \forall v \in K. \tag{9.7}$$

has a unique solution $u \in K$. Hence, equation (9.7) defines a solution mapping

$$T : E^* \to K, \quad b \mapsto Tb = u,$$

which is Lipschitz continuous with Lipschitz constant $\frac{1}{c_2}$, where c_2 is the coercivity constant of a.

Remark 9.2. It follows from the above considerations that if a satisfies the conditions of Theorem 9.1 except that it is not necessarily symmetric, and if inequality (9.7) has a solution for every $b \in E^*$, then the solution mapping T, above is well-defined and satisfies the Lipschitz condition (9.6).

9.2. **Bilinear forms continued.** Let $a\colon E \times E \to \mathbb{R}$ be a continuous, coercive, bilinear form, $b \in E^*$, and K a closed convex set.

9.3. **The problem.** We pose the following problem: Find (prove the existence of) $u \in K$ such that

$$a(u, v-u) \ge \langle b, v-u\rangle, \quad \forall v \in K. \tag{9.8}$$

In case a is symmetric, this problem has been solved above and Theorem 9.1 provides its solution. Thus, it remains to be shown that the theorem remains true in case a is not necessarily symmetric.

The development in this section follows closely the development in [51] and [44].

Uniqueness of the solution. Using properties of bilinear forms, one concludes (see Remark 9.2) that for all $b \in E^*$, problem (9.8) has at most one solution and if $b_1, b_2 \in E^*$ and solutions u_1, u_2 exist, then

$$\|u_1 - u_2\| \le \frac{1}{c_2}\|b_1 - b_2\|_*,$$

where c_2 is a coercivity constant of a.

Existence of the solution. We write $a = a_e + a_o$, where

$$a_e(u,v) := \frac{1}{2}(a(u,v) + a(v,u)),$$
$$a_o(u,v) := \frac{1}{2}(a(u,v) - a(v,u)),$$

then a_e is a continuous, symmetric, coercive, bilinear form and a_o is continuous and bilinear.

Consider the family of problems

$$a_e(u, v-u) + ta_o(u, v-u) \geq \langle b, v-u \rangle, \quad \forall v \in K, \quad 0 \leq t \leq 1, \tag{9.9}$$

and let us denote by

$$a_t := a_e + ta_0.$$

We have the following lemma.

Lemma 9.3. *Let $t \in [0,\infty)$ be such that the problem*

$$a_t(u, v-u) \geq \langle b, v-u \rangle, \quad \forall v \in K, \tag{9.10}$$

has a unique solution for all $b \in E^*$. Then there exists a constant $c > 0$, depending only on the continuity and coercivity constants of a, such that problem

$$a_{t+\tau}(u, v-u) \geq \langle b, v-u \rangle, \quad \forall v \in K, \tag{9.11}$$

has a unique solution for all $b \in E^*$ and $0 \leq \tau \leq c$.

Proof. For $w \in K$ and $t \geq 0$, consider

$$a_t(u, v-u) \geq \langle b, v-u \rangle - \tau a_o(w, v-u), \quad \forall v \in K. \tag{9.12}$$

Note that for fixed $w \in K$,

$$b_w := b - \tau a_o(w, \cdot) \in E^*,$$

hence, there exists a unique $u = Tw$ solving (9.12) and

$$\|Tw_1 - Tw_2\| \leq \frac{1}{c_2}\|b_{w_1} - b_{w_2}\|_*.$$

On the other hand

$$\|b_{w_1} - b_{w_2}\|_* = \sup_{\|u\|=1} \tau|a_o(w_1,u) - a_o(w_2,u)| \leq \tau c_1\|w_1 - w_2\|,$$

and hence

$$\|Tw_1 - Tw_2\| \leq \frac{\tau c_1}{c_2}\|w_1 - w_2\|,$$

and $T\colon K \to K$ is a contraction mapping provided $\frac{\tau c_1}{c_2} < 1$. Therefore, there is a unique solution of (9.11) as long as $\tau < \frac{c_2}{c_1}$, and we may choose $c = \frac{c_2}{2c_1}$, for example. $\square$

We may apply the above lemma with $t = 0$, since $a_0 = a_e$, and a_e is symmetric, and obtain that

$$a_t(u, v-u) \geq \langle d, v-u \rangle, \quad \forall v \in K \tag{9.13}$$

has a unique solution for all $d \in E^*$, for $0 \leq t \leq c$. Hence by the lemma, we obtain that (9.13) has a unique solution for $0 \leq t \leq 2c$, and continuing in this manner we obtain a unique solution of (9.13) for all $t \in [0,\infty)$, and in particular for $t = 1$, and we have shown that problem (9.8) is uniquely solvable.

We therefore have the following theorem.

Theorem 9.4. *Let $a : E \times E \to \mathbb{R}$ be a continuous, bilinear, and coercive form and let K be a closed convex subset of E. Then for any $b \in E^*$ the variational inequality*

$$a(u, v - u) \geq \langle b, v - u \rangle, \quad \forall v \in K. \tag{9.14}$$

has a unique solution $u \in K$. Hence equation (9.7) defines a solution mapping

$$T : E^* \to K, \quad b \mapsto Tb = u,$$

which is Lipschitz continuous with Lipschitz constant $\frac{1}{c_2}$, where c_2 is a coercivity constant of a.

Using this result one may immediately obtain an existence result for solutions of nonlinearly perturbed variational inequalities of the form

$$a(u, v - u) \geq \langle F(u), v - u \rangle, \quad \forall v \in K, \tag{9.15}$$

where $F : E \to E^*$, is a Lipschitz continuous mapping, say,

$$\|F(u_1) - F(u_2)\|_* \leq k\|u_1 - u_2\|.$$

We have the following result.

Theorem 9.5. *Let a, K, F be as above. Then the variational inequality* (9.15) *has a unique solution, provided that*

$$k < c_2,$$

where k is the Lipschitz constant for F and c_2 is the coercivity constant for a.

Proof. It follows from Theorem 9.4 that the variational inequality (9.15) is equivalent to the fixed point problem

$$u = TF(u). \tag{9.16}$$

Since

$$TF : K \to K,$$

and K is a closed convex subset of E, hence a complete metric space, the result then follows from the contraction mapping principle and Theorem 9.4, once we observe that for any $u_1, u_2 \in K$

$$\|TF(u_1) - TF(u_2)\| \leq \frac{1}{c_2}\|F(u_1) - F(u_2)\|_* \leq \frac{k}{c_2}\|u_1 - u_2\|.$$

□

9.4. **Some examples.**

An obstacle problem. Let Ω be a bounded domain in $\mathbb{R}^N$ and let $E = L^2(\Omega)$. Let $\psi \in E$ be given and let

$$K := \{u \in E : u(x) \geq \psi(x), \text{ a.e. in } \Omega\}.$$

Then K is a closed, convex subset of E. We let $a : E \times E \to \mathbb{R}$ be defined by

$$a(u, v) := \int_\Omega uv dx = \langle u, v \rangle.$$

Then

$$a(u, u) = \|u\|^2,$$

where $\|\cdot\|$ is the norm in the space E. Thus we see that a is a continuous, symmetric, coercive, and bilinear form.

Let $b \in E$ and define $f : E \to \mathbb{R}$ as

$$f(u) = \frac{1}{2}\|u\|^2 - \langle b, u\rangle.$$

Then there exists a unique $u \in K$ such that

$$f(u) = \min_{v \in K} f(v)$$

and, furthermore, u solves the variational inequality

$$a(u, v-u) - \langle b, v-u\rangle \geq 0, \quad \forall v \in K;$$

i.e.,

$$\int_\Omega (u-b)(v-u)dx \geq 0, \quad \forall v \in K, \tag{9.17}$$

and the latter must have a unique solution. The natural candidate for this solution is $u = \max(\psi, b)$, as one easily verifies by substituting into (9.17).

Another example. Let $E := L^2(0,1)$, $K := \{u : \int_0^1 u dx = 1\}$. Then K is closed and convex (hence weakly closed). Let

$$a(u, v) := \int_0^1 uv\, dx = \langle u, v\rangle,$$

then, as above, a is a continuous, coercive, symmetric, and bilinear form. Hence there exists a unique $u \in K$ such that

$$a(u, v-u) \geq 0, \quad \forall v \in K,$$

i.e.

$$\langle u, v-u\rangle \geq 0, \quad \forall v \in K,$$

or

$$\langle u, v\rangle \geq \langle u, u\rangle, \quad \forall v \in K,$$

i.e.

$$\int_0^1 uvdx \geq \int_0^1 u^2 dx, \quad \forall v \in K.$$

On the other hand

$$\Big|\int_0^1 udx\Big| \leq \Big(\int_0^1 u^2 dx\Big)^{1/2},$$

and hence

$$\int_0^1 uvdx \geq 1, \quad \forall v \in K.$$

Clearly $u = 1$ solves the inequality.

9.5. **A second order boundary value problem.** In this section we shall provide an example of a boundary value problem for a second order ordinary differential equation on the interval $(0,\infty)$ which may be solved by the methods developed here. It furnishes an example where the associated quadratic form is not symmetric.

Let us denote by $E = H_0^1(0,\infty)$ (the closure of the space $C_0^\infty(0,\infty)$ in the Sobolev space of functions $u : (0,\infty) \to \mathbb{R}$ which together with their first distributional derivatives are square integrable on $(0,\infty)$; see Chapter 2, section 2.3). The norm in E is given by

$$\|u\|^2 := \int_0^\infty u^2 dx + \int_0^\infty (u')^2 dx.$$

Let the quadratic form $a : E \times E \to \mathbb{R}$, be given by

$$a(u,v) := \int_0^\infty u'v' dx + \int_0^\infty uv' dx + \int_0^\infty uv dx. \tag{9.18}$$

One quickly may check that a is continuous, bilinear, and coercive (with coercivity constant $\frac{1}{2}$) but, it is clearly not symmetric. It follows that for any $b \in L^2(0,\infty)$ the variational inequality

$$a(u, v-u) - \langle b, v-u\rangle \geq 0, \quad \forall v \in E, \tag{9.19}$$

has a unique solution, and, hence, since in this problem the convex set K is the whole space E, the equation

$$a(u,v) - \langle b, v\rangle = 0, \quad \forall v \in E, \tag{9.20}$$

has a unique solution. I.e., there exists a unique $u \in E$ such that

$$\int_0^\infty u'v' dx + \int_0^\infty uv' dx + \int_0^\infty uv dx = \int_0^\infty bv dx, \quad \forall v \in E. \tag{9.21}$$

Since $C_0^\infty(0,\infty)$ (the infinitely smooth functions with compact support, or test functions) is dense in E, we may interpret (9.21) in the sense of distributions and obtain

$$-\partial^2 u(v) - \partial u(v) + \int_0^\infty uv dx = \int_0^\infty bv dx, \quad \forall v \in C_0^\infty(0,\infty), \tag{9.22}$$

and hence

$$-\partial^2 u - \partial u + u = b, \tag{9.23}$$

has a unique solution $u \in E$ for any $b \in L^2(0,\infty)$ (here $\partial^2 u, \partial u$ are the second, respectively, first distributional derivatives of the function u (see again Chapter 2, section 2.3)).

9.6. **An obstacle problem.** Let us consider here once more the quadratic form a of the previous section and pose the obstacle problem

$$a(u, v-u) - \langle b, v-u\rangle \geq 0, \quad \forall v \in K, \tag{9.24}$$

where K is the closed convex set

$$K := \{u \in H_0^1(0,\infty) : u(x) \geq 0,\ 0 \leq x \leq 1\}.$$

Again this problem will have a a unique solution for any $b \in L^2(0,\infty)$. Rewriting (9.24) as

$$\int_0^\infty u'v' dx + \int_0^\infty uv' dx + \int_0^\infty uv dx \geq \int_0^\infty bv dx, \quad \forall v \in K. \tag{9.25}$$

we may conclude that

$$\int_0^\infty u'v'dx + \int_0^\infty uv'dx + \int_0^\infty uvdx = \int_0^\infty bvdx, \quad \forall v \in C_0^\infty(0,\infty), \tag{9.26}$$

with

$$v(x) = 0, \quad 0 \le x \le 1$$

(note that, for such v, both v and $-v$ belong to K). This implies

$$\int_1^\infty u'v'dx + \int_1^\infty uv'dx + \int_1^\infty uvdx = \int_1^\infty bvdx, \quad \forall v \in C_0^\infty(1,\infty), \tag{9.27}$$

and, as above, we conclude that u is a solution of

$$-\partial^2 u - \partial u + u = b, \quad 1 \le x \le \infty.$$

The latter equation will also be satisfied at those points $x \in (0,1)$, for which $u(x) > 0$. Combining these statements, we conclude that $u \in K$ solves

$$-\partial^2 u - \partial u + u = b, \quad 1 \le x \le \infty$$
$$u(-\partial^2 u - \partial u + u - b) = 0, \quad 0 \le x \le 1.$$

9.7. **Elliptic boundary value problems.** Let Ω be a bounded open set (with smooth boundary) in $\mathbb{R}^N$, let $\{a_{ij}\}_{i,j=1}^N \subset L^\infty(\Omega)$ satisfy

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \ge c_0|\xi|^2, \ \forall \xi \in \mathbb{R}^N, \quad \forall x \in \Omega, \ c_0 > 0 \text{ a constant},$$

where $|\cdot|$ is a norm in $\mathbb{R}^N$. Let $E := H_0^1(\Omega)$ with

$$\|u\|^2 = \|u\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u|^2 dx,$$

(this is a norm, equivalent to the H^1 norm, as follows from an inequality due to Poincaré, see [1]), and let $a(u,v)$ be given by

$$a(u,v) = \sum_{i,j} \int_\Omega a_{ij}(x)\partial_i u \partial_j v dx = \int_\Omega A\nabla u \cdot \nabla v dx,$$

where A is the $N \times N$ matrix whose ij entry is a_{ij} and $\partial_i u, \ i = 1, \dots, N$ are the partial distributional derivatives of u and ∇ is the distributional gradient. Then

$$|a(u,v)| \le c_1\|v\|\|u\|, \quad c_1 = \max_{ij} \|a_{ij}\|_{L^\infty(\Omega)},$$
$$|a(u,u)| \ge c_0\|u\|^2.$$

For $b \in L^2(\Omega) \subset H_0^1(\Omega)^*$ we obtain the existence of a unique $u \in H_0^1(\Omega)$ such that

$$a(u, v-u) \ge \int_\Omega b(v-u), \quad \forall v \in H_0^1(\Omega),$$

hence,

$$a(u,v) = \int_\Omega bvdx, \quad \forall v \in H_0^1(\Omega).$$

In particular, this will hold for all $v \in C_0^\infty(\Omega)$, and therefore the partial differential equation

$$-\sum_{i,j} \partial_j(a_{ij}\partial_i u) = b$$

has a unique solution $u \in H_0^1(\Omega)$, in the sense of distributions (see also [14], [29], [34], [50]). As a special case we obtain that the partial differential equation

$$-\Delta u = b, \tag{9.28}$$

has, in the sense of distributions, a unique solution $u \in H_0^1(\Omega)$ for every $b \in L^2(\Omega)$ and

$$\int_\Omega |\nabla u|^2 dx \le \int_\Omega b^2 dx. \tag{9.29}$$

If Ω is a not necessarily bounded open set, one may, using arguments like the above establish the unique solvability in $H_0^1(\Omega)$ of the elliptic equation

$$-\Delta u + u = b, \tag{9.30}$$

for every $b \in L^2(\Omega)$ and

$$\int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 dx \le \int_\Omega b^2 dx. \tag{9.31}$$

For additional and more detailed examples see the following: [13, 18, 19, 26, 33, 44, 47, 50].

10. Semilinear elliptic equations

In this chapter we shall discuss how the contraction mapping principle may be used to deduce the existence of solutions of Dirichlet problems for semilinear elliptic partial differential equations. Such results have their origin in work of Picard [59], Lettenmeyer [49]. Our derivation is based on work contained in [35], and [54], using the results established in Chapter 9.

10.1. **The boundary value problem.** Let Ω be a bounded open set (with smooth boundary) in $\mathbb{R}^N$, and let

$$f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$$

be a continuous function which satisfies the Lipschitz condition

$$|f(x, u_1, v_1) - f(x, u_2, v_2)| \le L_1 |u_1 - u_2| + L_2 |v_1 - v_2|, \tag{10.1}$$

for all $(x, u_1, v_1),\ (x, u_2, v_2) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, here $|\cdot|$ denotes both absolute value and the Euclidean norm in $\mathbb{R}^N$. We shall also assume that $f(\cdot, 0, 0) \in L^2(\Omega)$.

We consider the following boundary value problem (in the sense of distributions)

$$-\Delta u = f(x, u, \nabla u),\ u \in H_0^1(\Omega) =: H. \tag{10.2}$$

We note that if $u \in H^1(\Omega)$, then

$$|f(x, u, \nabla u)| \le |f(x, 0, 0)| + L_1 |u| + L_2 |\nabla u|.$$

Hence f may be thought of as a mapping

$$f : H^1(\Omega) \to L^2(\Omega),$$

which, because of the Lipschitz condition (10.1) is, in fact, a continuous mapping.

Let us denote by T, the solution operator

$$\begin{gathered} T : L^2(\Omega) \to H_0^1(\Omega) \\ T(w) = u, \end{gathered} \tag{10.3}$$

where $u \in H_0^1(\Omega)$ solves $-\Delta u = w$ (see Chapter 9). Inequality (9.31) implies

$$\|u\|_H^2 = \int_\Omega |\nabla u|^2 dx \le \int_\Omega w^2 dx = \|w\|_{L^2}^2. \tag{10.4}$$

We, therefore, find that problem (10.2) is equivalent to the fixed point problem in $L^2(\Omega)$

$$w = f(\cdot, T(w), \nabla T(w)). \tag{10.5}$$

Define the operator

$$S : L^2(\Omega) \to L^2(\Omega)$$

by

$$S(w) = f(\cdot, T(w), \nabla T(w)). \tag{10.6}$$

Using the Lipschitz condition imposed on f, we find

$$|S(w_1)(x) - S(w_2(x)| \le L_1|T(w_1)(x) - T(w_2(x)| + L_2|\nabla T(w_1)(x) - \nabla T(w_2(x)|$$

and thus

$$\|S(w_1) - S(w_2\|_{L^2} \le L_1\|T(w_1) - T(w_2\|_{L^2} + L_2\||\nabla T(w_1) - \nabla T(w_2)|\|_{L^2} \tag{10.7}$$

We now recall the Poincaré inequality for $H_0^1(\Omega)$ (see [29])

$$\|u\|_{L^2} \le \frac{1}{\lambda_1}\|u\|_H, \quad \forall u \in H_0^1(\Omega),$$

where λ_1^2 is the smallest eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ (see [1], [34]). Inequalities (10.7) and (10.4) imply

$$\begin{aligned} \|T(w_1) - T(w_2)\|_{L^2} &= \|T(w_1 - w_2)\|_{L^2} \\ &\le \frac{1}{\lambda_1}\|T(w_1 - w_2)\|_H \\ &\le \frac{1}{\lambda_1}\|w_1 - w_2\|_{L^2}. \end{aligned} \tag{10.8}$$

We next use Green's identity (see [34]) to compute

$$\begin{aligned} \||\nabla T(w_1) - \nabla T(w_2)|\|_{L^2}^2 &= -\langle w_1 - w_2, T(w_1) - T(w_2)\rangle \\ &\le \|w_1 - w_2\|_{L^2}\|T(w_1 - w_2)\|_{L^2} \\ &\le \frac{1}{\lambda_1}\|w_1 - w_2\|_{L^2}^2, \end{aligned} \tag{10.9}$$

where $\langle \cdot, \cdot \rangle$ is the L^2 inner product. Combining (10.8) and (10.9) we obtain

$$\|S(w_1) - S(w_2\|_{L^2} \le \Big(\frac{L_1}{\lambda_1} + \frac{L_2}{\sqrt{\lambda_1}}\Big)\|w_1 - w_2\|_{L^2}. \tag{10.10}$$

The operator $S : L^2(\Omega) \to L^2(\Omega)$, therefore, has a unique fixed point provided that

$$\frac{L_1}{\lambda_1} + \frac{L_2}{\sqrt{\lambda_1}} < 1. \tag{10.11}$$

As observed earlier, this fixed point is a solution of (10.5) and setting $u = T(w)$ we obtain the solution of (10.2).

We summarize the above in the following theorem.

Theorem 10.1. *Let Ω be a bounded open set (with smooth boundary) in $\mathbb{R}^N$, and let*

$$f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$$

be a continuous function which satisfies the Lipschitz condition

$$|f(x, u_1, v_1) - f(x, u_2, v_2)| \leq L_1|u_1 - u_2| + L_2|v_1 - v_2|,$$

for all $(x, u_1, v_1),\ (x, u_2, v_2) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, further assume that $f(\cdot, 0, 0) \in L^2(\Omega)$. Then the boundary value problem

$$-\Delta u = f(x, u, \nabla u),\ u \in H_0^1(\Omega).$$

has a unique solution provided that

$$\frac{L_1}{\lambda_1} + \frac{L_2}{\sqrt{\lambda_1}} < 1,$$

where λ_1^2 is the principal eigenvalue of $-\Delta$ with respect to $H_0^1(\Omega)$.

In case f is independent of ∇u such a result, under the assumption that L_1 be sufficiently small, was already established for the two-dimensional case by Picard [59].

10.2. **A particular case.** In this section we shall consider problem (10.2) in the case of one space dimension, $N = 1$. We shall study this problem for the case

$$\Omega = (0, \pi).$$

The case of an arbitrary finite interval (a, b) may easily be deduced for this one.

In this case $\lambda_1 = 1$ and condition (10.11) becomes

$$L_1 + L_2 < 1. \tag{10.12}$$

Given the boundary value problem

$$u'' = f(x, u, u'), \quad u \in H_0^1(0, \pi), \tag{10.13}$$

with f satisfying the above assumptions, it is natural, also, to seek a solution $u \in C^1[0, \pi]$ and to formulate an integral equation equivalent to (10.13) in this space, rather than $L^2(0, 1)$. This may be accomplished by using the Green's function G given by formula (8.8) of Chapter 8 (see e.g [36]). I.e., we have that problem (10.13) is equivalent to the integral equation problem

$$u(x) = \int_0^\pi G(x, s) f(s, u(s), u'(s))ds,\ u \in E := C^1[0, \pi]. \tag{10.14}$$

As usual, we use

$$\|u\|_E := \|u\| + \|u'\|,$$

where

$$\|u\| := \max_{[0,\pi]} |u(x)|.$$

We define $T : E \to E$ by

$$T(u)u(x) := \int_0^\pi G(x, s) f(s, u(s), u'(s))ds, \quad u \in E = C^1[0, \pi].$$

An easy computation shows that

$$\|T(u_1) - T(u_2\|_E \leq \frac{\pi^2 L_1}{8}\|u_1 - u_2\| + \frac{\pi L_2}{2}\|u_1' - u_2'\|,$$

and therefore

$$\|T(u_1) - T(u_2\|_E \le \Big(\frac{\pi^2 L_1}{8} + \frac{\pi L_2}{2}\Big)\|u_1 - u_2\|_E, \tag{10.15}$$

i.e., T is a contraction mapping, whenever

$$\frac{\pi^2 L_1}{8} + \frac{\pi L_2}{2} < 1. \tag{10.16}$$

Clearly the requirement (10.16) is different from the requirement (10.12). Thus, for a given problem, several different metric space settings may be possible, with the different settings yielding different requirements.

(The condition (10.16) was already derived by Picard [59] and later by Lettenmeyer [49]; many different types of requirements using different approaches may be found, e.g., in [20] and [35].)

We remark that in the above considerations we could equally have assumed that f is a mapping

$$f : (0, 1) \times E \times E \to E,$$

where E is a Banach space.

10.3. **Monotone solutions.** In this section we shall discuss some recent work in [24] concerning the nonlinear second order equation

$$u'' + F(t, u) = 0, \; t \in [0, \infty), \tag{10.17}$$

where $F : [0, \infty)\times[0, \infty) \to [0, \infty)$ is continuous and satisfies the Lipschitz condition

$$|F(t, u) - F(t, v)| \le k(t)|u - v|, \tag{10.18}$$

with

$$k : [0, \infty) \to [0, \infty)$$

a continuous function, satisfying

$$\int_0^\infty tk(t)dt < 1. \tag{10.19}$$

We have the following theorem on monotone solutions of (10.17).

Theorem 10.2. *Let the above conditions hold and assume that there exists $M > 0$ such that for any $u \in X$, where*

$$X := \{u \in C[0, \infty) : 0 \le u(t) \le M, \; t \in [0, \infty)\},$$

we have

$$\int_0^\infty tF(t, u(t))dt \le M. \tag{10.20}$$

Then (10.17) *has a monotone solution $u : [0, \infty) \to [0, M]$ such that*

$$\lim_{t\to\infty} u(t) = M.$$

Proof. Let

$$E := \{u \in C[0, \infty) : \|u\| := \sup_{t\in[0,\infty)} |u(t)| < \infty\},$$

then E is a Banach space and X is a closed subset of E and hence a complete metric space with respect to the metric defined by the norm in E.

Next consider the mapping T on X defined by

$$(Tu)(t) := M - \int_t^\infty (\tau - t)F(\tau, u(\tau))d\tau. \tag{10.21}$$

Then, since for $u \in X$

$$0 \le \int_t^\infty (\tau - t)F(\tau, u(\tau))d\tau \le \int_0^\infty \tau F(\tau, u(\tau))d\tau \le M, \tag{10.22}$$

it follows that $T : X \to X$.

On the other hand, for $u, v \in X$ we have

$$\begin{aligned}
|(Tu)(t) - T(v)(t)| &\le \int_t^\infty (\tau - t)|F(\tau, u(\tau)) - F(\tau, v(\tau))|d\tau \\
&\le \int_t^\infty (\tau - t)k(\tau)|u(\tau) - v(\tau)|d\tau \\
&\le \int_0^\infty \tau k(\tau)d\tau \|u - v\|.
\end{aligned}$$

Hence T is a contraction on X and therefore has a unique fixed point. If $u \in X$ is the fixed point of T, then it easily follows that u is monotone and $\lim_{t\to\infty} u(t) = M$. □

For applications of this result to nonoscillation theory of second order differential equations, see [24]

11. A mapping theorem in Hilbert space

In this chapter we shall discuss a result of Mawhin [53] on nonlinear mappings in Hilbert spaces which has, among others, several interesting applications to existence questions about periodic solutions of nonlinear conservative systems (see [53] and [74] for many references to this interesting problem area; see also [3] and [67] for further directions).

11.1. **A mapping theorem.** Let H be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We assume that

$$L : \mathrm{dom}L \subset H \to H,$$

is a linear, self-adjoint operator (dom$L$ is the domain of L) and

$$N : H \to H$$

is a nonlinear mapping which is Fréchet differentiable there, with symmetric Fréchet derivative N' (see [63], [66]).

For a given linear operator

$$A : \mathrm{dom}A \subset H \to H,$$

we denote by $\rho(A)$, $\sigma(A)$, $r(A)$, respectively the *resolvent set*, *spectrum*, and the *spectral radius* of the operator A (see [43]). Also one writes, for a given linear operator A,

$$A \ge 0, \text{ if, and only if, } \langle Au, u\rangle \ge 0, \ \forall u \in \mathrm{dom}A.$$

and

$$A \ge B, \text{ if, and only if, } A - B \ge 0,$$

(here, of course, $A - B$ is an operator defined on the intersection of the domains of A and B).

We establish the following surjectivity theorem.

Theorem 11.1. *Suppose L and N are as above and there exist real numbers λ, μ, p, q such that*
$$\lambda < q \le p < \mu, \quad [\lambda, \mu] \subset \rho(L),$$
and that
$$qI \le N'(u) \le pI, \quad \forall u \in H,$$
where $I : H \to H$ is the identity mapping. Then
$$L - N : \mathrm{dom} L \to H$$
is a bijection.

Proof. For any $\nu \in (\lambda, \mu)$, and $v \in H$ the equation
$$Lu - N(u) = v,$$
is equivalent to the equation
$$(L - \nu I)u - (N - \nu I)(u) = v,$$
or
$$Au - B(u) = v, \tag{11.1}$$
where
$$A := L - \nu I, \quad B := N - \nu I.$$
It follows from the assumptions that B has the symmetric Fréchet derivative given by
$$B'(u) = N'(u) - \nu I.$$
Since $\nu \in \rho(L)$, it follows that
$$A^{-1} = (L - \nu I)^{-1}$$
exists and is a bounded operator and further that
$$[\lambda - \nu, \mu - \nu] \subset \rho(A).$$
Hence, since A is self-adjoint,
$$\sigma(A) \subset (-\infty, \lambda - \nu) \cup (\mu - \nu, \infty).$$
One may further deduce that
$$\sigma(A^{-1}) \subset \left((\lambda - \nu)^{-1}, (\mu - \nu)^{-1}\right).$$
Hence
$$\|A^{-1}\| = r(A^{-1}) \le \max\{(\nu - \lambda)^{-1}, (\mu - \nu)^{-1}\} =: \alpha. \tag{11.2}$$
All of the above follow from properties of linear operators (see [43]). Next, we note that
$$\begin{aligned} \|B(u) - B(v)\| &\le \sup_{\tau \in [0,1]} \|B'(u + \tau(v - u)\|\|u - v\| \\ &\le \sup_{w \in H} \|N'(w) - \nu I\|\|u - v\|. \end{aligned}$$
On the other hand
$$(q - \nu)I \le N'(u) - \nu I = B'(u) \le (p - \nu)I,$$
and hence for each $u \in H$,
$$\langle (q - \nu)Iv, v\rangle \le \langle B'(u)v, v\rangle \le \langle (p - \nu)Iv, v\rangle, \forall v \in H,$$

or

$$(q-\nu)\|v\|^2 \le \langle B'(u)v, v\rangle \le (p-\nu)\|v\|^2.$$

The latter implies (see again [43])

$$\|B'(u)\| \le \max(|q-\nu|, |p-\nu|) =: \beta. \tag{11.3}$$

Now, equation (11.1) is equivalent to the equation

$$u = A^{-1}(B(u)+v). \tag{11.4}$$

We next note that

$$\begin{aligned}\|A^{-1}(B(u)+v) - A^{-1}(B(w)+v)\| &\le \|A^{-1}\|\|B(u)-B(w)\| \\ &\le \alpha\beta\|u-w\|\end{aligned} \tag{11.5}$$

(see (11.2), (11.3)), and hence, for every v, the mapping

$$u \mapsto A^{-1}(B(u)+v)$$

will be a contraction mapping provided $\alpha\beta < 1$, will be satisfied, whenever

$$\frac{p+\lambda}{2} < \nu < \frac{q+\mu}{2}.$$

The latter will hold, for example if we choose

$$\nu = \frac{p+q}{2} \quad \text{or} \quad \nu = \frac{\lambda+\mu}{2}.$$

□

We remark that it follows from the results in [43] that

$$\|A^{-1}\| = \frac{1}{\operatorname{dist}(\nu, \sigma(L))}$$

and hence (11.1) may be rewritten as

$$\|A^{-1}(B(u)+v) - A^{-1}(B(w)+v)\| \le \|A^{-1}\|\|N(u)-N(v)-\nu(u-v)\|. \tag{11.6}$$

Using this, one obtains the following more general version of Theorem 11.1.

Theorem 11.2. *Suppose L and N are as above and there exists a real number $\nu \in \rho(L)$, such that $0 < \operatorname{dist}(\nu, \sigma(L))$ and*

$$\|N(u)-N(v)-\nu(u-v)\| \le k\|u-v\|, \quad \forall u, v \in H,$$

where

$$k < \operatorname{dist}(\nu, \sigma(L)).$$

Then $L - N : \operatorname{dom}L \to H$ is a bijection.

11.2. **Periodic solutions of conservative systems.** In this section we shall discuss the existence of 2π-periodic solutions of the system of second order differential equations

$$u'' + n(u) = h(t), \tag{11.7}$$

where $n : \mathbb{R}^N \to \mathbb{R}^N$ is a C^1 function such that its Fréchet derivative $n'(u)$ satisfies

$$rI \le n'(u) \le sI,$$

where, for some m,

$$m^2 < r \le s < (m+1)^2, \quad m \in \{0, 1, 2, \dots\},$$

r and s are given and

$$h : (-\infty, \infty) \to \mathbb{R}^N$$

is a $2\pi-$ periodic function with $h \in L^2(0, 2\pi)$.

The following result (for more general versions; see, e.g. [53] or [74]) is valid:

Theorem 11.3. *Let the above assumptions hold. Then there exists a unique $2\pi-$ periodic solution of* (11.7).

Proof. We let $H = L^2(0, 2\pi)$ with inner product

$$\langle u, v\rangle := \frac{1}{2\pi}\int_0^{2\pi} uv dx.$$

Let

$$\text{dom}L = \{u \in H : u,\ u' \text{ absolutely continuous, periodic, } u'' \in H\}$$

$$L : \text{dom}L \to H,\ u \mapsto u''.$$

Then L is self-adjoint and it follows, using elementary ordinary differential equations results, that

$$\sigma(L) = \{-n^2,\ n = 0, 1, \dots\}.$$

It is also an easy exercise to verify that, if we define by

$$Nu := -n(u(\cdot)),$$

then $N : H \to H$, and

$$(N'(u))v(t) = -n'(u(t))v(t).$$

Using the other assumptions, one sees that all hypotheses of Theorem 1 hold, proving the theorem. □

12. The theorem of Cauchy-Kowalevsky

In this chapter we provide one of the fundamental theorems of the theory of partial differential equations, the existence theorem of Cauchy-Kowalevsky. It marks the starting point for the general theory of partial differential equations, both from the point of view of analysis and as well as from the historical point of view. It basically asserts that the noncharacteristic Cauchy problem with holomorphic coefficients and holomorphic initial data is well-posed. The proof is based on an elegant paper of Walter [73] and follows the treatment provided in [68].

12.1. **The setting.** Let $\mathbb{C}$ denote the set of complex numbers and $\mathbb{C}^n$ denote the n-dimensional space of n-tuples of complex numbers $z = (z_1, \dots z_n)$. For $z \in \mathbb{C}^n$, we define the norm of z by $|z| = \max_{1 \le j \le n} |z_j|$.

Definition 12.1. Let G be a domain in $\mathbb{C}^n$. A function $f : G \to \mathbb{C}$ will be said to be *holomorphic* in G, if f and $\partial f / \partial z_j = f_{z_j}$, $j = 1, \dots n$, are continuous in G.

Let Ω be an open set in $\mathbb{C}^n$ such that the boundary $\Gamma = \partial\Omega$ is nonempty. Let

$$d(z) := \operatorname{dist}(z, \Gamma)$$

denote the distance from z to the boundary of Ω,

$$\operatorname{dist}(z, \Gamma) := \inf_{\zeta \in \Gamma} |z - \zeta|.$$

Let η be a positive real number and define the set Ω^η by

$$\Omega^\eta := \{(t, z) : z \in \Omega, |t| < \eta d(z)\},$$

and either $t \in \mathbb{R}$ (real case) or $t \in \mathbb{C}$ (complex case). In geometrical terms in the real case, i.e., Ω^η is the double cone, with base Ω whose sides have slope η. We denote by $\Omega_t \subset \mathbb{C}^n$ the set

$$\Omega_t = \{z \in \Omega : (t, z) \in \Omega^\eta\},$$

and by Γ_t the boundary of Ω_t. If $z \in \Omega_t$, then $d(z) > |t|/\eta$. Geometrically, in the real case, the set Ω_t is the projection onto $\mathbb{C}^n$ of the base of that part of Ω which lies above t ($t > 0$) or below t ($t < 0$). For $z \in \Omega_t$, we define $d(t, z)$ by

$$d(t, z) := d(z) - \frac{|t|}{\eta}. \tag{12.1}$$

The function $d(t, z)$ is positive and represents the distance from $z \in \Omega_t$ to Γ_t. The following property of $d(t, z)$ will be needed later in the proof of the theorem.

Lemma 12.2. *If $z' \in \mathbb{C}^n$ satisfies $|z - z'| = r < d(t, z)$ for some $z \in \Omega_t$, then $z' \in \Omega_t$ and*

$$d(t, z') \ge d(t, z) - r. \tag{12.2}$$

12.2. **The linear case.** We will first give a proof of the Cauchy-Kowalevsky Theorem in the case where the equation is linear. I.e., we consider the initial value problem

$$\begin{gathered} u_t = A(t, z)u + \sum_{j=1}^{n} B_j(t, z)u_{z_j} + C(t, z), \quad z \in \Omega, \\ u(0, z) = f(z) \quad z \in \Omega. \end{gathered} \tag{12.3}$$

Here, $z \in \Omega$, $t \in \mathbb{R}$ (real case), or $t \in \mathbb{C}$ (complex case) and u, u_t, u_{z_j}, and $C(t, z)$ are complex valued column vectors in $\mathbb{C}^m$; and $A(t, z)$ and $B_j(t, z)$ are complex valued $m \times m$ matrices. The set $G := \Omega^\eta$ is the set defined above. If we integrate both sides of equation (12.3) with respect to t and use the initial condition to evaluate the integral of the left hand side, we obtain the equivalent integral formulation of the problem

$$\begin{aligned} u(t, z) = f(z) &+ \int_0^t C(\tau, z)\, d\tau \\ &+ \int_0^t \Big[A(\tau, z)u(\tau, z) + \sum_{j=1}^{n} B_j(\tau, z)u_{z_j}(\tau, z)\Big]\, d\tau. \end{aligned} \tag{12.4}$$

Remark 12.3. In the complex case ($t \in \mathbb{C}$), the integration in equation (12.4) is taken along the straight line which connects 0 to t in $\mathbb{C}$.

Definition 12.4. By a solution to equation (12.4), we mean a function $u(t,z)$ which is continuous in G, holomorphic in z for fixed t (real case) or holomorphic in (t,z) (complex case), and which satisfies (12.4).

Under suitable assumptions on the coefficients $A(t,z)$, $B_j(t,z)$, $C(t,z)$, and $f(z)$, we shall see that if $u(t,z)$ is a solution to one of the two problems (12.4) or (12.3), then u is of class C^1 and is a solution to the other of the two problems, i.e., the two formulations of the problem are equivalent. Throughout we shall use the operator norm for matrices, $|A| = \max_{1\le k\le m}\sum_{j=1}^n |a_{jk}|$, induced by the vector norm $|\cdot|$.

Before stating the main result of the section, we will first prove a lemma which gives a crucial bound on the derivatives of holomorphic functions. The result is usually referred to as Nagumo's lemma.

Lemma 12.5. *Let Ω be a domain in $\mathbb{C}^n$ and let $f:\Omega\to\mathbb{C}^m$ be holomorphic and let $p\ge 0$. If, for $z\in\Omega$,*
$$|f(z)| \le \frac{c}{d^p(z)},$$
then
$$|f_{z_j}(z)| \le C_p \frac{c}{d^{p+1}(z)},$$
where
$$C_p = (1+p)\Big(1+\frac{1}{p}\Big)^p < e(p+1), \quad C_0 = 1.$$

Proof. Consider first the case of a single function of a single complex variable. Let $\psi:\mathbb{C}\to\mathbb{C}$ be holomorphic in the disk $\{\zeta' : |\zeta-\zeta'|\le r\}$. Then by the Cauchy integral formula, see [63]
$$\psi'(\zeta) = \frac{1}{2\pi i}\oint_{|\zeta-\zeta'|=r}\frac{\psi(\zeta')}{(\zeta'-\zeta)^2}\,d\zeta'.$$
Thus, we obtain
$$\begin{aligned}
|\psi'(\zeta)| &= \frac{1}{2\pi}\Big|\oint_{|\zeta-\zeta'|=r}\frac{\psi(\zeta')}{(\zeta'-\zeta)^2}\,d\zeta'\Big| \\
&\le \frac{1}{2\pi}\oint_{|\zeta-\zeta'|=r}\frac{|\psi(\zeta')|}{|\zeta'-\zeta|^2}\,|d\zeta'| \\
&\le \frac{1}{2\pi r^2}\max_{|\zeta-\zeta'|=r}|\psi(\zeta')|\cdot(2\pi r) \\
&= \frac{1}{r}\max_{|\zeta-\zeta'|=r}|\psi(\zeta')|.
\end{aligned}$$
Now let $f:\Omega\to\mathbb{C}^m$ be holomorphic. We apply the result that we have just obtained to $f(z)$, with $\zeta = z_j$. If $0<r<d(z)$, then
$$\begin{aligned}
|f_{z_j}(z)| &\le \frac{1}{r}\max_{|z-z'|=r}|f(z')| \\
&\le \frac{c}{r}\max_{|z-z'|=r}\frac{1}{d^p(z')} \\
&\le \frac{c}{r(d(z)-r)^p}.
\end{aligned}$$

Where $d(z') \geq d(z) - r$ because of Lemma 12.2. The choice $r = d(z)/(p+1)$ gives the (optimal) value stated in the lemma. $\square$

We next establish the linear Cauchy-Kowalevsky theorem.

Theorem 12.6. *Let the functions $A(t,z)$, $B_j(t,z)$ and $C(t,z)$ be continuous in (t,z) and holomorphic in z for fixed t and let $f(z)$ be holomorphic in z. Suppose that there exist constants α, β_j, γ, δ and p such that for every (t,z) in G,*

$$\begin{gathered} |A(t,z)| \leq \frac{\alpha}{d(t,z)}, \quad |B_j(t,z)| \leq \beta_j, \\ |C(t,z)| \leq \frac{\gamma}{d^{p+1}(t,z)}, \quad |f(z)| \leq \frac{\delta}{d^p(z)}. \end{gathered} \tag{12.5}$$

Suppose further that $\eta > 0$ is such that

$$\frac{\alpha}{p} + \Big(1 + \frac{1}{p}\Big)^{p+1} \sum_{j=1}^{n} \beta_j < \frac{1}{\eta}. \tag{12.6}$$

Then (12.4) *has a unique solution $u(t,z)$ in G and for some constant c this solution satisfies*

$$|u(t,z)| \leq \frac{c}{d^p(t,z)}.$$

Note that condition (12.6) is a smallness condition on η and thus makes the theorem a local one, i.e., guarantees the existence of solutions for small time. The condition, as will be seen, is imposed in order for the contraction mapping principle to be applicable.

Proof. Let E be the normed linear space defined by

$$E := \{u \in C^0(G, \mathbb{C}^m) : \ u \text{ is holomorphic in } z \text{ and } \|u\| < \infty\},$$

where the norm on E is defined by

$$\|u\| := \sup_G \{d^p(t,z)|u(t,z)|\}.$$

Note that convergence in this norm implies uniform convergence on compact subsets of G; hence limit functions are holomorphic in z and E is complete.

We write equation (12.4) in the form

$$u = g + Tu,$$

where g is given by the equation

$$g(t,z) := f(z) + \int_0^t C(\tau,z)\,d\tau,$$

and T is the linear operator given by

$$(Tu)(t,z) := \int_0^t \Big[A(\tau,z)u(\tau,z) + \sum_{j=1}^{n} B_j(\tau,z)u_{z_j}(\tau,z)\Big]\,d\tau.$$

We first show that $g + Tu \in E$ if $u \in E$. Consider each of the terms in turn. For the function $f(z)$ we have, using (12.5),

$$d^p(t,z)|f(z)| \leq d^p(t,z)\frac{\delta}{d^p(z)} \leq \delta.$$

The last inequality follows from the fact that $d(z) \geq d(z) - |t|/\eta = d(t,z)$. Taking the supremum of the left-hand side of this inequality, we get $\|f\| \leq \delta$.

Before estimating the next term in g, we note that a direct integration gives

$$\Big| \int_0^t \frac{d\tau}{d^{p+1}(\tau z)} \Big| \leq \int_0^{|t|} \frac{ds}{\big(d(z) - \frac{s}{\eta}\big)^{p+1}} \leq \frac{\eta}{pd^p(t,z)}. \tag{12.7}$$

Therefore, for the second term in g, we have

$$\begin{aligned}
\Big| \int_0^t C(\tau,z)\,d\tau \Big| &\leq \Big| \int_0^t |C(\tau,z)|\,d\tau \Big| \\
&\leq \gamma \Big| \int_0^t \frac{d\tau}{d^{p+1}(\tau,z)} \Big| \\
&< \frac{\gamma\eta}{pd^p(t,z)}.
\end{aligned}$$

Again, we have used the bound from (12.5). Multiplying both sides of this inequality by $d^p(t,z)$ and taking the supremum over G, we obtain

$$\Big\| \int_0^t C(\tau,z)\,d\tau \Big\| \leq \frac{\gamma\eta}{p}.$$

It follows from this last inequality and the bound on f, that $g \in E$.

According to the definition of the norm on E, we have the inequality

$$|u(t,z)| \leq \frac{\|u\|}{d^p(t,z)}.$$

Starting from this inequality, we may now apply Nagumo's lemma, Lemma 12.5, to the region Ω_t, using the distance function $d(t,z)$. As a result, we get the estimate

$$|u_{z_j}(t,z)| \leq C_p \frac{\|u\|}{d^{p+1}(t,z)}.$$

Combining (12.5) with the bounds just obtained on u and u_{z_j}, we get the estimates

$$\begin{gathered}
|A(t,z)u(t,z)| \leq \frac{\alpha\|u\|}{d^{p+1}(t,z)}, \\
|B_j(t,z)u_{z_j}(t,z)| \leq \frac{\|u\|}{d^{p+1}(t,z)} \beta_j C_p.
\end{gathered}$$

Hence, with $\beta := \sum_{j=1}^n \beta_j$ and using the estimate (12.7), we have

$$\begin{aligned}
|(Tu)(t,z)| &\leq \|u\|(\alpha + \beta C_p) \Big| \int_0^t \frac{d\tau}{d^{p+1}(\tau,z)} \Big| \\
&\leq \frac{1}{p}(\alpha + \beta C_p)\eta \frac{\|u\|}{d^p(t,z)}.
\end{aligned}$$

If we multiply both sides of this inequality by $d^p(t,z)$ and take the supremum of the left hand side, we get the final estimate

$$\|Tu\| \leq q\|u\|, \tag{12.8}$$

where

$$q = \Big(\frac{\alpha}{p} + \beta \big(1 + \frac{1}{p}\big)^{p+1} \Big) \eta.$$

This shows that $Tu \in E$. Furthermore, the constant q satisfies $q < 1$ by the hypotheses of the theorem, hence, the contraction mapping principle may be applied to obtain a unique solution of the equation $u = g + Tu$. $\square$

Example 12.7. Consider the equation

$$u_t = bu_z,$$

(here $n = 1$, b =constant) subject to the initial condition

$$u(0, z) = \phi(z).$$

In this example $\alpha = 0$ and $\beta = |b|$. The solution of this initial value problem is given by $u(t, z) = \phi(bt + z)$. Suppose that Ω is the disk $|z| < 1$ and that ϕ is holomorphic in Ω. Then the solution exists for $|bt + z| < 1$, and if ϕ cannot be continued analytically beyond the unit circle, then this is best possible. If we vary b, but keep $|b| = \beta$ fixed, then the largest region common to all those regions of existence is the circular cone $\beta|t| < 1 - |z|$, i.e., $\eta = 1/\beta$ is best possible.

An added advantage to the approach that has been adopted for the Cauchy-Kowalevsky Theorem, is that the fixed point of a contractive mapping S is not only unique, it also depends continuously on the operator S. Thus we can easily obtain results on continuous dependence of the solution on the coefficients and the initial data.

12.3. **The quasilinear case.** We are now prepared to present the result for the general case. It may be shown that the Cauchy problem for a general nonlinear system with holomorphic coefficients can be reduced, by a change of variables, to an equivalent initial value problem for a first order system which is quasilinear. Therefore, it is sufficient to state and prove the result for the initial value problem for a first order quasilinear system. Furthermore, if u satisfies an initial condition of the form $u(0, z) = f(z)$, then the substitution of the function $u(t, z) = f(z) + v(t, z)$ gives a new differential equation for v which is again a first order quasi-linear equation of the same type and satisfies the initial condition $v(t, z) = 0$. Taking these observations into account, we shall consider the following initial value problem

$$\begin{gathered} u_t = \sum_{j=1}^{n} B_j(t, z, u) u_{z_j} + C(t, z, u), \quad (t, z) \in G, \\ u(0, z) = 0, \quad z \in \Omega. \end{gathered} \tag{12.9}$$

As was the case for linear equations, the B_j are $m \times m$ matrices and C is a column vector of length m. The set $G = \Omega^\eta$ is again defined by the inequality $|t| < \eta d(z)$, where η is defined later in the theorem and $d(z)$ is an appropriately determined "distance" function. The set $B_R := B_R(0)$ is the open ball $\{z : |z| < R\}$ in $\mathbb{C}^n$. We shall assume that $d(z)$ is bounded on Ω and satisfies the inequalities

$$0 < d(z) \le \operatorname{dist}(z, \Gamma), \quad |d(z) - d(z')| \le |z - z'|, \quad z, z' \in \Omega.$$

If G, Ω_t, and $d(t, z)$ are defined as before, using the new function $d(z)$, then Lemma 12.2 remains valid. Moreover, $0 < d(t, z) < \operatorname{dist}(z, \Gamma_t)$, for $z \in \Omega_t$. As a consequence, Lemma 12.5 remains true. Although the region G of existence obtained is less than optimal under this assumption, we derive the benefit that the proof of the theorem is greatly simplified. We now state and prove the Cauchy-Kowalevsky theorem for the quasilinear case.

Theorem 12.8. *Let the functions $B_j(t,z,u)$ and $C(t,z,u)$ be continuous in $\Omega^\eta \times B_R$ and holomorphic with respect to z and u (real case), or holomorphic in $\Omega^\eta \times B_R$ (complex case). Suppose further, that the following estimates hold on $\Omega^\eta \times B_R$:*

$$\begin{gathered}|B_j(t,z,u)| \le \beta_j,\\ \sqrt{d(t,z)}|B_j(t,z,u) - B_j(t,zv)| \le \beta_j'|u-v|, \quad j = 1,\dots,n,\\ |C(t,z,u)| \le \frac{\gamma}{\sqrt{d(t,z)}}, \quad d(t,z)|C(t,z,u) - C(t,z,v)| \le \gamma'|u-v|.\end{gathered} \tag{12.10}$$

If $\eta > 0$ is such that

$$2\eta\sqrt{d_0}(\beta+\gamma) < R, \quad \eta(3\sqrt{3}(\beta+\gamma)+2\beta) \le 1, \quad \eta e\beta' < 1,$$

where $d_0 = \sup d(z) < \infty$, $\beta = \sum_{j=1}^n \beta_j$, and $\beta' = \sum_{j=1}^n \beta_j'$. Then the initial value problem (12.9) *has a unique solution which exists in Ω^η.*

Proof. The proof proceeds along lines similar to the linear case. We consider an equivalent integral formulation of (12.9) given by

$$u(t,z) = \int_0^t \Big[\sum_{j=1}^n B_j(\tau,z,u(\tau,z))u_{z_j}(\tau,z) + C(\tau,z,u(\tau,z))\Big]\,d\tau \tag{12.11}$$

and treat it as a fixed point equation of the form $u = Su$ to which the contraction mapping principle, can be applied. As the underlying Banach space E, we use the same space as in the proof of Theorem 12.6. However, in contrast to the linear case, the operator S is not globally defined on E and it will be necessary to define a proper closed subset F of E which is mapped into itself by S.

Let F be the closed subset of E defined by

$$F := \big\{u \in E : |u(t,z)| \le \rho,\ |u_{z_j}(t,z)| \le 1/\sqrt{d(t,z)}\big\},$$

where $\rho := 2\eta\sqrt{d_0}(\beta+\gamma) < R$. Let $u \in F$ and set $v = Su$. Then, using the bounds in (12.10), we obtain

$$\begin{aligned}|v_t(t,z)| &= \big|\sum_{j=1}^n B_j(t,z,u(t,z))u_{z_j}(t,z) + C(t,z,u(t,z))\big|\\ &\le \sum_{j=1}^n |B_j(t,z,u(t,z))||u_{z_j}(t,z)| + |C(t,z,u(t,z))|\\ &\le \frac{\beta+\gamma}{\sqrt{d(t,z)}}.\end{aligned}$$

From this inequality and the fundamental theorem of calculus, it follows that

$$|v(t,z)| = \Big|\int_0^t v_t(\tau,z)\,d\tau\Big| \le (\beta+\gamma)\int_0^{|t|} \frac{ds}{\sqrt{d(s,z)}}.$$

A direct integration then yields

$$\begin{aligned}
\int_0^{|t|} \frac{ds}{\sqrt{d(s,z)}} &= \int_0^{|t|} \frac{ds}{\sqrt{d(z)-\frac{s}{\eta}}} \\
&= -2\eta\sqrt{d(z)-\frac{s}{\eta}}\Big|_0^{|t|} \\
&\le 2\eta\sqrt{d(z)} \\
&\le 2\eta\sqrt{d_0}.
\end{aligned}$$

Therefore,

$$|v(t,z)| \le 2\eta(\beta+\gamma)\sqrt{d_0} = \rho,$$

which is the first inequality in the definition of the set F. In order to verify the second inequality in the definition of F, we estimate the derivatives of v. Using the inequalities in (12.10) and applying Nagumo's Lemma, Lemma 12.5, we obtain the bounds

$$\begin{aligned}
\Big|\frac{\partial}{\partial z_k} C(t,z,u(t,z))\Big| &\le \frac{\gamma C_{1/2}}{d^{3/2}(t,z)} \\
\Big|\frac{\partial}{\partial z_k} B_j(t,z,u(t,z))\Big| &\le \frac{\beta_j}{d^{3/2}(t,z)},
\end{aligned}$$

and from the second inequality in the definition of F and Nagumo's Lemma,

$$|u_{z_j z_k}| \le \frac{C_{1/2}}{d(t,z)}.$$

Here $C_{1/2} = 3\sqrt{3}/2$. The product formula for derivatives, leads to the the inequality

$$\begin{aligned}
|v_{t,z_k}| &= \Big|\sum_{j=1}^{n} [(B_j)_{z_k} u_{z_j} + B_j u_{z_j z_k}] + C_{z_k}\Big| \\
&\le \sum_{j=1}^{n} |(B_j)_{z_k}||u_{z_j}| + |B_j||u_{z_j z_k}| + |C_{z_k}| \\
&\le \frac{(\beta+\gamma)C_{1/2}+\beta}{d^{3/2}(t,z)}.
\end{aligned}$$

A direct integration with respect to s establishes the inequality

$$\int_0^{|t|} \frac{ds}{d^{3/2}(s,z)} \le \frac{2\eta}{\sqrt{d(t,z)}}.$$

Therefore, using the fundamental theorem of calculus, we have the estimate

$$\begin{aligned}
|v_{z_j}| &\le \Big|\int_0^t v_{t,z_j}(\tau,z)\,d\tau\Big| \\
&\le ((\beta+\gamma)C_{1/2}+\beta)\int_0^{|t|} \frac{1}{(d^{3/2}(\tau,z))}\,d\tau \\
&\le \frac{1}{\sqrt{d(t,z)}}.
\end{aligned}$$

This last inequality verifies that v satisfies the second inequality in the definition of F, as well. It follows from the above that $Su \in F$ whenever $u \in F$, i.e., S maps F into itself.

To complete the proof, we need to show that S is a contraction mapping. Let u and v be in F and consider the t derivative of the difference

$$\begin{aligned}(Su - Sv)_t = \sum_{j=1}^{n} &[B_j(t,z,u) - B_j(t,z,v)]u_{z_j} \\ &+ \sum_{j=1}^{n} B_j(t,z,v)[u_{z_j} - v_{z_j}] + C(t,z,u) - C(t,z,v).\end{aligned}$$

The definition of the Banach space E gives us the inequality

$$|u - v| \le \frac{\|u - v\|}{d^p(t,z)}.$$

Applying Nagumo's lemma to this inequality, gives the estimate

$$|u_{z_j} - v_{z_j}| \le \frac{C_p\|u - v\|}{d^{p+1}(t,z)}.$$

Combining these results with the hypotheses (12.10), we obtain the following inequality for the difference in the derivatives

$$\begin{aligned}|(Su - Sv)_t| &\le \Big(\sum_{j=1}^{n} \beta_j'\Big)\frac{|u - v|}{d(t,z)} + \sum_{j=1}^{n} \beta_j|u_{z_j} - v_{z_j}| + \gamma'\frac{|u - v|}{d(t,z)} \\ &\le \big(\beta' + \beta C_p + \gamma'\big)\frac{\|u - v\|}{d^{p+1}(t,z)}.\end{aligned}$$

Thus, making use of (12.7), and the fundamental theorem of calculus, we get

$$|Su - Sv| \le \Big|\int_0^t (Su - Sv)_t\,dt\Big| \le \frac{\eta}{p}(\beta' + C_p\beta + \gamma')\frac{\|u - v\|}{d^p(t,z)}.$$

Hence, when we multiply by $d^p(t,z)$ and then take the supremum over Ω^η, we get

$$\|Su - Sv\| \le \frac{\eta}{p}(\beta' + C_p\beta + \gamma')\|u - v\|.$$

It follows from our hypotheses on η that S is a contraction mapping for p sufficiently large and S has a unique fixed point $u \in F$ which is the solution to (12.11). $\square$

An example to which the above, for example may be applied, is the the initial value problem for a Burger's type equation

$$\begin{gathered}u_t = -(1 + z^2 + u)u_z - 2z(1 + z^2 + u), \quad z \in \Omega,\ 0 < t \\ u(0,z) = 0, \quad z \in \Omega,\end{gathered}$$

where $\Omega = \{z : |z| < 1\}$ is the unit disk, centered at the origin. In the terms of the notation of Theorem 12.8, we have $B(t,z,u) = -(1 + z^2 + u)$ and $C(t,z,u) = -2z(1 + z^2 + u)$, which are entire functions. Note also that $B(t,z,u)$ and $C(t,z,u)$ are real valued for real values of t, z, and u. The distance function $d(z)$ is given by

$d(z) = 1 - |z|$, $d_0 = 1$, and $d(t, z) = 1 - |z| - |t|\eta$. Let $R > 0$ be given. Then the conditions (12.10) are satisfied with the constants

$$\beta = 2 + R, \quad \beta' = 1,$$
$$\gamma = 2(2 + R), \quad \gamma' = 2.$$

If η is chosen such that

$$\eta \le \min\big\{\frac{R}{6(2+r)}, \frac{1}{(9\sqrt{3}+2)(2+R)}, \frac{1}{e}\big\},$$

then the hypotheses of Theorem 12.8 are satisfied and the existence of a unique holomorphic solution defined on Ω^η follows.

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