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EXISTENCE AND ULAM STABILITIES FOR HADAMARD FRACTIONAL INTEGRAL EQUATIONS WITH RANDOM EFFECTS

SAÏD ABBAS, WAFAA A. ALBARAKATI, MOUFFAK BENCHOHRA, JOHNNY HENDERSON

Abstract. This article concerns the existence and Ulam stabilities for a class of random integral equations via Hadamard's fractional integral. Our main tools is a random fixed point theorem with stochastic domain.

1. Introduction

Fractional calculus is a powerful tool in applied mathematics to study many problems from fields of science and engineering. It has produced many break-through results in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [21, 43]. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Abbas et al [4, 5], Kilbas et al [28], Miller and Ross [33], and the papers of Abbas et al [1, 2, 6], Benchohra et al [9], Vityuk et al [45, 46], and the references therein.

Abbas et al [3] obtained some existence and uniqueness results for determinist integral equations involving the Hadamard fractional integral of two independent variables. Butzer et al [12] investigate properties of the Hadamard fractional integral and the derivative. In [13], they obtained the Mellin transforms of the Hadamard fractional integral and differential operators and Pooseh et al [36] obtained expansion formulas of the Hadamard operators in terms of integer order derivatives. Many other interesting properties of those operators and others are summarized in [40] and the references therein.

The nature of a dynamic system in engineering or natural sciences depends on the accuracy of the information we have concerning the parameters that describe that system. If the knowledge about a dynamic system is precise then a deterministic dynamical system arises. Unfortunately in most cases the available data for the description and evaluation of parameters of a dynamic system are inaccurate, imprecise or confusing. In other words, evaluation of parameters of a dynamical system is not without uncertainties. When our knowledge about the parameters of

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a dynamic system are of statistical nature, that is, the information is probabilistic, the common approach in mathematical modeling of such systems is the use of random differential equations or stochastic differential equations. Random differential equations, as natural extensions of deterministic ones, arise in many applications and have been investigated by many mathematicians; see [29, 30, 31, 42, 52] and references therein. Between them differential equations with random coefficients (see, [42, 15]) offer a natural and rational approach (see [41], Chapter 1), since sometimes we can obtain the random distributions of some main disturbances by historical experiences and data rather than take all random disturbances into account and assume the noise to be white noise.

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University. The problem posed by Ulam was the following: "Under what conditions does there exist an additive mapping near an approximately additive mapping?" (for more details see [44]). The first answer to Ulam's question was given by Hyers in 1941 in the case of Banach spaces in [22]. Thereafter, this type of stability is called the Ulam-Hyers stability. In 1978, Rassias [37] provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations becomes, "How do the solutions of the inequality differ from those of the given functional equation?" Considerable attention has been given to the study of the Ulam-Hyers and Ulam-Hyers-Rassias stability for all kinds of functional equations; one can see the monographs of [23, 25]. Bota-Boriceanu and Petrusel [11], Petru et al [35], and Rus [38, 39] who discussed the Ulam-Hyers stability for operatorial equations and inclusions. Castro and Ramos [14] and Jung [27] considered the Hyers-Ulam-Rassias stability for a class of Volterra integral equations. More details from a historical point of view, and recent developments of such stabilities are reported in [26, 38] and [47]-[51].

This article concerns the existence and Ulam stability of solutions to the Hadamard fractional integral equation

$$\begin{aligned} u(x,y,w) = \mu(x,y,w) + \int_1^x \int_1^y \Big(\log\frac{x}{s}\Big)^{r_1-1} \Big(\log\frac{y}{t}\Big)^{r_2-1} \\ \times \frac{f(s,t,u(s,t,w),w)}{st}\,dt\,ds, \quad (x,y)\in J,\ w\in\Omega, \end{aligned} \tag{1.1}$$

where $J := [1,a]\times[1,b]$, $a,b>1$, $r_1,r_2>0$, $(\Omega,\mathcal{A})$ is a measurable space, $\mu : J\times\Omega\to\mathbb{R}$ and $f : J\times\mathbb{R}\times\Omega\to\mathbb{R}$ are given continuous functions. In this article we obtain the existence and Ulam stabilities of random solutions via fixed point techniques.

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts which are used throughout this article. Denote by $L^1(J,\mathbb{R})$ the Banach space of functions $u : J\to\mathbb{R}$ that are Lebesgue integrable, with norm

$$\|u\|_{L^1} = \int_1^a \int_1^b |u(x,y)|dydx.$$

Let $L^\infty(J)$ be the Banach space of functions $u : J \to \mathbb{R}$ which are measurable and essentially bounded. As usual, by $C := C(J, \mathbb{R})$ we denote the Banach space of all continuous functions $u : J \to \mathbb{R}$ with the norm

$$\|u\|_C = \sup_{(x,y)\in J} |u(x, y)|.$$

Let β_E be the σ-algebra of Borel subsets of E. A mapping $v : \Omega \to E$ is said to be measurable if for any $B \in \beta_E$, one has

$$v^{-1}(B) = \{w \in \Omega : v(w) \in B\} \subset \mathcal{A}.$$

To define integrals of sample paths of random process, it is necessary to define a jointly measurable map.

Definition 2.1. A mapping $T : \Omega \times E \to E$ is called jointly measurable if for any $B \in \beta_E$, one has

$$T^{-1}(B) = \{(w, v) \in \Omega \times E : T(w, v) \in B\} \subset \mathcal{A} \times \beta_E,$$

where $\mathcal{A} \times \beta_E$ is the direct product of the σ-algebras $\mathcal{A}$ and β_E those defined in Ω and E respectively.

Lemma 2.2 ([16])**.** *Let $T : \Omega \times E \to E$ be a mapping such that $T(., v)$ is measurable for all $v \in E$, and $T(w, \cdot)$ is continuous for all $w \in \Omega$. Then the map $(w, v) \mapsto T(w, v)$ is jointly measurable.*

Definition 2.3 ([20])**.** A function $f : J \times E \times \Omega \to E$ is called random Carathéodory if the following conditions are satisfied:

(i) The map $(x, y, w) \to f(x, y, u, w)$ is jointly measurable for all $u \in E$, and
(ii) The map $u \to f(x, y, u, w)$ is continuous for almost all $(x, y) \in J$ and $w \in \Omega$.

Let $T : \Omega \times E \to E$ be a mapping. Then T is called a random operator if $T(w, u)$ is measurable in w for all $u \in E$ and it is expressed as $T(w)u = T(w, u)$. In this case we also say that $T(w)$ is a random operator on E. A random operator $T(w)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $T(w, u)$ is continuous (resp. compact, totally bounded and completely continuous) in u for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [24].

Definition 2.4 ([17])**.** Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y and C be a mapping from Ω into $\mathcal{P}(Y)$. A mapping $T : \{(w, y) : w \in \Omega,\ y \in C(w)\} \to Y$ is called random operator with stochastic domain C if C is measurable (i.e., for all closed $A \subset Y$, $\{w \in \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $y \in Y$, $\{w \in \Omega : y \in C(w), T(w, y) \in D\}$ is measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $y : \Omega \to Y$ is called random (stochastic) fixed point of T if for P-almost all $w \in \Omega$, $y(w) \in C(w)$ and $T(w)y(w) = y(w)$ and for all open $D \subset Y$, $\{w \in \Omega : y(w) \in D\}$ is measurable.

Let $\mathcal{M}_X$ denote the class of all bounded subsets of a metric space X.

Definition 2.5. Let X be a complete metric space. A map $\alpha : \mathcal{M}_X \to [0, \infty)$ is called a measure of noncompactness on X if it satisfies the following properties for all $B, B_1, B_2 \in \mathcal{M}_X$.

(1) $\alpha(B) = 0$ if and only if B is precompact (Regularity),
(2) $\alpha(B) = \alpha(\overline{B})$ (Invariance under closure),

(3) $\alpha(B_1 \cup B_2) = \alpha(B_1) + \alpha(B_2)$ (Semi-additivity).

For more details on measure of noncompactness and its properties see [7].

Example 2.6. In every metric space X, the map $\phi : \mathcal{M}_X \to [0,\infty)$ with $\phi(B) = 0$ if B is relatively compact and $\phi(B) = 1$ otherwise is a measure of noncompactness, the so-called discrete measure of noncompactness [8, Example 1, p. 19], .

Definition 2.7 ([19, 28])**.** The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1([1,a],\mathbb{R})$, is defined as

$$({}^H I_1^r g)(x) = \frac{1}{\Gamma(q)} \int_1^x \Big(\log \frac{x}{s}\Big)^{q-1} \frac{g(s)}{s} ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.8. Let $r_1,\ r_2 \geq 0$, $\sigma = (1,1)$ and $r = (r_1, r_2)$. For $w \in L^1(J,\mathbb{R})$, define the Hadamard partial fractional integral of order r by the expression

$$({}^H I_\sigma^r w)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \Big(\log \frac{x}{s}\Big)^{r_1-1} \Big(\log \frac{y}{t}\Big)^{r_2-1} \frac{w(s,t)}{st}\, dt\, ds.$$

Now, we consider the Ulam stability for the Hadamard random integral equation (1.1). Consider the operator $N : \Omega \times C \to C$ defined by:

$$\begin{aligned} &(N(w)u)(x,y) \\ &= \mu(x,y,w) + \int_1^x \int_1^y \Big(\log \frac{x}{s}\Big)^{r_1-1} \Big(\log \frac{y}{t}\Big)^{r_2-1} \frac{f(s,t,u(s,t,w),w)}{st}\, dt\, ds. \end{aligned} \tag{2.1}$$

Let ϵ be a positive real number and $\Phi : J \times \Omega \to [0,\infty)$ be a measurable and bounded function. We consider the following inequalities:

$$|u(x,y,w) - (N(w)u)(x,y)| \leq \epsilon; \quad \text{for a.a. } (x,y) \in J,\ w \in \Omega, \tag{2.2}$$

$$|u(x,y,w) - (N(w)u)(x,y)| \leq \Phi(x,y,w); \quad \text{for a.a. } (x,y) \in J,\ w \in \Omega, \tag{2.3}$$

$$|u(x,y,w) - (N(w)u)(x,y)| \leq \epsilon\Phi(x,y,w); \quad \text{for a.a. } (x,y) \in J,\ w \in \Omega. \tag{2.4}$$

Definition 2.9 ([4, 38])**.** Equation (1.1) is Ulam-Hyers stable if there exists a real number $c_N > 0$ such that for each $\epsilon > 0$ and for each random solution $u : \Omega \to C$ of the inequality (2.2) there exists a random solution $v : \Omega \to C$ of the equation (1.1) with

$$|u(x,y,w) - v(x,y,w)| \leq \epsilon c_N; \quad (x,y) \in J,\ w \in \Omega.$$

Definition 2.10 ([4, 38])**.** Equation (1.1) is generalized Ulam-Hyers stable if there exists $c_N \in C([0,\infty),[0,\infty))$ with $c_N(0) = 0$ such that for each $\epsilon > 0$ and for each random solution $u : \Omega \to C$ of inequality (2.2) there exists a random solution $v : \Omega \to C$ of the equation (1.1) with

$$|u(x,y,w) - v(x,y,w)| \leq c_N(\epsilon); \quad (x,y) \in J,\ w \in \Omega.$$

Definition 2.11 ([4, 38])**.** Equation (1.1) is Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{N,\Phi} > 0$ such that for each $\epsilon > 0$ and for each random solution $u : \Omega \to C$ of inequality (2.4) there exists a random solution $v : \Omega \to C$ of equation (1.1) with

$$|u(x,y,w) - v(x,y,w)| \leq \epsilon c_{N,\Phi}\Phi(x,y,w); \quad (x,y) \in J,\ w \in \Omega.$$

Definition 2.12 ([4, 38])**.** Equation (1.1) is generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{N,\Phi} > 0$ such that for each random solution $u : \Omega \to C$ of inequality (2.3) there exists a random solution $v : \Omega \to C$ of equation (1.1) with

$$|u(x, y, w) - v(x, y, w)| \leq c_{N,\Phi}\Phi(x, y, w); \quad (x, y) \in J,\ w \in \Omega.$$

Remark 2.13. It is clear that: (i) Definition 2.9 implies Definition 2.10; (ii) Definition 2.11 implies Definition 2.12; (iii) Definition 2.11 for $\Phi(.,.,.) = 1$ implies Definition 2.9.

One can have similar remarks for the inequalities (2.2) and (2.4). So, the Ulam stabilities of the fractional random differential equations are some special types of data dependence of the solutions of fractional differential equations.

Lemma 2.14 ([10])**.** *If Y is a bounded subset of a Banach space X, then for each $\epsilon > 0$, there is a sequence $\{y_k\}_{k=1}^{\infty} \subset Y$ such that*

$$\alpha(Y) \leq 2\alpha(\{y_k\}_{k=1}^{\infty}) + \epsilon.$$

Lemma 2.15 ([34, 53])**.** *If $\{u_k\}_{k=1}^{\infty} \subset L^1(J)$ is uniformly integrable, then the function $\alpha(\{u_k\}_{k=1}^{\infty})$ is measurable and for each $(x, y) \in J$,*

$$\alpha\Big(\Big\{\int_1^x \int_1^y u_k(s, t)\,dt\,ds\Big\}_{k=1}^{\infty}\Big) \leq 2\int_1^x \int_1^y \alpha(\{u_k(s, t)\}_{k=1}^{\infty})\,dt\,ds.$$

Lemma 2.16 ([32])**.** *Let F be a closed and convex subset of a real Banach space, let $G : F \to F$ be a continuous operator and $G(F)$ be bounded. If there exists a constant $k \in [0, 1)$ such that for each bounded subset $B \subset F$,*

$$\alpha(G(B)) \leq k\alpha(B),$$

then G has a fixed point in F.

3. Existence and Ulam stability results

In this section, we discuss the existence of solutions and we present conditions for the Ulam stability for the Hadamard integral equation (1.1). The following hypotheses will be used in the sequel.

- (H1) The function $w \mapsto \mu(x, y, w)$ is measurable and bounded for a.e. $(x, y) \in J$,
- (H2) The function f is random Carathéodory on $J \times \mathbb{R} \times \Omega$,
- (H3) There exist functions $p_1, p_2 : J \times \Omega \to [0, \infty)$ with $p_i(w) \in C(J, \mathbb{R}_+);\ i = 1, 2$ such that for each $w \in \Omega$,
$$|f(x, y, u, w)| \leq p_1(x, y, w) + \frac{p_2(x, y, w)}{1 + |u(x, y)|}|u(x, y, w)|,$$
for all $u \in \mathbb{R}$ and a.e. $(x, y) \in J$,
- (H4) There exists a function $q : J \times \Omega \to [0, \infty)$ with $q(w) \in L^{\infty}(J, [0, \infty))$ for each $w \in \Omega$ such that for any bounded $B \subset \mathbb{R}$,
$$\alpha(f(x, y, B, w)) \leq q(x, y, w)\alpha(B),\ \textit{for a.e. } (x, y) \in J,$$
- (H5) There exists a random function $R : \Omega \to (0, \infty)$ such that
$$R(w) \geq \mu^*(w) + \frac{(p_1^*(w) + p_2^*(w))(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)},$$

where

$$\mu^*(w)=\sup_{(x,y)\in J}|\mu(x,y,w)|,\ p_i^*(w)=\operatorname{ess\,sup}_{(x,y)\in J}p_i(x,y,w);\ i=1,2,$$

(H6) There exist $q_1,q_2:J\times\Omega\to[0,\infty)$ with $q_i(.,w)\in L^\infty(J,[0,\infty));\ i=1,2$ such that for each $w\in\Omega$, and a.e. $(x,y)\in J$, we have

$$p_i(x,y,w)\le q_i(x,y,w,w)\Phi(x,y,w),$$

(H7) $\Phi(w)\in L^1(J,[0,\infty))$ for all $w\in\Omega$, and there exists $\lambda_\Phi>0$ such that for each $(x,y)\in J$, we have

$$({}^HI^r_\sigma\Phi)(x,y,w)\le\lambda_\Phi\Phi(x,y,w).$$

Set

$$q^*=\operatorname{ess\,sup}_{(x,y,w)\in J\times\Omega}q(x,y,w).$$

Theorem 3.1. *Assume that hypotheses* (H1)–(H5) *hold. If*

$$\ell:=\frac{4q^*(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}<1,$$

then the integral equation (1.1) *has a random solution defined on J. Furthermore, if the hypotheses* (H6) *and* (H7) *hold, then the random equation* (1.1) *is generalized Ulam-Hyers-Rassias stable.*

Proof. Let N be the operator defined in (2.1). From the hypotheses (H2) and (H3), for each $w\in\Omega$ and almost all $(x,y)\in J$, we have that $f(x,y,u(x,y,w),w)$ is in L^1. Since the function f is continuous, then the indefinite integral is continuous for all $w\in\Omega$ and almost all $(x,y)\in J$. Again, as the map μ is continuous for all $w\in\Omega$ and the indefinite integral is continuous on J, then $N(w)$ defines a mapping $N:\Omega\times C\to C$. Hence u is a solution for the integral equation (1.1) if and only if $u=(N(w))u$. We shall show that the operator N satisfies all conditions of Lemma 2.16. The proof will be given in several steps.

Step 1: $N(w)$ is a random operator with stochastic domain on C. Since $f(x,y,u,w)$ is random Carathéodory, the map $w\to f(x,y,u,w)$ is measurable in view of Definition 2.1. Similarly, the product $\left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t,u(s,t,w),w)}{st}$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$w\mapsto\mu(x,y,w)+\int_1^x\int_1^y\Big(\log\frac{x}{s}\Big)^{r_1-1}\Big(\log\frac{y}{t}\Big)^{r_2-1}\frac{f(s,t,u(s,t,w),w)}{st\Gamma(r_1)\Gamma(r_2)}\,dt\,ds$$

is measurable. As a result, N is a random operator on $\Omega\times C$ into C.

Let $W:\Omega\to\mathcal{P}(C)$ be defined by

$$W(w)=\{u\in C:\|u\|_C\le R(w)\},$$

with $W(w)$ bounded, closed, convex and solid for all $w\in\Omega$. Then W is measurable by [17, Lemma 17]. Let $w\in\Omega$ be fixed, then from (H4), for any $u\in w(w)$, we obtain

$$\begin{aligned}&|(N(w)u)(x,y)|\\&\le|\mu(x,y,w)|+\int_1^x\int_1^y\Big|\log\frac{x}{s}\Big|^{r_1-1}\Big|\log\frac{y}{t}\Big|^{r_2-1}\frac{|f(s,t,u(s,t,w),w)|}{st\Gamma(r_1)\Gamma(r_2)}\,dt\,ds\end{aligned}$$

$$
\begin{aligned}
&\le |\mu(x,y,w)| + \int_1^x \int_1^y \big|\log\frac{x}{s}\big|^{r_1-1}\big|\log\frac{y}{t}\big|^{r_2-1}\frac{|p_1(s,t,w)+p_2(s,t,w)|}{\Gamma(r_1)\Gamma(r_2)}\,dt\,ds\\
&\le \mu^*(w) + \frac{(p_1^*(w)+p_2^*(w))(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\\
&\le R(w).
\end{aligned}
$$

Therefore, N is a random operator with stochastic domain W and $N(w): W(w) \to N(w)$. Furthermore, $N(w)$ maps bounded sets into bounded sets in C.

Step 2: $N(w)$ is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $\mathcal{C}$. Then, for each $(x,y) \in J$ and $w \in \Omega$, we have

$$
\begin{aligned}
&|(N(w)u_n)(x,y) - (N(w)u)(x,y)|\\
&\le \int_1^x \int_1^y \big|\log\frac{x}{s}\big|^{r_1-1}\big|\log\frac{y}{t}\big|^{r_2-1}\\
&\quad\times \frac{|f(s,t,u_n(s,t,w),w) - f(s,t,u(s,t,w),w)|}{\Gamma(r_1)\Gamma(r_2)}\,dt\,ds.
\end{aligned}
$$

Using the Lebesgue Dominated Convergence Theorem, we obtain

$$
\|N(w)u_n - N(w)u\|_C \to 0 \quad \text{as } n \to \infty.
$$

As a consequence of Steps 1 and 2, we can conclude that $N(w): W(w) \to N(w)$ is a continuous random operator with stochastic domain W, and $N(w)(W(w))$ is bounded.

Step 3: For each bounded subset B of $W(w)$ we have $\alpha(N(w)B) \le \ell\alpha(B)$. Let $w \in \Omega$ be fixed. From Lemmas 2.14 and 2.15, for any $B \subset W$ and any $\epsilon > 0$, there exists a sequence $\{u_n\}_{n=0}^\infty \subset B$, such that for all $(x,y) \in J$, we have

$$
\begin{aligned}
&\alpha((N(w)B)(x,y))\\
&= \alpha\Big(\Big\{\mu(x,y) + \int_1^x \int_1^y \Big(\log\frac{x}{s}\Big)^{r_1-1}\Big(\log\frac{y}{t}\Big)^{r_2-1}\frac{f(s,t,u(s,t,w),w)}{st\Gamma(r_1)\Gamma(r_2)}\,dt\,ds;\\
&u \in B\Big\}\Big)\\
&\le 2\alpha\Big(\Big\{\int_1^x \int_1^y \Big(\log\frac{x}{s}\Big)^{r_1-1}\Big(\log\frac{y}{t}\Big)^{r_2-1}\frac{f(s,t,u_n(s,t,w),w)}{st\Gamma(r_1)\Gamma(r_2)}\,dt\,ds\Big\}_{n=1}^\infty\Big) + \epsilon\\
&\le 4\int_1^x \int_1^y \alpha\Big(\Big\{\Big(\log\frac{x}{s}\Big)^{r_1-1}\Big(\log\frac{y}{t}\Big)^{r_2-1}\\
&\quad\times \frac{f(s,t,u(s,t,w),w)}{st\Gamma(r_1)\Gamma(r_2)}\,dt\,ds\Big\}_{n=1}^\infty\Big)\,dt\,ds + \epsilon\\
&\le 4\int_1^x \int_1^y \Big(\log\frac{x}{s}\Big)^{r_1-1}\Big(\log\frac{y}{t}\Big)^{r_2-1}\\
&\quad\times \frac{1}{\Gamma(r_1)\Gamma(r_2)}\alpha\left(\{f(s,t,u_n(s,t,w),w)\}_{n=1}^\infty\right)\,dt\,ds + \epsilon\\
&\le 4\int_1^x \int_1^y \Big(\log\frac{x}{s}\Big)^{r_1-1}\Big(\log\frac{y}{t}\Big)^{r_2-1}\\
&\quad\times \frac{1}{\Gamma(r_1)\Gamma(r_2)}q(s,t,w)\alpha\left(\{u_n(s,t,w)\}_{n=1}^\infty\right)\,dt\,ds + \epsilon\\
&\le \Big(4\int_1^x \int_1^y \Big(\log\frac{x}{s}\Big)^{r_1-1}\Big(\log\frac{y}{t}\Big)^{r_2-1}\frac{1}{\Gamma(r_1)\Gamma(r_2)}q(s,t,w)dsdt\Big)\alpha\left(\{u_n\}_{n=1}^\infty\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\epsilon\\
&\le \Big(4\int_1^x\int_1^y \Big(\log\frac{x}{s}\Big)^{r_1-1}\Big(\log\frac{y}{t}\Big)^{r_2-1}\frac{1}{\Gamma(r_1)\Gamma(r_2)}q(s,t,w)\,dt\,ds\Big)\alpha(B)+\epsilon\\
&\le \frac{4q^*(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\alpha(B)+\epsilon\\
&=\ell\alpha(B)+\epsilon.
\end{aligned}
$$

Since $\epsilon>0$ is arbitrary, $\alpha(N(B))\le\ell\alpha(B)$. Hence, from Lemma 2.16 it follows that for each $w\in\Omega$, N has at least one fixed point in W. Since $\bigcap_{w\in\Omega} intW(w)\neq\emptyset$ the hypothesis that a measurable selector of $intW$ exists holds. By Lemma 2.16, N has a stochastic fixed point, i.e., the integral equation (1.1) has at least one random solution on C.

Step 4: The generalized Ulam-Hyers-Rassias stability. Set

$$
q_i^*=\operatorname{ess\,sup}_{(x,y,w)\in J\times\Omega} q_i(x,y,w);\quad i=1,2.
$$

Let $u:\Omega\to C$ be a solution of inequality (2.3). By Theorem 3.1, there exists v which is a solution of the random equation (1.1). Hence

$$
\begin{aligned}
v(x,y,w)=\mu(x,y,w)+\int_1^x\int_1^y\Big(\log\frac{x}{s}\Big)^{r_1-1}\Big(\log\frac{y}{t}\Big)^{r_2-1}\\
\times\frac{f(s,t,v(s,t,w),w)}{st\Gamma(r_1)\Gamma(r_2)}\,dt\,ds;\quad (x,y)\in J,\ w\in\Omega.
\end{aligned}
$$

From inequality (2.3) and hypotheses (H6), (H7), for each $(x,y)\in J$ and $w\in\Omega$, it follows that

$$
\begin{aligned}
&|u(x,y,w)-v(x,y,w)|\le|u(x,y,w)-N(w)(u)|+|N(w)(u)-N(w)(v)|\\
&\le\Phi(x,y,w)+\int_1^x\int_1^y\big|\log\frac{x}{s}\big|^{r_1-1}\big|\log\frac{y}{t}\big|^{r_2-1}\\
&\quad\times\frac{|f(s,t,u(s,t,w))-f(s,t,v(s,t,w))|}{\Gamma(r_1)\Gamma(r_2)}\,dt\,ds\\
&\le\Phi(x,y,w)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y\Big|\log\frac{x}{s}\Big|^{r_1-1}\Big|\log\frac{y}{t}\Big|^{r_2-1}\\
&\quad\times\Big(2q_1^*+\frac{q_2^*|u(s,t,w)|}{1+|u|}+\frac{q_2^*|v(s,t,w)|}{1+|v|}\Big)\frac{\Phi(s,t,w)}{st}\,dt\,ds\\
&\le\Phi(x,y,w)+2(q_1^*+q_2^*)(^HI_\sigma^r\Phi)(x,y,w)\\
&\le[1+2(q_1^*+q_2^*)\lambda_\phi]\Phi(x,y,w)\\
&:=c_{N,\Phi}\Phi(x,y,w).
\end{aligned}
$$

Hence, the random equation (1.1) is generalized Ulam-Hyers-Rassias stable. □

4. An example

Let $E=\mathbb{R}$ and $\Omega=(-\infty,0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty,0)$. Given a measurable function $u:\Omega\to C([1,e]\times[1,e])$, consider the following partial random Hadamard integral equation of the form

$$\begin{aligned} u(x,y,w) = \mu(x,y,w) + \int_1^x \int_1^y \Big(\log\frac{x}{s}\Big)^{r_1-1} \Big(\log\frac{y}{t}\Big)^{r_2-1} \\ \times \frac{f(s,t,u(s,t,w),w)}{st\Gamma(r_1)\Gamma(r_2)}\,dt\,ds, \end{aligned} \tag{4.1}$$

for $(x,y) \in [1,e]\times[1,e]$, $w \in \Omega$, where $r_1, r_2 > 0$, $\mu(x,y,w) = x\sin w + y^2\cos w$; $(x,y) \in [1,e]\times[1,e]$, and

$$f(x,y,u(x,y)) = \frac{w^2xy^2}{(1+w^2+u(x,y,w)|)e^{x+y+3}}, \quad (x,y) \in [1,e]\times[1,e],\ w \in \Omega.$$

The function $w \mapsto \mu(x,y,w) = x\sin w + y^2\cos w$ is measurable and bounded with

$$|\mu(x,y,w)| \le e + e^2,$$

hence, condition (H1) is satisfied.

The mapping $(x,y,w) \mapsto f(x,y,u,w)$ is jointly continuous for all $u \in \mathbb{R}$ and hence jointly measurable for all $u \in \mathbb{R}$. Also the map $u \mapsto f(x,y,u,w)$ is continuous for all $(x,y) \in [1,e]\times[1,e]$ and $w \in \Omega$. So the function f is Carathéodory on $[1,e]\times[1,e]\times\mathbb{R}\times\Omega$. For each $u \in \mathbb{R}$, $(x,y) \in [1,e]\times[1,e]$ and $w \in \Omega$, we have

$$|f(x,y,u,w)| \le w^2xy^2(1 + \frac{1}{e^3}|u|).$$

Hence the condition (H3) is satisfied with $p_1^* = e^3$ and $p_1(x,y,w) = p_2^* = 1$.

We shall show that the condition $\ell < 1$ holds with $a = b = e$ and $q^* = \frac{1}{e^3}$. Indeed, for each $r_1, r_2 > 0$ we obtain

$$\ell = \frac{4q^*(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \le \frac{4}{e^3\Gamma(1+r_1)\Gamma(1+r_2)} < 1.$$

Also, the hypothesis (H6) is satisfied with

$$\Phi(x,y,w) = w^2w^2xy^2, \quad \text{and} \quad \lambda_\Phi = \frac{1}{\Gamma(1+r_1)\Gamma(1+r_2)}.$$

Indeed, for each $(x,y) \in [1,e]\times[1,e]$ we obtain

$$({}^HI_\sigma^r\Phi)(x,y,w) \le \frac{w^2e^3}{\Gamma(1+r_1)\Gamma(1+r_2)} = \lambda_\Phi\Phi(x,y,w).$$

Finally, we can see that the hypothesis (H7) is satisfied with

$$q_1(x,y,w) = 1 \quad \text{and} \quad q_2(x,y,w) = \frac{1}{e^3}.$$

Consequently Theorem 3.1 implies that the Hadamard integral equation (4.1) has a solution defined on $[1,e]\times[1,e]$, and (4.1) is generalized Ulam-Hyers-Rassias stable.

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Saïd Abbas
Laboratory of Mathematics, University of Saïda, P.O. Box 138, 20000 Saïda, Algeria
E-mail address: abbasmsaid@yahoo.fr

Wafaa A. Albarakati
Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: wbarakati@kau.edu.sa

Mouffak Benchohra
Laboratory of Mathematics, University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria.
Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: benchohra@univ-sba.dz

Johnny Henderson
Department of Mathematics, Baylor University, Waco, Texas 76798-7328, USA
E-mail address: Johnny_Henderson@baylor.edu