

EXISTENCE OF SOLUTIONS TO FRACTIONAL DIFFERENTIAL INCLUSIONS WITH p -LAPLACIAN OPERATOR

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ABSTRACT. In this article, we prove the existence of solutions for three-point fractional differential inclusions with p -Laplacian operator. We use fixed point theory for set valued upper semi-continuous maps for obtaining the solutions.

1. INTRODUCTION

Since fractional derivatives provide an excellent tool for the description of the memory and hereditary properties of various materials and processes, the differential equations/inclusions of fractional-order are more suitable to describe a model in some real-life problems than integer-order equations [6]. The most widespread areas whose mathematical models involves derivatives of fractional order are viscoelasticity, electrochemistry, control, electromagnetism, aerodynamics, electrodynamics of complex media, polymer rheology, and so forth [5, 8, 20]. Because of these wide range of application areas, the fractional differential equations gain importance and attention day by day. Due to this importance several monographs are written. For the detailed information about differential equations involving fractional derivatives, we refer to the monographs of Kilbas et al [13], Podlubny [16], Lakshmikantham et al [14] and Samko et al [17] and the references therein.

Besides, integer order p -Laplacian boundary-value problems have been studied in terms of their importance in theory and applications in mathematics, physics and so on, see for example, [9, 23] and the references therein.

By unifying the ideas of fractional differential equations and p -laplacian operator which are mentioned above, Liu, Jia and Xiang [15] studied the fractional differential equations with p -Laplacian operator (for the first time in the literature as the authors claim). They studied the existence and uniqueness of solutions of Caputo fractional differential equation involving the p -Laplacian operator

$$(\varphi_p({}^c D^\alpha x(t)))' = f(t, x(t)),$$

with the boundary conditions

$$x(0) = r_0 x(1), \quad x'(0) = r_1 x'(1), \quad x^{(i)}(0) = 0, \quad i = 2, 3, \dots [\alpha] - 1.$$

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However there are some older studies in this area [9, 20, 21]. Chai [9] studied the existence and multiplicity of the solutions of

$$\begin{aligned} D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha}u))(t) + f(t, u(t)) &= 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) + \sigma D_{0+}^{\gamma}u(1) &= 0, \quad D_{0+}^{\alpha}u(0) = 0 \end{aligned}$$

by using fixed point theorem on cones. A similar problem with different boundary conditions

$$\begin{aligned} D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}u))(t) + f(t, u(t)) &= 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) = au(\xi), \quad D_{0+}^{\alpha}u(0) &= 0, \quad D_{0+}^{\alpha}u(1) = bD_{0+}^{\alpha}u(\mu) \end{aligned}$$

is studied by Wang and Xiang [21]. The upper and lower solutions method is used for the existence of solutions. In another article, Wang et al [20] studied the above equation with the boundary conditions

$$u(0) = 0, \quad u(1) = au(\xi), \quad D_{0+}^{\alpha}u(0) = 0$$

where Krasnoselskii's and Legget-Williams fixed point theorems are used to obtain the main results.

On the other hand, realistic problems arising from economics and optimal control can be modeled as differential inclusions which are the generalization of the concept of ordinary differential equations. Therefore, differential inclusions have been widely studied by many authors, see [1, 2, 3, 5, 9, 12, 16] and the references therein.

The differential inclusions with fractional derivatives have been studied by many authors in the literature. Among these studies one of the principle one belongs to Chang and Nieto [10]. Authors deal with the existence of solutions for the following fractional differential inclusion

$$\begin{aligned} {}_0^c D_t^{\delta} y(t) &\in F(t, y(t)) \quad t \in [0, 1], \quad \delta \in (1, 2) \\ y(0) = \alpha, \quad y(1) &= \beta, \quad \alpha, \beta \neq 0, \end{aligned}$$

where ${}_0^c D_t^{\delta} y(t)$ is the Caputo's derivative and $F : [0, 1] \times \mathbb{R} \rightarrow 2^{\mathbb{R}} \setminus \emptyset$.

It is worthwhile to emphasize that the framawork presented in this article has the following features which are different from those mentioned above. As stated above, in the literature the unification of fractional calculus with p -laplacian differential equations was studied as well as its unification with differential inclusions was also analyzed. However, to our knowledge, there are no contributions in the literature addressing the unification of fractional calculus, p -laplacian operator and differential inclusions. Motivated by this gap the intrinsic feature of the present article is to analyze the existence of solutions for the following *fractional differential inclusions with p -Laplacian operator*

$$D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u))(t) \in F(t, u(t)), \quad t \in [0, 1], \quad (1.1)$$

$$u(0) = 0, \quad u(1) = \gamma u(\eta), \quad D_{0+}^{\alpha}u(0) = 0 \quad (1.2)$$

where D_{0+}^{β} and D_{0+}^{α} are the standard Riemann-Liouville derivatives of order α and β with $\alpha \in (1, 2]$, $\beta \in (0, 1]$. Moreover $\eta \in (0, 1)$ with $1 - \gamma\eta^{\alpha-1} > 0$ and φ_p is p -Laplacian operator; i.e., $\varphi_p(s) = |s|^{p-2}s$, $p > 1$ such that $(\varphi_p)^{-1} = \varphi_q$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Also $F : [0, 1] \times \mathbb{R} \rightarrow 2^{\mathbb{R}} \setminus \emptyset$ is a multi-valued function with compact and convex values such that $|F(t, u)| = \sup\{|v| : v \in F(t, u)\}$. By $F(t, u) > 0$, we mean $v > 0$ for each $v \in F(t, u)$.

By u being a solution of (1.1)-(1.2), we mean that there exists a function $v \in C^1([0, 1], \mathbb{R})$ such that $v(t) \in F(t, u(t))$ on $[0, 1]$ satisfying the equation

$$D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u))(t) = v(t), \quad t \in [0, 1] \quad (1.3)$$

and the boundary conditions (1.2).

This article is organized as follows: Section 2 is devoted to preliminary definitions of multi-valued maps and a very brief introduction to fractional calculus. In Section 3, we first state the fixed point results. Consequently, we prove the existence results for the integral inclusion corresponding to the main problem (1.1)-(1.2). Finally, we illustrate our main result by an example.

2. PRELIMINARIES

In this section, we list some preliminary definitions, notation and results that will be used in the rest of the article.

Let $C(I)$ denote the Banach space of continuous functions from I into \mathbb{R} with the supremum norm $\|y\| = \sup_{t \in I} \{|y(t)|\}$.

Let $(X, \|\cdot\|)$ be a Banach space and $CK(X)$ denote the family of nonempty, closed and convex subsets of X . A multi-valued map $H : X \rightarrow CK(X)$ is said to be *upper semi-continuous (u.s.c.)* provided that $\{u_k\}_{k \in \mathbb{N}}, \{v_k\}_{k \in \mathbb{N}} \subset X$ with $u_k \rightarrow u, v_k \rightarrow v$ (while $k \rightarrow \infty$) and $v_k \in H(u_k)$ for all $k \in \mathbb{N}$ imply $v \in H(u)$. Moreover a multi-valued map H is said to be *completely continuous* if $H(B)$ is relatively compact for every bounded subset B of X . Furthermore, we say that H has a fixed point if there exists $x \in X$ such that $x \in H(x)$.

Throughout this article, we impose the following condition on the multi-valued function F :

(H1) $F : [0, 1] \times \mathbb{R} \rightarrow CK(\mathbb{R})$ is a multi-valued map such that $F(t, \cdot)$ is u.s.c. for all $t \in [0, 1]$.

Definition 2.1 ([13, 16]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (a, b) \rightarrow \mathbb{R}$ is given by

$$I_{a+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}y(s)ds, \quad t \in (a, b].$$

Definition 2.2 ([13, 16]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : (a, b) \rightarrow \mathbb{R}$ is given by

$$D_{a+}^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds, \quad t \in (a, b],$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α .

Lemma 2.3. *Let $\alpha > 0$. If $y \in C(0, 1) \cap L(0, 1)$ possesses a fractional derivative of order α that belongs to $C(0, 1) \cap L(0, 1)$, then*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}y(t) = y(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \dots + c_nt^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, where $n = [\alpha] + 1$.

To find the form of the a solution of the problem (1.3)-(1.2), we first consider the fractional boundary-value problem

$$\begin{aligned} D_{0+}^{\alpha}u(t) &= \phi(t), \quad t \in [0, 1], \\ u(0) &= 0, \quad u(1) = \gamma u(\eta), \end{aligned}$$

where $\phi \in C([0, 1], \mathbb{R})$. Ahmad and Nieto [4] presented the unique solution of the above problem by

$$u(t) = \int_0^1 G(t, s)\phi(s)ds,$$

where $G(t, s)$ is the Green's function given by

$$G(t, s) = \frac{1}{\Gamma(\alpha)(1 - \gamma\eta^{\alpha-1})} \begin{cases} G_1(t, s), & 0 \leq t \leq \eta, \\ G_2(t, s), & \eta < t \leq 1. \end{cases} \quad (2.1)$$

Here $G_1(t, s)$ and $G_2(t, s)$ are given by

$$G_1(t, s) = \begin{cases} (t-s)^{\alpha-1}(1 - \gamma\eta^{\alpha-1}) - t^{\alpha-1}[(1-s)^{\alpha-1} - \gamma(\eta-s)^{\alpha-1}], & 0 \leq s \leq t, \\ -t^{\alpha-1}[(1-s)^{\alpha-1} - \gamma(\eta-s)^{\alpha-1}], & t < s \leq \eta, \\ -(t(1-s))^{\alpha-1}, & \eta < s \leq 1, \end{cases}$$

$$G_2(t, s) = \begin{cases} (t-s)^{\alpha-1}(1 - \gamma\eta^{\alpha-1}) - t^{\alpha-1}[(1-s)^{\alpha-1} - \gamma(\eta-s)^{\alpha-1}], & 0 \leq s \leq \eta, \\ (t-s)^{\alpha-1}(1 - \gamma\eta^{\alpha-1}) - (t(1-s))^{\alpha-1}, & \eta < s \leq t, \\ -(t(1-s))^{\alpha-1}, & t < s \leq 1 \end{cases}$$

respectively.

Substituting $D_{0+}^\alpha u = \phi$, $\varphi_p(\phi) = \omega$ in (1.3), we obtain the equation $D_{0+}^\beta \omega(t) = v(t)$. Lemma 2.3 implies that the solution of initial value problem

$$D_{0+}^\alpha \omega(t) = v(t), \quad t \in [0, 1], \omega(0) = 0,$$

is of the form $\omega(t) = c_1 t^{\beta-1} + I_{0+}^\beta v(t)$. It follows from the initial condition and $\beta \in (0, 1]$ that $c_1 = 0$. Hence

$$\omega(t) = I_{0+}^\beta v(t), \quad t \in [0, 1].$$

Taking $D_{0+}^\alpha u = \phi$ and $\phi = \varphi_p^{-1}(\omega) = \varphi_q(\omega)$ into account, we establish that the solution of (1.3)-(1.2) satisfies the boundary-value problem

$$D_{0+}^\alpha u(t) = \varphi_q(I_{0+}^\beta v)(t), \quad t \in [0, 1], \quad (2.2)$$

$$u(0) = 0, \quad u(1) = \gamma u(\eta), \quad D_{0+}^\alpha u(0) = 0. \quad (2.3)$$

Using the result of Ahmad and Nieto [4], we obtain the solution of the problem (2.2)-(2.3) as

$$u(t) = \int_0^1 G(t, s)\varphi_q(I_{0+}^\beta v)(s)ds, \quad t \in [0, 1].$$

Since $v(t) > 0$ for $t \in [0, 1]$, we have $\varphi_q(I_{0+}^\beta v)(s) = (I_{0+}^\beta v)^{q-1}(s)$ and therefore

$$\begin{aligned} u(t) &= \int_0^1 G(t, s)(I_{0+}^\beta v)^{q-1}(s)ds \\ &= \frac{1}{(\Gamma(\beta))^{q-1}} \int_0^1 G(t, s) \left(\int_0^s (s-\tau)^{\beta-1} v(\tau) d\tau \right)^{q-1} ds. \end{aligned}$$

Hence we have the following lemma.

Lemma 2.4. *Let $v(t) > 0$ for $t \in [0, 1]$. Then the solution of (1.3)-(1.2) is given by*

$$u(t) = \frac{1}{(\Gamma(\beta))^{q-1}} \int_0^1 G(t, s) \left(\int_0^s (s-\tau)^{\beta-1} v(\tau) d\tau \right)^{q-1} ds.$$

3. EXISTENCE RESULTS

This section is devoted to the existence results regarding solutions for the differential inclusion (1.1)-(1.2). For this purpose, we first give an existence result for the integral inclusion corresponding to (1.1)-(1.2). Consequently, we derive an existence result for the differential inclusion (1.1)-(1.2). Finally, we illustrate our result by an example.

Assume that $G : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is a single-valued function. By a solution u of the integral inclusion

$$u(t) \in \frac{1}{(\Gamma(\beta))^{q-1}} \int_0^1 G(t, s) \left(\int_0^s (s - \tau)^{\beta-1} F(\tau, u(\tau)) d\tau \right)^{q-1} ds, \quad t \in [0, 1] \quad (3.1)$$

we mean that there exists a function $v \in C^1([0, 1], \mathbb{R})$ such that $v(t) \in F(t, u(t))$ on $[0, 1]$ satisfying the integral equation

$$u(t) = \frac{1}{(\Gamma(\beta))^{q-1}} \int_0^1 G(t, s) \left(\int_0^s (s - \tau)^{\beta-1} v(\tau) d\tau \right)^{q-1} ds, \quad t \in [0, 1].$$

The proofs of our main existence results are based on the following two fixed point results.

Theorem 3.1 ([2]). *Let C be a nonempty closed convex subset of a Banach space E and $H : C \rightarrow CK(C)$ an u.s.c. compact map, then H has a fixed point in C .*

Theorem 3.2 ([2]). *Let E be a Banach space, U an open subset of E and $0 \in U$. If $H : \bar{U} \rightarrow CK(E)$ is an u.s.c. compact map, then either*

- (1) H has a fixed point in \bar{U} or
- (2) there exists $u \in \partial U$ and $\lambda \in (0, 1)$ such that $u \in \lambda H(u)$.

Our main result is based on the following existence principle.

Lemma 3.3. *Assume that (H1) holds and $G(t, s) : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is continuous. Then we have the following existence results:*

(a) *For any $r > 0$, suppose that there exists a continuous $h_r \in C([0, 1])$ with $|F(t, u)| \leq h_r(t)$ for all $t \in [0, 1]$ and all $|u| \leq r$. If there exists a constant M with $\|u\| \neq M$ for all solutions u of integral inclusion*

$$u(t) \in \frac{\lambda}{(\Gamma(\beta))^{q-1}} \int_0^1 G(t, s) \left(\int_0^s (s - \tau)^{\beta-1} F(\tau, u(\tau)) d\tau \right)^{q-1} ds, \quad (3.2)$$

for each $\lambda \in (0, 1)$, then the inclusion (3.1) has a solution.

(b) *Suppose that there exists a continuous function $h \in C([0, 1])$ with $|F(t, u)| \leq h(t)$ for all $t \in [0, 1]$ and all $u \in \mathbb{R}$, then the inclusion (3.1) has a solution.*

Proof. (a) We define a linear and continuous operator $H : C([0, 1]) \rightarrow C([0, 1])$ by

$$Hu(t) := \frac{1}{(\Gamma(\beta))^{q-1}} \int_0^1 G(t, s) \left(\int_0^s (s - \tau)^{\beta-1} u(\tau) d\tau \right)^{q-1} ds, \quad t \in [0, 1].$$

Let

$$\mathcal{F}(u) := \{v \in C([0, 1]) : v(t) \in F(t, u(t)) \text{ for } t \in [0, 1]\}.$$

Clearly $F : [0, 1] \times \mathbb{R} \rightarrow CK(\mathbb{R})$ implies that $\mathcal{F} : C([0, 1]) \rightarrow CK(C([0, 1]))$. Note that the integral inclusion (3.2) is equivalent to the fixed point problem

$$u \in \lambda(H \circ \mathcal{F})(u),$$

where $H \circ \mathcal{F} : C([0, 1]) \rightarrow CK(C([0, 1]))$.

Let $U := \{u \in C([0, 1]) : \|u\| < M\}$ and $E = C([0, 1])$. Now we apply Theorem 3.2 to the function $H \circ \mathcal{F}$ for the existence of solutions for the inclusion (3.1).

Assume there exists $u \in \partial U$ and $\lambda \in (0, 1)$ with $u \in \lambda(H \circ \mathcal{F})(u)$. Then $\|u\| = M$ and so the second possibility given in Theorem 3.2 is ruled out. Hence, if $H \circ \mathcal{F} : \bar{U} \rightarrow CK(E)$ is u.s.c. and compact then Theorem 3.2 guarantees that $H \circ \mathcal{F}$ has a fixed point in \bar{U} , i.e. (3.1) has a solution.

First, we show that $H \circ \mathcal{F} : \bar{U} \rightarrow CK(E)$ is u.s.c. For this purpose we let $\{u_k\}_{k \in \mathbb{N}}, \{w_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$ with $u_k \rightarrow u_0, w_k \rightarrow w_0$ in $C([0, 1])$ as $k \rightarrow \infty$ and $w_k \in H \circ \mathcal{F}(u_k)$ for $k \in \mathbb{N}$. Thus there exist $v_k \in \mathcal{F}(u_k)$ with $w_k = H v_k$. Since $u_k \in \bar{U}$ for all $k \in \mathbb{N}$, the condition $|F(t, u)| \leq h_r(t)$ for all $|u| < r$ with $h_r \in C([0, 1])$ guarantees (see the proof of [3, Remark 2.1]) that there exists a compact set $\Omega \subset E$ with $\{v_k\}_{k \in \mathbb{N}} \subseteq \Omega$. Therefore there exists a convergent subsequence $\{v_{k_n}\}_{n \in \mathbb{N}}$ of $\{v_k\}_{k \in \mathbb{N}}$, such that $v_{k_n} \rightarrow v_0$ as $n \rightarrow \infty$. Now $v_{k_n} \rightarrow v_0$ and $u_{k_n} \rightarrow u_0$ as $n \rightarrow \infty$ and $v_{k_n}(t) \in F(t, u_{k_n}(t))$ for all $t \in [0, 1]$. Thus, since $F(t, \cdot)$ is u.s.c. for all $t \in [0, 1]$, we conclude with $v_0(t) \in F(t, u_0(t))$ which results $v_0 \in \mathcal{F}(u_0)$. Since $v_{k_n} \rightarrow v_0$ as $n \rightarrow \infty$ and H is continuous, we see that $w_{k_n} = H v_{k_n} \rightarrow H v_0$ as $n \rightarrow \infty$, and hence $w_0 = H v_0 \in (H \circ \mathcal{F})(u_0)$. As a result $H \circ \mathcal{F} : \bar{U} \rightarrow CK(E)$ is u.s.c. The compactness of $H \circ \mathcal{F}$ follows from the Arzela-Ascoli theorem. Therefore the proof of part (a) is complete.

(b) Let H and \mathcal{F} be as in part (a). Clearly proving the existence a solution for the integral inclusion (3.1) is equivalent to the fixed point problem

$$u \in H \circ \mathcal{F}(u).$$

The argument in part (a) guarantees that $H \circ \mathcal{F}$ is u.s.c. and compact and hence the claim follows by using Theorem 3.1. \square

With the help of Lemma 3.3, we present our existence result for fractional differential inclusion (1.1) with the boundary conditions (1.2).

Theorem 3.4. *Assume that (H1) holds, and there exist a continuous nondecreasing function $\psi : [0, \infty) \rightarrow [0, \infty)$ with $\psi(u) > 0$ for $u > 0$ and a function $r : [0, 1] \rightarrow [0, \infty)$ such that $|F(t, u)| \leq r(t)\psi(|u|)$ for all $u \in \mathbb{R}$ and $t \in [0, 1]$. If Q defined by*

$$Q := \frac{1}{(\Gamma(\beta))^{q-1}} \max_{t \in [0, 1]} \left(\int_0^1 |G(t, s)| \left(\int_0^s (s - \tau)^{\beta-1} r(\tau) d\tau \right)^{q-1} ds \right),$$

where $G(t, s)$ is defined by (2.1), satisfies the property

$$\sup_{c \in (0, \infty)} \frac{c}{(\psi(c))^{q-1}} > Q,$$

then the problem (1.1)-(1.2) has a solution.

Proof. Suppose $M > 0$ satisfies $M/(\psi(M))^{q-1} > Q$. Consider

$$D_{0+}^{\beta} (\varphi_p(D_{0+}^{\alpha} u))(t) \in \lambda F(t, u(t)), \quad t \in [0, 1], \quad (3.3)$$

with the boundary condition (1.2) and $\lambda \in (0, 1)$. Solving the inclusion (3.3) is equivalent to finding a function $u \in C([0, 1])$ which satisfies the equation (3.2) for $t \in [0, 1]$. Therefore by Lemma 3.3, it is enough to show that there exists a constant $M > 0$ with $\|u\| = \max_{t \in [0, 1]} |u(t)| \neq M$ for any solution u of the inclusion (3.2). Let u be any solution of (3.2) for $\lambda \in (0, 1)$.

For $t \in [0, 1]$, we have

$$\begin{aligned} |u(t)| &\leq \frac{1}{(\Gamma(\beta))^{q-1}} \int_0^1 |G(t, s)| \left(\int_0^s (s-\tau)^{\beta-1} |F(\tau, u(\tau))| d\tau \right)^{q-1} ds \\ &\leq \frac{1}{(\Gamma(\beta))^{q-1}} \int_0^1 |G(t, s)| \left(\int_0^s (s-\tau)^{\beta-1} |r(\tau)| \psi(|u(\tau)|) d\tau \right)^{q-1} ds \\ &\leq \left(\frac{\psi(\|u\|)}{\Gamma(\beta)} \right)^{q-1} \int_0^1 |G(t, s)| \left(\int_0^s (s-\tau)^{\beta-1} r(\tau) d\tau \right)^{q-1} ds. \end{aligned}$$

Therefore,

$$\frac{\|u\|}{(\psi(\|u\|))^{q-1}} \leq Q.$$

If $\|u\| = M$, then

$$\frac{M}{(\psi(M))^{q-1}} \leq Q,$$

which contradicts our assumption. Hence we obtain $\|u\| \neq M$ which is the desired result. \square

We illustrate our result with the following example. Consider the problem

$$D_{0+}^{\frac{1}{2}} (\varphi_{\frac{3}{2}}(D_{0+}^{\frac{4}{3}} u))(t) \in F(t, u(t)), \quad t \in [0, 1], \quad (3.4)$$

$$u(0) = 0, \quad u(1) = \frac{1}{4^{\frac{2}{3}}} u\left(\frac{1}{4}\right), \quad D_{0+}^{\frac{4}{3}} u(0) = 0 \quad (3.5)$$

where $F : [0, 1] \times \mathbb{R} \rightarrow 2^{\mathbb{R}} \setminus \emptyset$ is a multi-valued map defined by

$$(t, u) \rightarrow F(t, u) := \left[\frac{u^4}{4(u^4 + 1)} + \frac{t^2 + 1}{8}, \frac{1}{4} \cos u + \frac{t^2 + 1}{4} \right].$$

For $v \in F$, we have

$$|v| \leq \max \left[\frac{u^4}{4(u^4 + 1)} + \frac{t^2 + 1}{8}, \frac{1}{4} \cos u + \frac{t^2 + 1}{4} \right] \leq \frac{3}{4}.$$

Thus,

$$|F(t, u)| = \sup\{|v| : v \in F(t, u)\} \leq r(t)\psi(|u|), \quad u \in \mathbb{R},$$

with $r(t) = 1$, $\psi(u) = \sqrt{u} + 1$. Clearly, $\sup_{c \in (0, \infty)} \frac{c}{(\psi(c))^2} = 1 > 0,0785502$ is satisfied. Hence the problem (3.4)-(3.5) has a solution on $[0, 1]$.

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