We investigate the existence of weak solutions to a class of quasi-linear elliptic equations with nonlinear Neumann boundary conditions in exterior domains. Problems of this kind arise in various areas of science and technology. An important model case related to the initial data problem in general relativity is presented. As an application of our main result, we deduce the existence of the conformal factor for the Hamiltonian constraint in general relativity in the presence of multiple black holes. We also give a proof for uniqueness in this case.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an exterior domain with smooth compact boundary. In this paper, we study the existence and uniqueness of solutions of the following elliptic boundary-value problem

\begin{align}
- \text{div} \left[ A(x, \nabla u) \right] &= F(x, u), \quad x \in \Omega, \quad \quad \quad (1.1) \\
A(x, \nabla u) \cdot n &= f(x, u), \quad x \in \partial \Omega, \quad (1.2)
\end{align}

where $n$ stands for the unit exterior normal to $\partial \Omega$. Here $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function satisfying the following conditions:

- There exist $p > 1$, $a_1(\cdot) \in L^{p'}(\Omega)$ ($p'$ is the conjugate of $p$, that is $1/p + 1/p' = 1$), and $b_1 > 0$ such that $|A(x, \xi)| \leq a_1(x) + b_1|\xi|^{p-1}$, for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$.
- $A(x, \xi)$ is strictly monotone in $\xi$, that is $[A(x, \xi_2) - A(x, \xi_1) \cdot (\xi_2 - \xi_1)] > 0$, for a.e. $x \in \Omega$ and all $\xi_1, \xi_2 \in \mathbb{R}^N$ with $\xi_1 \neq \xi_2$.
- There exist $a_2 \in L^1(\Omega)$ and $b_2 > 0$ such that the following coercivity property holds $A(x, \xi) \cdot \xi \geq b_2|\xi|^p - a_2(x)$, for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$.

Problems of this type arise in many and diverse contexts like differential geometry (e.g., in the scalar curvature problem and the Yamabe problem), nonlinear elasticity, non–Newtonian fluid mechanics, mathematical biology, general relativity, and elsewhere. In Section 3 we address one of these applications related to the

\textbf{2000 Mathematics Subject Classification.} 35J65, 83C05.

\textbf{Key words and phrases.} Quasilinear equation; exterior domain; general relativity; nonlinear boundary conditions; initial data problem; conformal factor.

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initial data problem in general relativity (more precisely, the existence of conformal factor to the Hamiltonian constraint equation in the case of multiple black holes).

Nonlinear boundary value problems related to (1.1)-(1.2) have been studied for some time by numerous authors and in various frameworks. For example, in [24], Pflüger considered the problem (1.1)-(1.2) for the $p$-Laplacian with polynomial nonlinearities on the right hand side and in the boundary condition. In this context, Pflüger showed the existence of a nontrivial, positive weak solution. Due to the unbounded domain, the lack of compactness was overcome through the use of weighted Sobolev spaces. For more recent work on this subject, see [3, 4, 12, 17, 20, 21, 25, 28, 30], and references therein.

We denote by $W^{1,p}(\Omega)$ the weighted Sobolev space (the suitable weight function in our case is $(1 + |x|^2)^{-1/2}$ for $x \in \Omega$)

$$W^{1,p}(\Omega) := \{ u \in L^p_{\text{loc}}(\Omega) : \frac{u}{(1 + |x|^2)^{1/2}} \in L^p(\Omega) \text{ and } \nabla u \in L^p(\Omega) \}.$$ 

Notice that on each bounded part of the open set $\Omega$, the space $W^{1,p}(\Omega)$ coincides with the usual Sobolev space $W^{1,p}_{\text{loc}}(\Omega)$. Functions in these two spaces differ only by their behaviour at infinity. For more on these spaces, see [22, 26], and references therein.

A variational formulation for the exterior boundary-value problem (1.1) and (1.2) is

$$\int_{\Omega} A(x, \nabla u) \cdot \nabla v \, dx - \int_{\Omega} F(x, u)v \, dx - \int_{\partial \Omega} f(x, u)v \, d\sigma = 0, \quad \forall v \in W^{1,p}(\Omega).$$

A function $u$ (resp. $\overline{u}$) in $W^{1,p}(\Omega)$ is called a (weak) subsolution (resp. supersolution) of (1.1) and (1.2) if

$$\int_{\Omega} A(x, \nabla u) \cdot \nabla v \, dx - \int_{\Omega} F(x, u)v \, dx - \int_{\partial \Omega} f(x, u)v \, d\sigma \leq 0, \quad (\text{resp. } \geq) \quad (1.3)$$

for each $v \in W^{1,p}(\Omega)$, $v \geq 0$ a.e. in $\Omega$.

Under the above conditions, our main result may be stated as follows.

**Theorem 1.1.** Assume there exist a pair of sub- and supersolution $u$ and $\overline{u}$ of (1.1)-(1.2) and that the functions $F$ and $f$ satisfy the following growth conditions:

- There exists $a_3 \in L^{p'}(\Omega)$ such that $|F(x, u)| \leq a_3(x)/(1 + |x|^2)^{1/2}$, for a.e. $x \in \Omega$ and all $u \in [u(x), \overline{u}(x)]$.
- There exist $a_4 \in L^p(\partial \Omega)$ and $b_3 \in L^p(\partial \Omega)$ such that $|f(x, u)| \leq a_4(x) + b_3(x)|u|^{p-1}$, for a.e. $x \in \partial \Omega$ and all $u \in [u(x), \overline{u}(x)]$.

Then, (1.1)-(1.2) has at least one (weak) solution $u \in W^{1,p}(\Omega)$ such that $u \leq u \leq \overline{u}$.

A proof of this theorem is given in Section 2. As an application of this result, we will discuss the existence of the conformal factor for the Hamiltonian constraint in general relativity in Section 3. We also provide a proof for the uniqueness of the conformal factor in the case of multiple black holes in Subsection 3.2.
2. Proof of Theorem 1.1

Let \( \rho := (1 + |x|^2)^{1/2} \). For \( u \in W^{1,p}(\Omega) \), define

\[
b(x, u) := \begin{cases} 
[u(x) - \underline{u}(x)]^{p-1}/\rho^p & \text{if } u(x) > \underline{u}(x) \\
0 & \text{if } \underline{u}(x) \leq u(x) \leq \overline{u}(x) \\
-[u(x) - \underline{u}(x)]^{p-1}/\rho^p & \text{if } u(x) < \underline{u}(x)
\end{cases}
\]

\[
(Tu)(x) := \begin{cases} 
\underline{u}(x) & \text{if } u(x) > \overline{u}(x) \\
u(x) & \text{if } \underline{u}(x) \leq u(x) \leq \overline{u}(x) \\
\overline{u}(x) & \text{if } u(x) < \underline{u}(x)
\end{cases}
\]

Next, consider the operators \( A, B, F, \) and \( G : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^* \) defined by:

\[
\langle A(u), v \rangle := \int_\Omega A(x, \nabla u) \cdot \nabla v \, dx, \quad \langle B(u), v \rangle := \int_\Omega b(x, u)v \, dx,
\]

\[
\langle F(u), v \rangle := -\int_\Omega F(x, Tu)v \, dx, \quad \langle G(u), v \rangle := -\int_{\partial \Omega} f(x, Tu)v \, d\sigma,
\]

\[\Gamma : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*, \quad \Gamma(u) := A(u) + B(u) + F(u) + G(u).\]

The following lemma states that solving \( \Gamma(u) = 0 \) in \( (W^{1,p}(\Omega))^* \) produces a weak solution \( u \) to problem \((1.1)-(1.2)\), with \( \underline{u} \leq u \leq \overline{u} \) a.e. in \( \Omega \). Its proof relies on arguments largely similar to the ones used in [15] [16] (see also [2]) in a different context.

**Lemma 2.1.** Assume that \( u \in W^{1,p}(\Omega) \) is a solution to \( \Gamma(u) = 0 \). Then \( u \) is also a weak solution to \((1.1)-(1.2)\), with \( \underline{u} \leq u \leq \overline{u} \) a.e. in \( \Omega \).

**Proof.** Since \( u \) and \( \underline{u} \) are elements of \( W^{1,p}(\Omega) \), it follows that \( (\underline{u} - u)^+ \in W^{1,p}(\Omega) \). Then

\[
\langle \Gamma u, (\underline{u} - u)^+ \rangle = \langle A(u) + B(u) + F(u) + G(u), (\underline{u} - u)^+ \rangle = 0,
\]

and so

\[
\int_\Omega A(x, \nabla u) \cdot \nabla (\underline{u} - u)^+ \, dx + \int_\Omega b(x, u)(\underline{u} - u)^+ \, dx 
- \int_\Omega F(x, Tu)(\underline{u} - u)^+ \, dx - \int_{\partial \Omega} f(x, Tu)(\underline{u} - u)^+ \, d\sigma = 0.
\]

Since \( u \) is a subsolution to \((1.1)-(1.2)\), we have

\[
\int_\Omega A(x, \nabla \underline{u}) \cdot \nabla (\underline{u} - u)^+ \, dx - \int_\Omega F(x, \underline{u})(\underline{u} - u)^+ \, dx - \int_{\partial \Omega} f(x, \underline{u})(\underline{u} - u)^+ \, d\sigma \leq 0.
\]

Subtracting \((2.2)\) from \((2.3)\) gives

\[
\int_\Omega [A(x, \nabla \underline{u}) - A(x, \nabla u)] \cdot \nabla (\underline{u} - u)^+ \, dx - \int_\Omega [F(x, \underline{u}) - F(x, Tu)](\underline{u} - u)^+ \, dx 
- \int_{\partial \Omega} [f(x, \underline{u}) - f(x, Tu)](\underline{u} - u)^+ \, d\sigma 
\leq \int_\Omega b(x, u)(\underline{u} - u)^+ \, dx.
\]
Observe that (from the hypotheses and Stampachia’s Theorem)
\[
\int_{\Omega} [A(x, \nabla u) - A(x, \nabla u)] \cdot \nabla (u - u)^+ \, dx
\]
\[
= \int_{\{u(x) > u(x)\}} [A(x, \nabla u) - A(x, \nabla u)] \cdot (\nabla u - \nabla u) \, dx \geq 0.
\]

Furthermore, from the definition of \( Tu \), it follows that \( Tu(x) = u(x) \) on \( \{u(x) > u(x)\} \), and so
\[
\int_{\Omega} [F(x, u) - F(x, Tu)](u - u)^+ \, dx
\]
\[
= \int_{\{u(x) > u(x)\}} [F(x, u) - F(x, Tu)](u - u) \, dx = 0.
\]

Also,
\[
\int_{\partial \Omega} [f(x, u) - f(x, Tu)](u - u)^+ \, d\sigma
\]
\[
= \int_{\{x \in \partial \Omega: u(x) > u(x)\}} [f(x, u) - f(x, Tu)](u - u) \, d\sigma = 0.
\]

By (2.4), (2.5), (2.6), and (2.7), we obtain
\[
0 \leq \int_{\Omega} b(x, u)(u - u)^+ \, dx = -\int_{\{u(x) > u(x)\}} (u - u)^p / p^p \, dx \leq 0,
\]
and thus \( u = u \) a.e. in \( \{u(x) > u(x)\} \). That is, the set \( \{u(x) > u(x)\} \) has measure 0. This shows that \( u(x) \leq u \) a.e. in \( \Omega \). For the inequality \( u(x) \leq u \) a.e. in \( \Omega \), we proceed similarly (by considering \( (u - u)^+ \) this time). Since \( u(x) \leq u(x) \) a.e. in \( \Omega \), we have both \( b(x, u(x)) = 0 \) and \( Tu(x) = 0 \) a.e. in \( \Omega \). Thus, \( u \) is a weak solution of (1.1)–(1.2).

\[\square\]

**Lemma 2.2.** The operator \( \Gamma \) is bounded.

**Proof.** Let \( M \) be a bounded subset of \( W^{1,p}(\Omega) \), that is, there exists a constant \( C_1 \geq 0 \) such that \( \|u\|_{W^{1,p}(\Omega)} \leq C_1 \), for all \( u \in M \). Our goal is to prove that there is a constant \( C_2 \geq 0 \) such that \( \|\Gamma(u)\|_{W^{1,p}(\Omega)} \leq C_2 \), for all \( u \in M \). Hereafter, the symbol \( \lesssim \) between two terms means that the first term is bounded from above by the second term up to a multiplicative positive constant that may depend on \( M \) but not on the individual elements of \( M \).

For \( u \in M \) and \( v \in W^{1,p}(\Omega) \), we have
\[
|\langle \Gamma(u), v \rangle| \leq |\langle A(u), v \rangle| + |\langle B(u), v \rangle| + |\langle F(u), v \rangle| + |\langle G(u), v \rangle|.
\]

Let us place an upper bound on the terms on the right-hand side of inequality (2.8).
\[
|\langle A(u), v \rangle| \leq \int_{\Omega} |A(x, \nabla u)| \cdot |\nabla u| \, dx
\]
\[
\leq \int_{\Omega} (a_1(x) + b_1) |\nabla u|^{p-1} \cdot |\nabla v| \, dx
\]
\[
\leq \|a_1\|_{L^{p'}(\Omega)} \|\nabla v\|_{L^p(\Omega)} + b_1 \|\nabla u\|_{L^p(\Omega)}^{p-1} \|\nabla v\|_{L^p(\Omega)}
\]
\[
\leq (\|a_1\|_{L^{p'}(\Omega)} + b_1 \|u\|_{W^{1,p}(\Omega)}^{p-1}) \|v\|_{W^{1,p}(\Omega)}
\]
\[
\lesssim \|v\|_{W^{1,p}(\Omega)}.
\]
Next, observe that
\[ |b(x, u)| \leq \rho^{-p}(|u(x)| + |f(x)| + |g(x)|) \]
\[ \lesssim \rho^{-p}(|u(x)| + |f(x)| + |g(x)|) \]
\[ \lesssim \rho^{-p}A_1(x) + \rho^{-p}u(x), \]
with \( A_1(x) := |\rho^{-1}f(x)|^{p-1} + |\rho^{-1}g(x)|^{p-1} \in L^p(\Omega). \) Thus,
\[ |\langle B(u), v \rangle| \leq \int_\Omega |b(x, u)| \cdot |v|dx \]
\[ \lesssim \left( |\langle \nabla u(x) \rangle|^{p-1} + |\rho^{-1}v| \right) \|\rho^{-1}v\|_{L^p(\Omega)} \]
\[ \lesssim |\langle \nabla u(x) \rangle|^{p-1} + |\rho^{-1}v| \]
\[ \lesssim \|\rho^{-1}v\|_{L^p(\Omega)}. \]
For the third term of the right-hand side of (2.8) we obtain the following upper bound
\[ |\langle F(u), v \rangle| \leq \int_\Omega |F(x, Tu)| \cdot |v|dx \]
\[ \leq \int_\Omega |\rho^{-1}v|dx \]
\[ \leq \|\rho^{-1}v\|_{L^p(\Omega)} \]
\[ \leq \|\rho^{-1}v\|_{L^p(\Omega)}. \]
Finally, for the last term of the right-hand side of (2.8) we have
\[ |\langle G(u), v \rangle| \leq \int_{\partial \Omega} |f(x, Tu)| \cdot |v|d\sigma \]
\[ \leq \int_{\partial \Omega} \left( a_4(x) + b_3(x) |Tu|^{p-1} \right) \cdot |v|d\sigma \]
\[ \leq \int_{\partial \Omega} \left( a_4(x) + b_3(x) \left( |u(x)|^{p-1} + |f(x)|^{p-1} \right) \right) \cdot |v|d\sigma \]
\[ \leq \|a_4(x) + b_3(x) \left( |u(x)|^{p-1} + |f(x)|^{p-1} \right) \|_{L^p(\partial \Omega)} \|v\|_{L^p(\partial \Omega)} \]
\[ \lesssim \|v\|_{W^{1,p}(\Omega)}, \]
where the last inequality is a consequence of the trace theorem. Returning to inequality (2.8), and using (2.9), (2.10), (2.11), and (2.12), it follows that there exists a positive constant \( C_2 \) such that \( |\langle \Gamma(u), v \rangle| \leq C_2\|v\|_{W^{1,p}(\Omega)} \), for all \( u \in M \) and all \( v \in W^{1,p}(\Omega) \); that is, \( \|\Gamma(u)\|_{W^{1,p}(\Omega)} \leq C_2 \), for all \( u \in M \). □

**Lemma 2.3.** The operator \( \Gamma \) is coercive; that is,
\[ \lim_{\|u\|_{W^{1,p}(\Omega)} \to \infty} \frac{\langle \Gamma(u), u \rangle}{\|u\|_{W^{1,p}(\Omega)}} = \infty. \] (2.13)

**Proof.** First of all, observe that
\[ \langle A(u), u \rangle \geq b_2\|\nabla u\|_{L^p(\Omega)}^p - \|a_2\|_{L^1(\Omega)}. \] (2.14)
It is easy to prove that, for \( a > b \) and \( p > 1 \), there are positive constants \( C_1, C_2, C_3, \) and \( C_4 \) (independent of \( a, b \)) such that \( (a - b)^{p-1} a \geq C_1 |a| - C_2 |b|^{p-1} |a| \) and
\[(a - b)^{p-1}b \leq C_4|a|^{p-1}|b| - C_4|b|^p.\]

\[
\langle B(u), u \rangle = \int_{\{u > \overline{u}\}} \rho^{-p}(u - \overline{u})^{p-1}u \, dx - \int_{\{u \leq \underline{u}\}} \rho^{-p}(\underline{u} - u)^{p-1}u \, dx \\
\geq \int_{\{u > \overline{u}\}} \rho^{-p}(C_1|u|^p - C_2|\overline{u}|^{p-1}|u|) \, dx \\
+ \int_{\{u \leq \underline{u}\}} \rho^{-p}(C_4|u|^p - C_3|\underline{u}|^{p-1}|u|) \, dx \\
\geq \min \{C_1, C_4\} \rho^{-1}u_{L^p(\Omega)}^{p-1} - C_5 \rho^{-1}u_{L^{p'}(\Omega)}^{p-1}.
\]

with \(C_5 := C_2|\rho^{-1}\overline{u}|_{L^p(\Omega)}^{p-1} + C_3|\rho^{-1}\underline{u}|_{L^p(\Omega)}^{p-1}\). Also,

\[
\langle F(u), u \rangle \geq -\|a_3\|_{L^{p'}(\Omega)} \rho^{-1}u_{L^p(\Omega)}^{p-1} \geq -\|a_3\|_{L^{p'}(\Omega)} \rho^{-1}u_{W^{1,p}(\Omega)}^{p-1}
\]

and

\[
\langle G(u), u \rangle \geq \int_{\partial \Omega} |a_4(x)| + |b_3(x)||(u(x)|^{p-1} + |\overline{u}||^{p-1})|u| \, d\sigma \\
\geq -\|a_4\|_{L^{p'}(\partial \Omega)} + |b_3||(u|^{p-1} + |\overline{u}||^{p-1})|u|_{L^p(\partial \Omega)} \\
\geq -C_6\|a_4\|_{L^{p'}(\partial \Omega)} + |b_3||(u|^{p-1} + |\overline{u}||^{p-1})|u|_{W^{1,p}(\Omega)}^{p-1},
\]

where the last inequality follows from the trace theorem. Combining (2.14), (2.15), (2.16), and (2.17), we get

\[
\langle \Gamma(u), u \rangle \geq C_7\|u\|_{W^{1,p}(\Omega)}^{p-1} - C_8\|u\|_{W^{1,p}(\Omega)}^{p-1} - C_9, \quad \forall u \in W^{1,p}(\Omega),
\]

with \(C_7, C_8, C_9 > 0\). Because \(p > 1\), this estimate implies (2.13). \(\square\)

Fix an integer \(n_0 > \max_{x \in \partial \Omega} |x|\). For any \(n \geq n_0\), we set \(\Omega_n = \{x \in \Omega : |x| < n\}\) and introduce the space \(W_n := \{u \in W^{1,p}(\Omega_n) : u = 0\text{ on }|x| = n\}\). Notice that we can consider that \(W_n \subset W^{1,p}(\Omega)\) by setting, for all \(w \in W_n\), \(w(x) = 0\) whenever \(x \in \Omega\) with \(|x| > n\) (which is possible since \(w = 0\) on \(|x| = n\)). For each \(n \geq n_0\), let \(i_n : W_n \rightarrow W^{1,p}(\Omega)\) denote the inclusion map and \(i_n^* : W^{1,p}(\Omega)^* \rightarrow W_n^*\) its adjoint operator. Fix \(n \geq n_0\) and introduce the nonlinear operator \(\Gamma_n : W_n \rightarrow W_n^*\) by

\[
\Gamma_n := \Gamma_{i_n^{-1}}i_n = i_n^*A_i + i_n^*B_i + i_n^*F_i + i_n^*G_i.
\]

**Lemma 2.4.** For every \(n \geq n_0\), the equation \(\Gamma_n(u) = 0\) has at least a solution (in \(W_n\)).

**Proof.** The operator \(\Gamma_n : W_n \rightarrow W_n^*\) is pseudomonotone because it is the sum of the strictly monotone operator \(i_n^*A_i\) and the completely continuous operators \(i_n^*B_i\), \(i_n^*F_i\), \(i_n^*G_i\) (which is true because the domain \(\Omega_n\) is bounded). The operator \(\Gamma_n\) is also bounded by Lemma 2.2 and the boundedness of the operators \(i_n\) and \(i_n^*\). Moreover, from Lemma 2.3 we see that \(\Gamma_n\) is coercive. The application of the abstract surjectivity result (see [34, Theorem 27.A]) completes the proof. \(\square\)

2.1. **Proof of Theorem 1.1** Lemma 2.4 ensures that there exists \(u_n \in W_n\) such that

\[
\int_{\Omega_n} A(x, \nabla u_n) \cdot \nabla v \, dx + \int_{\Omega_n} b(x, u_n)v \, dx \\
- \int_{\Omega_n} F(x, Tu_n)v \, dx - \int_{\partial \Omega} f(x, Tu_n)v \, d\sigma = 0
\]

(2.18)
for all $v \in W_n$. Setting $v = u_n$ in (2.18) and using the coercivity of the operator $\Gamma_n$ lead to the conclusion that the sequence $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$.

Thus, up to a subsequence, we may suppose that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, $u_n \rightarrow u$ in $L^p_{\text{loc}}(\Omega)$ and a.e. in $\Omega$ and $\nabla u_n \rightharpoonup \nabla u$ in $L^p_{\text{loc}}(\Omega, \mathbb{R}^N)$, for some $u \in W^{1,p}(\Omega)$. Let $v \in C_0^\infty(\mathbb{R}^N) \cap W^{1,p}(\Omega)$. We note that $v \in W_n$ for $n$ sufficiently large, so we can make use of (2.18) which gives

$$
\int_{\text{supp}(v)} A(x, \nabla u_n) \cdot \nabla v \, dx + \int_{\text{supp}(v)} b(x, u_n) v \, dx \\
- \int_{\text{supp}(v)} F(x, Tu_n) v \, dx - \int_{\partial \Omega} f(x, Tu_n) v \, d\sigma = 0.
$$

We may pass to the limit in (2.19) as $n \rightarrow \infty$. Since $C_0^\infty(\mathbb{R}^N) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$, we arrive at $\Gamma u = 0$. Now it suffices to invoke Lemma 2.1 for concluding that $u$ is a weak solution of problem [1.1]–[1.2] with $u(x) \leq u(x) \leq \overline{u}(x)$ a.e. in $\Omega$.

**Remark 2.5.** One can show the uniqueness of the solution by assuming additional conditions, such as $F(x, \cdot)$ and $f(x, \cdot)$ are nonincreasing on the interval $[u(x), \overline{u}(x)]$ for a.e. $x \in \Omega$. Let $u_1, u_2 \in W^{1,p}(\Omega)$ be two weak solutions to [1.1]–[1.2] belonging to the ordered interval $[u, \overline{u}]$. Then we can write

$$
\int_{\Omega} (A(x, \nabla u_1) - A(x, \nabla u_2)) \cdot \nabla (u_1 - u_2)^+ \, dx
\quad = \quad \int_{\Omega} (F(x, u_1) - F(x, u_2))(u_1 - u_2)^+ \, dx + \int_{\partial \Omega} (f(x, u_1) - f(x, u_2))(u_1 - u_2)^+ \, d\sigma.
$$

In view of our hypothesis and since the operator $A(x, \cdot)$ is strictly monotone, we derive that $u_1 \leq u_2$ (and similarly that $u_2 \leq u_1$) a.e. in $\Omega$, and so $u_1 = u_2$ a.e. in $\Omega$.

## 3. Application to the Initial Data Problem in General Relativity

In this section, we indicate an example where we apply Theorem [1.1] to the existence of the conformal factor in general relativity. We mention that this section contains just an example of application of Theorem 1.1: it is in no way intended to give a deep or extensive analysis of the complicated initial data problem in general relativity. The interested reader can find important advances on various aspects of this subject in [5, 7, 10, 11, 17, 18, 19, 27, 29], among many others.

In Subsection 3.1, we briefly review York-Lichnerowicz’s formalism for decomposing the constraint equations. We then discuss the existence of the conformal factor under certain assumptions. Finally, in Subsection 3.2, we present an elementary proof for the uniqueness in the case of multiple black holes.

### 3.1. York-Lichnerowicz conformal decomposition method.

In general relativity, spacetime is a 4-dimensional manifold of events endowed with a pseudo-Riemannian metric $g_{\alpha\beta}$. Einstein’s equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ connect the spacetime curvature represented by the Einstein tensor $G_{\alpha\beta}$ with the stress-energy tensor $T_{\alpha\beta}$. In fact, these are equations for geometries, that is, their solutions are equivalent classes under spacetime diffeomorphisms of metric tensors. To break this diffeomorphism invariance, Einstein’s equations must first be transformed into a system having a well-posed Cauchy problem. That is, the spacetime is foliated and each slice $\Sigma_t$ is characterized by its intrinsic geometry $\gamma_{ij}$ and extrinsic curvature $K_{ij}$,
which is essentially the “velocity” of $\gamma_{ij}$ in the unit normal direction to the slice. Subsequent slices are connected via the lapse function $N$ and shift vector $\beta^i$ corresponding to the Arnowitt–Deser–Misner (ADM) 3+1 formulation of the line element $ds^2 = -N^2 dt^2 - \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$. This decomposition allows one to express six of the ten components of Einstein’s equations in vacuum ($T_{\alpha\beta} = 0$) as a constrained system of evolution equations for the metric $\gamma_{ij}$ and the extrinsic curvature $K_{ij}$ (repeated subscript-superscript indices means summation):

$$
\dot{\gamma}_{ij} = -2NK_{ij} + 2\nabla_{(i}\beta_{j)} ,
$$

$$
K_{ij} = N(R_{ij} + K^i_lK_{lj} - 2K_{il}K^i_j) + \beta^i\nabla_lK_{ij} + K_{il}\nabla_j\beta^l + K_{ij}\nabla_l\beta^l - \nabla_i\nabla_jN ,
$$

$$
R_i^j + (K^i_l)^2 - K_{ij}K^{ij} = 0 ,
$$

$$
\nabla^j K_{ij} - \nabla_i K^i_j = 0 ,
$$

where we use a dot to denote time differentiation and $\nabla_j$ for the covariant derivative associated to $\gamma_{ij}$. The spatial Ricci tensor $R_{ij}$ has components given by second order spatial differential operators applied to the spatial metric components $\gamma_{ij}$. Indices are raised and traces taken with respect to the spatial metric $\gamma_{ij}$, and parenthesized indices are used to denote the symmetric part of a tensor (e.g., $\nabla_{(i}\beta_{j)} := (\nabla_i\beta_j + \nabla_j\beta_i)/2$).

To evolve Einstein’s equations in the standard ADM 3+1 formulation, one needs to specify the 3-metric $\gamma_{ij}$ and the extrinsic curvature $K_{ij}$ on the initial time slice $\Sigma_0$. This is a difficult task, as these quantities must satisfy the constraint equations \((3.1)\) and \((3.2)\). We outline here the conformal decomposition method of York-Lichnerowicz (see [6, 29, 31, 32, 33] for the vacuum constraint equations. The base of the method consists of specifying the physical data only up to conformal equivalence, under the assumption that the trace of $K_{ij}$, $K^i_i := \gamma^{ij}K_{ij}$, is given and fixed. In essence, this means that we look for a metric $\hat{\gamma}_{ij}$ conformally related to a given metric $\tilde{\gamma}_{ij}$ by $\gamma_{ij} = \psi^4\hat{\gamma}_{ij}$, where the conformal factor $\psi$ is a strictly positive function to be determined. We will denote by $\hat{\gamma}^{ij}$, $\hat{\nabla}_j$, and $\hat{R}$ the inverse metric, covariant derivative operator, and scalar curvature associated to the metric $\hat{\gamma}_{ij}$. We now relate these to quantities based on the original metric $\gamma_{ij}$. The inverse metric $\hat{\gamma}^{ij}$ and the covariant derivative $\hat{\nabla}_j$ of scalars are easy: $\hat{\gamma}^{ij} = \psi^{-4}\hat{\gamma}^{ij}$ and $\hat{\nabla}_jK = \hat{\nabla}_jK$ and $\hat{\nabla}^jK = \psi^{-4}\hat{\nabla}^jK$ for any scalar function $K$. For the covariant derivative of tensors and for the scalar curvature, we need to relate the Christoffel symbols $\hat{\Gamma}^{ij}_{kl}$ formed with respect to $\hat{\gamma}^{ij}$ to the Christoffel symbols $\Gamma^{ij}_{kl}$ formed with respect to $\gamma^{ij}$. By direct calculation

$$
\Gamma^{ij}_{kl} = \frac{1}{2}\hat{\gamma}^{il}(\frac{\partial\hat{\gamma}_{jk}}{\partial x^k} - \frac{\partial\hat{\gamma}_{jk}}{\partial x^l}) = \hat{\Gamma}^{ij}_{jk} + 2\psi^{-1}(\frac{\partial\psi}{\partial x^k}\hat{\delta}_j^i - \frac{\partial\psi}{\partial x^l}\hat{\delta}_k^i) ,
$$

and so

$$
\Gamma^{ij}_{kl} = \hat{\Gamma}^{ij}_{jk} + 6\psi^{-1}\frac{\partial\psi}{\partial x^k} .
$$

Now, let us relate the extrinsic curvature $K^{ij}$ corresponding to $\gamma_{ij}$ to a given symmetric $(2, 0)$ tensor $\tilde{K}^{ij}$ by $K^{ij} = \psi^{-s}\tilde{K}^{ij}$ for some $s$. Then, by direct calculation,

$$
\nabla_jK^{ij} = \frac{\partial K^{ij}}{\partial x^j} + \Gamma^j_lK^{jl} + \Gamma^i_lK^{il}
$$

$$
= \psi^{-s}\nabla_j\tilde{K}^{ij} - 2\psi^{-s-1}\frac{\partial\psi}{\partial x^m}\hat{\gamma}^{im}\tilde{K} + (10 - s)\psi^{-s-1}\frac{\partial\psi}{\partial x^l}\tilde{K}^{il} .
$$
where \( \hat{K} = \hat{\gamma}_{ij}\hat{K}^{ij} \). This motivates the choice \( s = 10 \). Moreover, we choose the tensor \( \hat{K}^{ij} \) to be trace-free, i.e., \( \hat{K} = 0 \). Then, the zero trace is preserved, i.e., \( K^{ij} \) is trace-free, and \( \nabla_j K^{ij} = \psi^{-10}\nabla_j \hat{K}^{ij} \). The scalar curvatures \( R = \gamma_{ij}R^{ij} \) and \( \hat{R} = \hat{\gamma}_{ij}\hat{R}^{ij} \) are related by \( R = \psi^{-4}\hat{R} - 8\psi^{-5}\Delta\psi \), where \( \Delta\psi := \hat{\gamma}_{ij}\hat{\nabla}^i\hat{\nabla}^j\psi \) is the Laplacian of \( \psi \) with respect to the metric \( \hat{\gamma}_{ij} \).

If we choose \( \hat{\gamma}_{ij} \) to be the flat metric \( \hat{\gamma}_{ij} := \delta_{ij} \), then the momentum constraints (3.2) are linear, decoupled from the Hamiltonian constraint (3.1) (as a consequence of the assumption \( \hat{K} = 0 \)), and solutions \( \hat{K}^{ij} \) to them can be determined analytically (see [13, 14, 23, 29, 31, 32, 33], among others). Moreover, the Hamiltonian constraint equation reduces to the relatively simple semilinear elliptic equation

\[
-\Delta\psi = \hat{H}\psi^{-7},
\]

where \( \Delta := \delta_{ij}\partial^i\partial^j \) is the usual 3D Laplacian and \( \hat{H} := \frac{1}{8}\hat{\nabla}^{ij}\hat{K}_{ij} \) is a positive function. It is also necessary to specify the domain on which this equation will be solved, and the boundary conditions that will be applied. In the case of multiple black holes, our goal is to solve equation (3.3) in the exterior domain \( \Omega := \{ x \in \mathbb{R}^3 : |x - O_i| > R_i, \ i = 1, 2, \ldots, N \} \), where \( O_i \) are the centers, respectively the radii, of the disjoint black holes. Because we are interested in asymptotically flat spacetimes, we would like that the conformal factor approach unity as the distance from any sources approaches infinity:

\[
\psi(x) \to 1, \quad \text{as } |x| \to \infty.
\]

Also, we invoke an inner boundary condition (see [6, 29, 31], and references therein)

\[
\frac{\partial\psi}{\partial n} + \frac{1}{2R_i}\psi = 0 \quad \text{on } \partial B(O_i, R_i), \ i = 1, 2, \ldots, N,
\]

where the normal \( n \) to \( \partial B(O_i, R_i) \) points into the domain \( \Omega \).

Let \( u := \psi - 1 \). For \( \psi \) to be a solution of (3.3)–(3.5), \( u \) must satisfy the following boundary value problem in the exterior domain \( \Omega \):

\[
-\Delta u = \hat{H}(1 + u)^{-7} \quad \text{in } \Omega,
\]

\[
u(x) \to 0 \quad \text{as } |x| \to \infty,
\]

\[
\frac{\partial u}{\partial n} = -\frac{1}{2R_i}(1 + u) \quad \text{on } \partial B(O_i, R_i), \ i = 1, 2, \ldots, N.
\]

**Theorem 3.1.** Suppose that \( \rho\hat{H} \in L^2(\Omega) \). Then there exists at least one (weak) solution \( u \in W^{1,2}(\Omega) \) to (3.6)–(3.8).

**Proof.** Observe that one can now apply Theorem 1.1 to the boundary-value problem (3.6)–(3.8) if a pair of sub- and supersolution \( \underline{u} \) and \( \overline{u} \) can be found. It is easy to see that \( u := 0 \) is a subsolution to (3.6)–(3.8). Furthermore, the solution \( \overline{u} \in W^{1,2}(\Omega) \) of the following Dirichlet boundary-value problem (whose existence is guaranteed by [22, Theorem 2.5.14])

\[
-\Delta \overline{u}(x) = \hat{H}(x) \quad \text{in } \Omega, \quad \overline{u}(x) = 0 \quad \text{on } \partial\Omega,
\]

is a supersolution to (3.6)–(3.8). Moreover, by the maximum principle one obtains \( \overline{u} > 0 \) in \( \Omega \). Then, by Theorem 1.1 it follows that there exists a weak solution \( u \in W^{1,2}(\Omega) \), with \( (u \equiv 0) \leq u \leq \overline{u} \), to the boundary-value problem (3.6)–(3.8). In fact, by the maximum principle again, \( u \) is strictly positive in \( \Omega \). In addition, since \( u \in W^{1,2}(\Omega) \), we also have that \( u(x) \to 0 \), as \( |x| \to \infty \), a.e. in \( \Omega \).
3.2. The uniqueness in the general case of multiple black holes. In 1989 York [31] proved that the solution for the boundary value problem (3.3), (3.4), and (3.5) is locally unique, that is, he proved that no other solutions lie in the neighborhood of a given solution, but this does not preclude the existence of other solutions which are “significantly different.” Here the normal \( n \) to \( \partial \Omega \) points into the domain \( \Omega \), and this interferes with an usual existence and uniqueness analysis for the problem. That is, even though (3.5) looks like a Robin boundary condition, it has the “wrong” sign between its two terms. Therefore, as observed in [31], one cannot give a standard uniqueness argument for the problem (3.3), (3.4), and (3.5). In what follows we give a simple proof for uniqueness in the case of multiple black-holes; it has some points in common with the proof pointed out by York in [31].

**Theorem 3.2.** There exists at most one solution to the elliptic exterior boundary-value problem (3.3), (3.4), and (3.5).

**Proof.** Arguing by contradiction, suppose that we have two distinct solutions for (3.3), (3.4), and (3.5). Denote by \( u \) and \( v \) these two solutions. For each \( i = 1, \ldots, N \), by passing to spheric coordinates with respect to \( O_i \), we define a related function

\[
\tilde{u}_i(r, \theta, \phi) = R_i r u(r, \theta, \phi),
\]

where \( r = R_i^2 / r \), \( 0 < r \leq R_i \). Note that \( \tilde{u}_i(R_i, \theta, \phi) = u(R_i, \theta, \phi) \). Moreover, the first derivatives of \( u \) and \( \tilde{u}_i \) agree at \( r = R_i \) (we need only check the radial derivatives)

\[
\frac{\partial \tilde{u}_i}{\partial r}(r, \theta, \phi) = -\frac{R_i}{r^2} u(\tau, \theta, \phi) - \frac{R_i^3}{r^3} \frac{\partial u}{\partial r}(\tau, \theta, \phi),
\]

and so

\[
\frac{\partial \tilde{u}_i}{\partial r}(R_i, \theta, \phi) = -\frac{1}{R_i} u(R_i, \theta, \phi) - \frac{\partial u}{\partial r}(R_i, \theta, \phi) = \frac{\partial u}{\partial r}(R_i, \theta, \phi), \quad (3.9)
\]

where the last equality in (3.9) follows from the boundary condition (3.5).

Likewise, one finds that the second derivatives of \( u \) and \( \tilde{u}_i \) also match at \( r = R_i \).

\[
\frac{\partial^2 \tilde{u}_i}{\partial r^2}(r, \theta, \phi) = \frac{2R_i}{r^3} u(\tau, \theta, \phi) + \frac{4R_i^3}{r^4} \frac{\partial u}{\partial r}(\tau, \theta, \phi) + \frac{R_i^5}{r^5} \frac{\partial^2 u}{\partial r^2}(\tau, \theta, \phi),
\]

and so

\[
\frac{\partial^2 \tilde{u}_i}{\partial r^2}(R_i, \theta, \phi) = \frac{4}{R_i} \left( \frac{\partial u}{\partial r}(R_i, \theta, \phi) + \frac{1}{2R_i^2} u(R_i, \theta, \phi) \right) + \frac{\partial^2 u}{\partial r^2}(R_i, \theta, \phi), \quad (3.10)
\]

where the first term of the right-hand side of (3.10) vanishes because of the boundary condition (3.5).

Furthermore, simple computations show that

\[
\Delta \tilde{u}_i(r, \theta, \phi) = \frac{R_i^5}{r^5} \Delta u(\tau, \theta, \phi). \quad (3.11)
\]

Hence we can extend \( u \) as follows

\[
U(x) = \begin{cases} 
  u(x) & \text{for } x \in \overline{\Omega} \\
  \tilde{u}_i(x) & \text{for } x \in J_i(\Omega) \subset B(O_i, R_i), \ i = 1, 2, \ldots, N, 
\end{cases}
\]

where

\[
J_i(x) = \frac{R_i^2}{|x - O_i|^2} (x - O_i) + O_i,
\]
for all \( x \neq O_i, i = 1, N \). Observe that \( U \) is in \( C^2(\Omega) \), \( \bar{\Omega} := \Omega \cup J_1(\Omega) \cup J_2(\Omega) \ldots \cup J_N(\Omega) \), and (as a consequence of\footnote{3.11}) it satisfies the following differential equation in the open set \( \Omega \)

\[
-\Delta U = \hat{H}U^{-7},
\]

where

\[
\hat{H}(x) = \begin{cases} 
\hat{H}(x) & \text{for } x \in \overline{\Omega} \\
R_i^{12}|x - O_i|^{-12}\hat{H}(J_i^{-1}(x)) & \text{for } x \in J_i(\Omega), i = 1, 2, \ldots, N.
\end{cases}
\]

Doing the same for \( v \), we get its extension \( V \) in \( \tilde{\Omega} \). Without restricting generality, we can assume \( U(x) > V(x) \) in a nonzero measure subset of \( \Omega \). Let \( w(x) = \ln |U(x)/V(x)| \). Since both \( u \) and \( v \) tend to 1 as \( |x| \to \infty \) and by the construction of \( U \) and \( V \), it follows that \( \lim_{|x| \to \infty} |U(x)/V(x)| = 1 \) and \( \lim_{x \to O_i} |U(x)/V(x)| = 1, i = 1, 2, \ldots, N \). Therefore, there exists \( x_0 \) in the closure of the set \( \Omega \) such that \( w(x_0) = \sup_{x \in \Omega} w(x) \). First, let us prove that \( x_0 \) must belong to \( \partial \Omega \). Arguing by contradiction, assume that \( x_0 \) belongs to the interior of \( \tilde{\Omega} \). Then, from \( \nabla w(x_0) = 0 \), it follows that

\[
\frac{1}{U(x_0)} \nabla U(x_0) = \frac{1}{V(x_0)} \nabla V(x_0),
\]

and so

\[
\Delta w(x_0) = \frac{1}{U(x_0)} \Delta U(x_0) - \frac{1}{V(x_0)} \Delta V(x_0)
\]

\[
+ \frac{1}{U(x_0)^2} |\nabla U(x_0)|^2 - \frac{1}{V(x_0)^2} |\nabla V(x_0)|^2
\]

\[
= -\hat{H}(x_0) \left( \frac{1}{U(x_0)^8} - \frac{1}{V(x_0)^8} \right) > 0,
\]

which is impossible. This forces \( x_0 \) to belong to \( \partial \Omega \).

Suppose that \( x_0 \in J_i(\partial B(O_j, R_j)) \) for some \( i \) and \( j \), with \( i \neq j \). Then, for \( \tilde{x}_0 := J_i^{-1}(x_0) \in \partial B(O_j, R_j) \) we have

\[
w(\tilde{x}_0) = \ln |U(\tilde{x}_0)/V(\tilde{x}_0)| = \ln |u(\tilde{x}_0)/v(\tilde{x}_0)| = \ln |\tilde{u}_i(x_0)/\tilde{v}_j(x_0)| = w(x_0),
\]

and so \( w(\tilde{x}_0) = \sup_{x \in \Omega} w(x) \). Since \( \tilde{x}_0 \) is an interior point of \( \tilde{\Omega} \), we get the same contradiction as in \footnote{3.12}. \( \square \)

\section*{References}

\begin{enumerate}
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\end{enumerate}


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