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Abstract. We determine the values of a parameter $\lambda$ for which there exist positive solutions to the system of dynamic equations
\[ u^{\Delta\Delta}(t) + \lambda p(t)f(v(\sigma(t))) = 0, \quad t \in [a, b]_T, \]
\[ v^{\Delta\Delta}(t) + \lambda q(t)g(u(\sigma(t))) = 0, \quad t \in [a, b]_T, \]
with the boundary conditions,
\[ \alpha u(a) - \beta u^{\Delta}(a) = 0, \quad \gamma u(\sigma^2(b)) + \delta u^{\Delta}(b) = 0, \]
\[ \alpha v(a) - \beta v^{\Delta}(a) = 0, \quad \gamma v(\sigma^2(b)) + \delta v^{\Delta}(b) = 0, \]
where $T$ is a time scale. To this end we apply a Guo-Krasnosel’skii fixed point theorem.

1. Introduction

Let $T$ be a time scale with $a, \sigma^2(b) \in T$. Given an interval $J$ of $\mathbb{R}$, we will use the interval notation
\[ J_T = J \cap T. \tag{1.1} \]
We are concerned with determining values of $\lambda$ (eigenvalues) for which there exist positive solutions for the system of dynamic equations
\[ u^{\Delta\Delta}(t) + \lambda p(t)f(v(\sigma(t))) = 0, \quad t \in [a, b]_T, \]
\[ v^{\Delta\Delta}(t) + \lambda q(t)g(u(\sigma(t))) = 0, \quad t \in [a, b]_T, \tag{1.2} \]
satisfying the boundary conditions
\[ \alpha u(a) - \beta u^{\Delta}(a) = 0, \quad \gamma u(\sigma^2(b)) + \delta u^{\Delta}(b) = 0, \]
\[ \alpha v(a) - \beta v^{\Delta}(a) = 0, \quad \gamma v(\sigma^2(b)) + \delta v^{\Delta}(b) = 0. \tag{1.3} \]
We will use the following assumptions:
(A1) $f, g \in C([0, \infty), [0, \infty))$;
(A2) $p, q \in C([a, \sigma(b)]_T, [0, \infty))$, and each function does not vanish identically on any closed subinterval of $[a, \sigma(b)]_T$;
(A3) the following limits exist as real numbers:
\[ f_0 := \lim_{x \to 0^+} f(x)/x, \quad g_0 := \lim_{x \to 0^+} g(x)/x, \]
\[ f_\infty := \lim_{x \to \infty} f(x)/x, \quad g_\infty := \lim_{x \to \infty} g(x)/x \]
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There is an ongoing flurry of research activities devoted to positive solutions of dynamic equations on time scales. This work entails an extension of the paper by Chyan and Henderson [7] to eigenvalue problem for system of nonlinear boundary value problems on time scales. Also, in that light, this paper is closely related to the works of Li and Sun [21, 23].

On a larger scale, there has been a great deal of study focused on positive solutions of boundary value problems for ordinary differential equations. Interest in such solutions is high from a theoretical sense [9, 11, 13, 19, 20] and as applications for which only positive solutions are meaningful [2, 10, 16, 25]. These considerations are cast primarily for scalar problems, but good attention has been given to boundary value problems for systems of differential equations [14, 15, 17, 18].

The main tool in this paper is an application of the Guo-Krasnoselskii fixed point theorem for operators leaving a Banach space cone invariant [9]. A Green function plays a fundamental role in defining an appropriate operator on a suitable cone.

2. Green’s Function and Bounds

In this section, we state the well-known Guo-Krasnosel’skii fixed point theorem which we will apply to a completely continuous operator whose kernel, $G(t, s)$ is the Green’s function for

$$
\begin{align*}
-\Delta^2 y &= 0, \\
\alpha u(a) - \beta u^\Delta(a) &= 0, \\
\gamma u(\sigma^2(b)) + \delta u^\Delta(b) &= 0
\end{align*}
$$

(2.1)

is given by

$$
G(t, s) = \frac{1}{d} \left\{ \begin{array}{ll}
\{ \alpha(t - a) + \beta \} \{ \gamma(\sigma^2(b) - \sigma(s)) + \delta \} : & a \leq t \leq s \leq \sigma^2(b) \\
\{ \alpha(\sigma(s) - a) + \beta \} \{ \gamma(\sigma^2(b) - t) + \delta \} : & a \leq \sigma(s) \leq t \leq \sigma^2(b)
\end{array} \right.
$$

(2.2)

where $\alpha, \beta, \gamma, \delta \geq 0$ and

$$
d := \gamma \beta + \alpha \delta + \alpha \gamma (\sigma^2(b) - a) > 0.
$$

One can easily check that

$$
G(t, s) > 0, \quad (t, s) \in (a, \sigma^2(b))_T \times (a, \sigma(b))_T
$$

(2.3)

and

$$
G(t, s) \leq G(\sigma(s), s) = \frac{[\alpha(\sigma(s) - a) + \beta][\gamma(\sigma^2(b) - \sigma(s)) + \delta]}{d}
$$

(2.4)

for $t \in [a, \sigma^2(b)]_T$, $s \in [a, \sigma(b)]_T$. Let $I = \left[ \frac{3a + \sigma^2(b)}{4}, \frac{a + 3\sigma^2(b)}{4} \right]_T$. Then

$$
G(t, s) \geq kG(\sigma(s), s) = k \frac{[\alpha(\sigma(s) - a) + \beta][\gamma(\sigma^2(b) - \sigma(s)) + \delta]}{d}
$$

(2.5)

for $t \in I$, $s \in [a, \sigma(b)]_T$, where

$$
k = \min \left\{ \frac{\gamma(\sigma^2(b) - a) + 4\delta}{4(\gamma(\sigma^2(b) - a) + \delta)}, \frac{\alpha(\sigma^2(b) - a) + 4\beta}{4(\alpha(\sigma^2(b) - a) + \beta)} \right\}.
$$

(2.6)
We note that a pair \((u(t), v(t))\) is a solution of the eigenvalue problem (1.2), (1.3) if and only if
\[
\begin{align*}
u(t) &= \lambda \int_a^{\sigma(b)} G(t, s) p(s) f \left( \lambda \int_a^{\sigma(b)} G(\sigma(s), r) q(r) g(u(\sigma(r))) \Delta r \right) \Delta s, \\
v(t) &= \lambda \int_a^{\sigma(b)} G(t, s) q(s) g(u(\sigma(s))) \Delta s,
\end{align*}
\]
(a ≤ t ≤ \(\sigma^2(b)\)).

Values of \(\lambda\) for which there are positive solutions (positive with respect to a cone) of (1.2), (1.3) will be determined via applications of the following fixed point theorem [19].

**Theorem 2.1 (Krasnosel’skii).** Let \(\mathcal{B}\) be a Banach space, and let \(\mathcal{P} \subset \mathcal{B}\) be a cone in \(\mathcal{B}\). Assume that \(\Omega_1\) and \(\Omega_2\) are open subsets of \(\mathcal{B}\) with \(0 \in \Omega_1 \subset \Omega_1 \subset \Omega_2\), and let
\[
T : \mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow \mathcal{P}
\]
be a completely continuous operator such that either
(i) \(\|Tu\| \leq \|u\|, u \in \mathcal{P} \cap \partial \Omega_1, \text{ and } \|Tu\| \geq \|u\|, u \in \mathcal{P} \cap \partial \Omega_2\); or
(ii) \(\|Tu\| \geq \|u\|, u \in \mathcal{P} \cap \partial \Omega_1, \text{ and } \|Tu\| \leq \|u\|, u \in \mathcal{P} \cap \partial \Omega_2\).

Then, \(T\) has a fixed point in \(\mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1)\).

### 3. Positive Solutions in a Cone

In this section, we apply Theorem 2.1 to obtain solutions in a cone (i.e., positive solutions) of (1.2), (1.3). Assume throughout that \([a, \sigma^2(b)]\) is such that
\[
\begin{align*}
\xi &= \min \left\{ t \in T : t \geq \frac{3a + \sigma^2(b)}{4} \right\}, \\
\omega &= \max \left\{ t \in T : t \leq \frac{a + 3\sigma^2(b)}{4} \right\};
\end{align*}
\]
both exist and satisfy
\[
\frac{3a + \sigma^2(b)}{4} \leq \xi < \omega \leq \frac{a + 3\sigma^2(b)}{4}.
\]

Next, let \(\tau \in [\xi, \omega]_T\) be defined by
\[
\int_\xi^\omega G(\tau, s)p(s)\Delta s = \max_{t \in [\xi, \omega]_T \setminus \xi} \int_\xi^\omega G(t, s)p(s)\Delta s.
\]

Finally, we define
\[
\begin{align*}
l &= \min_{s \in [a, \sigma^2(b)]_T} \frac{G(\sigma(s), s)}{G(\sigma(s), s)}, \\
\gamma &= \min \{k, l\}.
\end{align*}
\]

For our construction, let \(\mathcal{B} = \{ x : [a, \sigma^2(b)]_T \rightarrow \mathbb{R} \} \) with supremum norm \(\|x\| = \sup \{ |x(t)| : t \in [a, \sigma^2(b)]_T \} \) and define a cone \(\mathcal{P} \subset \mathcal{B}\) by
\[
\mathcal{P} = \left\{ x \in \mathcal{B} | x(t) \geq 0 \text{ on } [a, \sigma^2(b)]_T, \text{ and } x(t) \geq \gamma \|x\|, \text{ for } t \in [\xi, \omega]_T \right\}.
\]
For our first result, define positive numbers $L_1$ and $L_2$, by
\[
L_1 := \max\left\{ \left[ \gamma \int_{\xi}^{\omega} G(\tau, s)p(s) \Delta s \right]^{-1}, \left[ \gamma \int_{\xi}^{\omega} G(\tau, s)q(s) \Delta s \right]^{-1} \right\},
\]
\[
L_2 := \min\left\{ \left[ \int_{a}^{\sigma(b)} G(\sigma(s), s)p(s) \Delta s f_0 \right]^{-1}, \left[ \int_{a}^{\sigma(b)} G(\sigma(s), s)q(s) \Delta s g_0 \right]^{-1} \right\}.
\]

**Theorem 3.1.** Assume that conditions (A1)–(A3) are satisfied. Then, for each $\lambda$ satisfying
\[
L_1 < \lambda < L_2, \tag{3.7}
\]
there exists a pair $(u, v)$ satisfying (1.2), (1.3) such that $u(x) > 0$ and $v(x) > 0$ on $(a, \sigma^2(b))$. 

**Proof.** Let $\lambda$ be as in (3.7). And let $\epsilon > 0$ be chosen such that
\[
\max\left\{ \left[ \gamma \int_{\xi}^{\omega} G(\tau, s)p(s) \Delta s (f_\infty - \epsilon) \right]^{-1}, \left[ \gamma \int_{\xi}^{\omega} G(\tau, s)q(s) \Delta s (g_\infty - \epsilon) \right]^{-1} \right\} \leq \lambda
\]
\[
\lambda \leq \min\left\{ \left[ \int_{a}^{\sigma(b)} G(\sigma(s), s)p(s) \Delta s (f_0 + \epsilon) \right]^{-1}, \left[ \int_{a}^{\sigma(b)} G(\sigma(s), s)q(s) \Delta s (g_0 + \epsilon) \right]^{-1} \right\}.
\]

Define an integral operator $T : \mathcal{P} \rightarrow \mathcal{B}$ by
\[
Tu(t) = \lambda \int_{a}^{\sigma(b)} G(t, s)p(s)f\left( \lambda \int_{a}^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r))) \Delta r \right) \Delta s. \tag{3.8}
\]

By the remarks in Section 2, we seek suitable fixed points of $T$ in the cone $\mathcal{P}$.

Notice from (A1), (A2), and (2.3) that, for $u \in \mathcal{P}, Tu(t) \geq 0$ on $[a, \sigma^2(b)]$. Also, for $u \in \mathcal{P}$, we have from (2.4) that
\[
Tu(t) := \lambda \int_{a}^{\sigma(b)} G(t, s)p(s)f\left( \lambda \int_{a}^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r))) \Delta r \right) \Delta s
\]
\[
\leq \lambda \int_{a}^{\sigma(b)} G(\sigma(s), s)p(s)f\left( \lambda \int_{a}^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r))) \Delta r \right) \Delta s. \tag{3.9}
\]

so that
\[
\|Tu\| \leq \lambda \int_{a}^{\sigma(b)} G(\sigma(s), s)p(s)f\left( \lambda \int_{a}^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r))) \Delta r \right) \Delta s. \tag{3.10}
\]

Next, if $u \in \mathcal{P}$, we have from (2.5), (3.5), and (3.8) that
\[
\min_{t \in [\xi, \omega]} Tu(t)
\]
\[
= \min_{t \in [\xi, \omega]} \lambda \int_{a}^{\sigma(b)} G(t, s)p(s)f\left( \lambda \int_{a}^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r))) \Delta r \right) \Delta s
\]
\[
\geq \lambda \gamma \int_{a}^{\sigma(b)} G(\sigma(s), s)p(s)f\left( \lambda \int_{a}^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r))) \Delta r \right) \Delta s
\]
\[
\geq \gamma \|Tu\|. \tag{3.11}
\]

Consequently, $T : \mathcal{P} \rightarrow \mathcal{P}$. In addition, standard arguments shows that $T$ is completely continuous.
Now, from the definitions of $f_0$ and $g_0$, there exists $H_1 > 0$ such that

$$f(x) \leq (f_0 + \epsilon)x, \quad g(x) \leq (g_0 + \epsilon)x, \quad 0 < x \leq H_1.$$  

Let $u \in \mathcal{P}$ with $\|u\| = H_1$. We first have from (2.4) and choice of $\epsilon$, for $a \leq s \leq \sigma(b)$, that

$$\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r \leq \lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)g(u(\sigma(r)))\Delta r$$

$$\leq \lambda \int_a^{\sigma(b)} G(\sigma(r), r)(g_0 + \epsilon)u(r)\Delta r$$

$$\leq \lambda \int_a^{\sigma(b)} G(\sigma(r), r)\Delta r(g_0 + \epsilon)\|u\|$$

$$\leq \|u\| = H_1.$$  

As a consequence, we next have from (2.4) and choice of $\epsilon$, for $a \leq t \leq \sigma^2(b)$, that

$$Tu(t) = \lambda \int_a^{\sigma(b)} G(t, s)p(s)f\left(\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r\right)\Delta s$$

$$\leq \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)(f_0 + \epsilon)\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r\Delta s$$

$$\leq \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)(f_0 + \epsilon)H_1\Delta s$$

$$\leq H_1 = \|u\|.$$  

So, $\|Tu\| \leq \|u\|$. If we set $\Omega_1 = \{x \in B \mid \|x\| < H_1\}$, then

$$\|Tu\| \leq \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial \Omega_1. \quad (3.12)$$

Next, from the definitions of $f_{\infty}$ and $g_{\infty}$, there exists $\overline{H}_2 > 0$ such that

$$f(x) \geq (f_{\infty} - \epsilon)x, \quad g(x) \geq (g_{\infty} - \epsilon)x, \quad x \geq \overline{H}_2. \quad (3.13)$$

Let $H_2 = \max\{2H_1, \overline{H}_2/\gamma\}$. Let $u \in \mathcal{P}$ and $\|u\| = H_2$. Then,

$$\min_{t \in [\xi, \omega]} u(t) \geq \gamma\|u\| \geq \overline{H}_2. \quad (3.14)$$

Consequently, from (2.5) and choice of $\epsilon$, for $a \leq s \leq \sigma(b)$, we have that

$$\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r \geq \lambda \int_\xi^{\omega} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r$$

$$\geq \lambda \int_\xi^{\omega} G(\tau, r)q(r)g(u(\sigma(r)))\Delta r$$

$$\geq \lambda \int_\xi^{\omega} G(\tau, r)q(r)(g_{\infty} - \epsilon)u(\sigma(r))\Delta r$$

$$\geq \gamma \lambda \int_\xi^{\omega} G(\tau, r)q(r)(g_{\infty} - \epsilon)\Delta r\|u\|$$

$$\geq \|u\| = H_2. \quad (3.15)$$
And so, we have from (2.5) and choice of \( \epsilon \) that
\[
Tu(\tau) = \lambda \int_a^{\tau(b)} G(\tau, s)p(s)f\left(\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r\right) \Delta s
\]
\[
\geq \lambda \int_a^{\sigma(b)} G(\tau, s)p(s)(f_\infty - \epsilon)\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r \Delta s
\]
\[
\geq \lambda \int_a^{\sigma(b)} G(\tau, s)p(s)(f_\infty - \epsilon)H_2 \Delta s
\]
\[
\geq \gamma H_2 > H_2 = \|u\|.
\]

Hence, \( \|Tu\| \geq \|u\|\). So if we set \( \Omega_2 = \{x \in \mathcal{B} : \|x\| < H_2\} \), then
\[
\|Tu\| \geq \|u\|, \quad u \in \mathcal{P} \cap \partial \Omega_2.
\] (3.16)

Applying Theorem 2.1 to (3.12) and (3.16), we obtain that \( T \) has a fixed point \( u \in \mathcal{P} \cap (\Omega_2 \setminus \Omega_1) \). As such, and with \( v \) being defined by
\[
v(t) = \lambda \int_a^{\sigma(b)} G(t, s)q(s)g(u(\sigma(s)))\Delta s,
\] (3.17)

the pair \((u, v)\) is a desired solution of (1.2), (1.3) for the given \( \lambda \). The proof is complete. \( \square \)

Prior to our next result, we introduce another hypothesis.

(A4) \( g(0) = 0 \), and \( f \) is an increasing function.

We now define positive numbers \( L_3 \) and \( L_4 \) by
\[
L_3 := \max \left\{ \left[ \gamma \int_{\xi}^{\omega} G(\tau, s)p(s)\Delta s f_0 \right]^{-1}, \left[ \gamma \int_{\xi}^{\omega} G(\tau, s)q(s)\Delta s g_0 \right]^{-1} \right\},
\]
\[
L_4 := \min \left\{ \left[ \int_a^{\sigma(b)} G(\sigma(s), s)p(s)\Delta s f_\infty \right]^{-1}, \left[ \int_a^{\sigma(b)} G(\sigma(s), s)q(s)\Delta s g_\infty \right]^{-1} \right\}.
\]

Theorem 3.2. Assume that conditions (A1)–(A4) are satisfied. Then, for each \( \lambda \) satisfying
\[
L_3 < \lambda < L_4,
\] (3.18)

there exists a pair \((u, v)\) satisfying (1.2), (1.3) such that \( u(x) > 0 \) and \( v(x) > 0 \) on \((a, \sigma^2(b))_\tau\).

Proof. Let \( \lambda \) be as in (3.18). And let \( \epsilon > 0 \) be chosen such that
\[
\max \left\{ \left[ \gamma \int_{\xi}^{\omega} G(\tau, s)p(s)\Delta s(f_0 - \epsilon) \right]^{-1}, \left[ \gamma \int_{\xi}^{\omega} G(\tau, s)q(s)\Delta s(g_0 - \epsilon) \right]^{-1} \right\} \leq \lambda,
\]
\[
\lambda \leq \min \left\{ \left[ \int_a^{\sigma(b)} G(\sigma(s), s)p(s)\Delta s(f_\infty + \epsilon) \right]^{-1}, \left[ \int_a^{\sigma(b)} G(\sigma(s), s)q(s)\Delta s(g_\infty + \epsilon) \right]^{-1} \right\}.
\]

Let \( T \) be the cone preserving, completely continuous operator that was defined by (3.8). From the definitions of \( f_0 \) and \( g_0 \), there exists \( H_1 > 0 \) such that
\[
f(x) \geq (f_0 - \epsilon)x, \quad g(x) \geq (g_0 - \epsilon)x, \quad 0 < x \leq H_1
\] (3.19)
Now, \( g(0) = 0 \), and so there exists \( 0 < H_2 < H_1 \) such that
\[
\lambda g(x) \leq \frac{H_1}{\int_a^{\sigma(b)} G(\sigma(s), s)q(s) \Delta s}, \quad 0 \leq x \leq H_2.
\] (3.20)

Choose \( u \in \mathcal{P} \) with \( \|u\| = H_2 \). Then, for \( a \leq s \leq \sigma(b) \), we have
\[
\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r))) \Delta r \leq \frac{\int_a^{\sigma(b)} G(\sigma(s), r)q(r)H_1 \Delta r}{\int_a^{\sigma(b)} G(\sigma(s), s)q(s) \Delta s} \leq H_1.
\] (3.21)

Then
\[
Tu(\tau) = \lambda \int_a^{\sigma(b)} G(\tau, s)p(s)f\left(\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r))) \Delta r\right) \Delta s
\]
\[
\geq \lambda \int_a^{\omega} G(\tau, s)p(s)(f_0 - \epsilon) \lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r))) \Delta r \Delta s
\]
\[
\geq \lambda \int_a^{\omega} G(\tau, s)p(s)(f_0 - \epsilon) \int_a^{\omega} G(\tau, r)q(r)g(u(\sigma(r))) \Delta r \Delta s
\]
\[
\geq \lambda \int_a^{\omega} G(\tau, s)p(s)(f_0 - \epsilon)\|u\| \Delta s
\]
\[
\geq \lambda \gamma \int_a^{\omega} G(\tau, s)p(s)(f_0 - \epsilon)\|u\| \Delta s \geq \|u\|.
\] (3.22)

So, \( \|Tu\| \geq \|u\| \). If we put \( \Omega_1 = \{x \in \mathcal{B} \mid \|x\| < H_2 \} \), then
\[
\|Tu\| \geq \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial \Omega_1.
\] (3.23)

Next, by definitions of \( f_\infty \) and \( g_\infty \), there exists \( \overline{H}_1 \) such that
\[
f(x) \leq (f_\infty - \epsilon)x, \quad g(x) \leq (g_\infty - \epsilon)x, \quad x \geq \overline{H}_1
\] (3.24)

There are two cases: (i) \( g \) is bounded, and (ii) \( g \) is unbounded.

For case (i), suppose \( N > 0 \) is such that \( g(x) \leq N \) for all \( 0 < x < \infty \). Then, for \( a \leq s \leq \sigma(b) \) and \( u \in \mathcal{P} \),
\[
\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r))) \Delta r \leq N\lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r) \Delta r.
\] (3.25)

Let
\[
M = \max \left\{ f(x) \mid 0 \leq x \leq N\lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r) \Delta r \right\},
\] (3.26)

and let
\[
H_3 > \max \left\{ 2H_2, M\lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s) \Delta s \right\}.
\] (3.27)

Then, for \( u \in \mathcal{P} \) with \( \|u\| = H_3 \),
\[
Tu(t) \leq \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)M \Delta s \leq H_3 = \|u\|
\] (3.28)
so that \( \|Tu\| \leq \|u\| \). If \( \Omega_2 = \{x \in \mathcal{B} \mid \|x\| < H_3 \} \), then
\[
\|Tu\| \leq \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial \Omega_2.
\] (3.29)
For case (ii), there exists $H_3 > \max\{2H_2, \overline{H_1}\}$ such that $g(x) \leq g(H_3)$, for $0 < x \leq H_3$. Similarly, there exists $H_4 > \max\{H_3, \lambda \int_a^\sigma (G(\sigma(r), r)q(r)g(H_3)\Delta r)\}$ such that $f(x) \leq f(H_4)$, for $0 < x \leq H_4$. Choosing $u \in P$ with $\|u\| = H_4$, we have by (A4) that

$$T u(t) \leq \lambda \int_a^\sigma G(t, s)p(s)f\left(\lambda \int_a^\sigma G(\sigma(r), r)q(r)g(H_3)\Delta r\right) \Delta s$$

$$\leq \lambda \int_a^\sigma G(t, s)p(s)f(H_4) \Delta s$$

$$\leq \lambda \int_a^\sigma G(\sigma(s), s)p(s)\Delta s(s_\infty + \epsilon)H_4$$

$$\leq H_4 = \|u\|,$$

and so $\|Tu\| \leq \|u\|$. For this case, if we let $\Omega_2 = \{x \in B : \|x\| < H_4\}$, then

$$\|Tu\| \leq \|u\|, \quad \text{for } u \in P \cap \partial \Omega_2.$$

In either case, application of part (ii) of Theorem 2.1 yields a fixed point $u$ of $T$ belonging to $P \cap (\overline{\Omega_2} \Omega_1)$, which in turn yields a pair $(u, v)$ satisfying (1.2), (1.3) for the chosen value of $\lambda$. The proof is complete.

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