Electronic Journal of Differential Equations, Vol. 2006(2006), No. 23, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXISTENCE THEORY FOR PERTURBED HYPERBOLIC DIFFERENTIAL INCLUSIONS

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Abstract. In this paper, the existence of solutions and extremal solutions for a perturbed hyperbolic differential inclusion is proved under the mixed generalized Lipschitz and Carathéodory's conditions.

1. Introduction

This paper concerns the existence of solutions and extremal solutions for a perturbed hyperbolic differential inclusion. First, we consider the following perturbed hyperbolic differential inclusion

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} \in F(x,y,u(x,y)) + G(x,y,u(x,y)), \quad (x,y) \in J_a \times J_b \tag{1.1}$$

$$u(x,0) = f(x), \quad u(0,y) = g(y), \tag{1.2}$$

where $J_a = [0,a]$, $J_b = [0,b]$, $F, G : J_a \times J_b \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ are compact valued multivalued maps, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $f : J_a \to \mathbb{R}$ and $g : J_b \to \mathbb{R}$ are continuous functions. Next, we consider the perturbed nonlocal hyperbolic problem

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} \in F(x,y,u(x,y)) + G(x,y,u(x,y)), \quad (x,y) \in J_a \times J_b \tag{1.3}$$

$$u(x,0) + Q(u) = f(x), \quad x \in J_a \tag{1.4}$$

$$u(0,y) + K(u) = g(y), \quad y \in J_b \tag{1.5}$$

where F, G and f, g are as in the problem (1.1)-(1.2) and $Q, K : C(J_a \times J_b, \mathbb{R}) \to \mathbb{R}$ are continuous functions.

The existence of solutions and the topological properties of the solutions set of hyperbolic differential equations have received much attention during the last two decades ; we refer for instance to the papers by Dawidowski and Kubiaczyk [11, 12], De Blasi and Myjak [14] and the references cited therein. Lakshmikantham and Pandit [20, 22] coupled the method of upper and lower solutions with the monotone method to obtain existence of extremal solutions for hyperbolic differential equations.

2000 *Mathematics Subject Classification.* 35L70, 35L20, 35R70.
Key words and phrases. Hyperbolic differential inclusion; fixed point; extremal solutions.

Submitted March 25, 2005. Published February 23, 2006.

Using a compactness type condition, involving the measure of noncompactness, Papageorgiou gave in [23] existence results for hyperbolic differential inclusions in Banach spaces. Other results with the same tools were given by Dawidowski *et al.* [10]. Recently, the method upper and lower solutions was applied to the particular problem (1.1)-(1.2) with $G \equiv 0$ by Benchohra and Ntouyas in [4]. The same problem was studied on unbounded domain by the same authors in [5] by using a fixed point theorem due to Ma which is an extension of Schaefer's theorem on locally convex topological spaces. By means of Martelli's fixed point theorem for condensing multivalued maps Benchohra [3] proved an existence theorem of solutions to the above cited problem.

Several papers have been devoted to study the existence of solutions for partial differential equations with nonlocal conditions. We refer for instance to the papers of Byszewski [6, 7, 8]. The nonlocal conditions of this type can be applied in the theory of elasticity with better effect that the initial or Darboux conditions.

In this paper, we shall prove the existence of solutions and extremal solutions for the problems (1.1)-(1.2) and (1.3)-(1.5) under the mixed generalized Lipschitz and Carathéodory's conditions. Our approach will be based, for the existence of solutions, on a fixed point theorem for the sum of a contraction multivalued map and a completely continuous map and, for the extremal solutions, on the concept of upper and lower solutions combined with a similar version of the above cited fixed point theorem on ordered Banach spaces established very recently by Dhage. These results extend some ones cited in the above literature devoted to the field.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

$C(J_a \times J_b, \mathbb{R})$ is the Banach space of all continuous functions from $J_a \times J_b$ into $\mathbb{R}$ with the norm

$$\|u\|_\infty = \sup\{|u(x,y)| : (x,y) \in J_a \times J_b\},$$

for each $u \in C(J_a \times J_b, \mathbb{R})$.

$L^1(J_a \times J_b, \mathbb{R})$ denotes the Banach space of measurable functions $u : J_a \times J_b \to \mathbb{R}$ which are Lebesgue integrable normed by

$$\|u\|_{L^1} = \int_0^a \int_0^b |u(x,y)| dy dx$$

for each $u \in L^1(J_a \times J_b, \mathbb{R})$. Let $(X, |\cdot|)$ be a normed space, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$ and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multivalued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x \in B} G(x)$ is bounded in X for all $B \in \mathcal{P}_b(X)$ (i.e. $\sup_{x \in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty closed subset of X and if for each open set N of X containing $G(x_0)$, there exists an open neighbourhood N_0 of x_0 such that $G(N_0) \subseteq N$. G is said to be completely continuous if $G(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in \mathcal{P}_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : J_a \times J_b \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $z \in \mathbb{R}$, the function $(x, y) \mapsto d(z, G(x, y)) = \inf\{|z - u| : u \in G(x, y)\}$ is measurable. For more details on multivalued maps see the books of Aubin and Cellina [1], Aubin and Frankowska [2], Deimling [13] and Hu and Papageorgiou [18].

Definition 2.1. A multivalued map $F : J_a \times J_b \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $(x, y) \mapsto F(x, y, z)$ is measurable for each $z \in \mathbb{R}$;
(ii) $z \mapsto F(x, y, z)$ is upper semi-continuous for almost each $(x, y) \in J_a \times J_b$.

For each $u \in C(J_a \times J_b, \mathbb{R})$, define the set of selections of F by

$$S_{F,u} = \{v \in L^1(J_a \times J_b, \mathbb{R}) : v(x, y) \in F(x, y, u(x, y)) \text{ a.e. } (x, y) \in J_a \times J_b\}.$$

Let $F : J_a \times J_b \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F} : C(J_a \times J_b, \mathbb{R}) \to \mathcal{P}(L^1(J_a \times J_b, \mathbb{R}))$$

by letting

$$\mathcal{F}(u) = \{w \in L^1(J_a \times J_b, \mathbb{R}) : w(x, y) \in F(x, y, u(x, y)) \text{ for a.e. } (x, y) \in J_a \times J_b\}.$$

The operator $\mathcal{F}$ is called the Niemytsky operator associated with F.

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\big\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\big\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $d(a, B) = \inf_{b\in B} d(a, b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [19]).

Definition 2.2. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x, y), \quad \text{for each } x,\ y \in X,$$

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

We apply the following form of the fixed point theorem of Dhage [15] in the sequel.

Theorem 2.3. *Let $B(0, r)$ and $B[0, r]$ denote respectively the open and closed balls in a Banach space E centered at origin and of radius r and let $A : E \to \mathcal{P}_{cl,cv,bd}(E)$ and $B : B[0, r] \to \mathcal{P}_{cp,cv}(E)$ be two multivalued operators satisfying*

(i) *A is multi-valued contraction, and*
(ii) *B is completely continuous.*

Then either

(a) *the operator inclusion $x \in Ax + Bx$ has a solution in $B[0, r]$, or*
(b) *there exists an $u \in E$ with $\|u\| = r$ such that $\lambda u \in Au + Bu$, for some $\lambda > 1$.*

The following lemma will be used in the sequel.

Lemma 2.4. [21]. *Let X be a Banach space. Let $F : J_a \times J_b \times X \to \mathcal{P}_{cp,c}(X)$ be a Carathéodory multivalued map, and let Γ be a linear continuous mapping from $L^1(J_a \times J_b, X)$ into $C(J_a \times J_b, X)$, then the operator*

$$\begin{aligned} \Gamma \circ S_F : C(J_a \times J_b, X) &\to \mathcal{P}_{cp,c}(C(J_a \times J_b, X)) \\ u &\mapsto (\Gamma \circ S_F)(u) := \Gamma(S_{F,u}) \end{aligned}$$

is a closed graph operator in $C(J_a \times J_b, X) \times C(J_a \times J_b, X)$.

3. Existence Result

In this section, we are concerned with the existence of solutions for the problem (1.1)-(1.2).

Let us start by defining what we mean by a solution of (1.1)-(1.2).

Definition 3.1. *A function $u(\cdot,\cdot) \in C(J_a \times J_b, \mathbb{R})$ is said to be a solution of* (1.1)-(1.2) *if there exist $v_1, v_2 \in L^1(J_a \times J_b, \mathbb{R})$ such that $v_1(t,s) \in F(t,s,u(t,s))$, $v_2(t,s) \in G(t,s,u(t,s))$ a.e. on $J_a \times J_b$, and*

$$u(x,y) = f(x)+g(y)-f(0)+\int_0^x \int_0^y (v_1(t,s)+v_2(t,s))dsdt \text{ for each } (x,y) \in J_a \times J_b.$$

The following hypotheses will be assumed hereafter.

- (H1) The function $(x,y) \to F(x,y,z)$ is measurable, convex and integrably bounded for each $z \in \mathbb{R}$.
- (H2) $H_d(F(x,y,z), F(x,y,\overline{z})) \le l(x,y)|z - \overline{z}|$ for almost each $(x,y) \in J_a \times J_b$ and all $z, \overline{z} \in \mathbb{R}$ where $l \in L^1(J_a \times J_b, \mathbb{R})$ and $d(0, F(x,y,0)) \le l(x,y)$ for almost each $(x,y) \in J_a \times J_b$.
- (H3) The multivalued map $G(x,y,z)$ has compact and convex values for each $(x,y,z) \in J_a \times J_b \times \mathbb{R}$.
- (H4) G is Carathéodory.
- (H5) There exist a function $q \in L^1(J_a \times J_b, \mathbb{R})$ with $q(x,y) > 0$ for a.e. $(x,y) \in J_a \times J_b$ and a continuous nondecreasing function $\psi : \mathbb{R}_+ \to (0,\infty)$ such that
$$\|G(x,y,z)\|_{\mathcal{P}} \le q(x,y)\psi(|z|) \text{ a.e. } (x,y) \in J_a \times J_b \text{ for all } z \in \mathbb{R}.$$
- (H6) There exists a real number $r > 0$ such that
$$r > \frac{\|f\|_\infty + \|g\|_\infty + |f(0)| + \|l\|_{L^1} + \psi(r)\|q\|_{L^1}}{1 - \|l\|_{L^1}}.$$

Theorem 3.2. *Suppose that hypotheses (H1)–(H6) are satisfied. If $\|l\|_{L^1} < 1$, then* (1.1)–(1.2) *has at least one solution on $J_a \times J_b$.*

Proof. Transform problem (1.1)–(1.2) into a fixed point problem. Consider the operator $N : C(J_a \times J_b, \mathbb{R}) \to \mathcal{P}(C(J_a \times J_b, \mathbb{R}))$ defined by

$$\begin{aligned} N(u) = \{h \in C(J_a \times J_b, \mathbb{R}) : h(x,y) &= f(x) + g(y) - f(0) \\ &+ \int_0^x \int_0^y (v_1(t,s) + v_2(t,s))dsdt,\ v_1 \in S_{F,u} \text{ and } v_2 \in S_{G,u}\}. \end{aligned}$$

Remark 3.3. Clearly, from Definition 3.1, the fixed points of N are solutions to (1.1)–(1.2).

Let $X = C(J_a \times J_b, \mathbb{R})$ and define an open ball $B(0,r)$ in X entered at origin of radius r, where the real number r satisfies the inequality in hypothesis (H6). Define two multivalued maps A, B on $B[0,r]$ by

$$A(u) = \{h \in X : h(x,y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v_1(t,s)dsdt,\ v_1 \in S_{F,u}\} \tag{3.1}$$

and

$$B(u) = \{h \in X : h(x,y) = \int_0^x \int_0^y v_2(t,s)dsdt,\ v_2 \in S_{G,u}\}. \tag{3.2}$$

We shall show that the operators A and B satisfy all the conditions of Theorem 2.3. The proof will be given in several steps.

Step 1: First, we show that $A(u)$ is a closed convex and bounded subset of X for each $u \in B[0,r]$. This follows easily if we show that the values of the Niemytsky operator associated are closed in $L^1(J_a \times J_b, \mathbb{R})$. Let $\{w_n\}$ be a sequence in $L^1(J_a \times J_b, \mathbb{R})$ converging to a point w. Then $w_n \to w$ in measure and so, there exists a subset S of positive integers with $\{w_n\}$ converging $a.e.$ to w as $n \to \infty$ through S. Now, since (H1) holds, the values of $S_{F,u}$ are closed in $L^1(J_a \times J_b, \mathbb{R})$. Thus, for each $u \in B[0,r]$, we have that $A(u)$ is a non-empty and closed subset of X.

We prove that $A(u)$ is a convex subset of X for each $u \in B[0,r]$. Let $h_1, h_2 \in A(u)$. Then there exist $v_1, v_2 \in S_{F,u}$ such that for each $(x,y) \in J_a \times J_b$ we have

$$h_i(x,y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v_i(t,s)dsdt,\ (i = 1,2)$$

Let $0 \le d \le 1$. Then, for each $(x,y) \in J_a \times J_b$ we have

$$(dh_1 + (1-d)h_2)(x,y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y (dv_1(t,s) + (1-d)v_2(t,s))dsdt.$$

Since $S_{F,u}$ is convex (because F has convex values), then

$$dh_1 + (1-d)h_2 \in A(u).$$

Step 2: We show that A is a multivalued contraction on $B[0,r]$. Let $u, \overline{u} \in B[0,r]$ and $h_1 \in A(u)$. Then, there exists $v_1(x,y) \in F(x,y,u(x,y))$ such that for each $(x,y) \in J_a \times J_b$,

$$h_1(x,y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v_1(t,s)dsdt.$$

From (H2) it follows that

$$H_d(F(x,y,u(x,y)), F(x,y,\overline{u}(x,y))) \le l(x,y)|u(x,y) - \overline{u}(x,y)|.$$

Hence, there exists $w \in F(x,y,\overline{u}(x,y))$ such that

$$|v_1(x,y) - w| \le l(x,y)|u(x,y) - \overline{u}(x,y)|.$$

Consider $U : J_a \times J_b \to \mathcal{P}(\mathbb{R})$ given by

$$U(x,y) = \{w \in \mathbb{R} : |v_1(x,y) - w| \le l(x,y)|u(x,y) - \overline{u}(x,y)|\}.$$

Since the multivalued operator $V(x,y) = U(x,y) \cap F(x,y,\overline{u}(x,y))$ is measurable (see Proposition III.4 in [9]), there exists a function $v_2(x,y)$ which is a measurable selection for V. So, $v_2(x,y) \in F(x,y,\overline{u}(x,y))$ and for each $(x,y) \in J_a \times J_b$

$$|v_1(x,y) - v_2(x,y)| \le l(x,y)|u(x,y) - \overline{u}(x,y)|.$$

Let us define for each $(x,y) \in J_a \times J_b$

$$h_2(x,y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v_2(t,s)dsdt.$$

We have

$$|h_1(x,y) - h_2(x,y)| \leq \int_0^x \int_0^y |v_1(t,s) - v_2(t,s)|dsdt.$$

Thus

$$\|h_1 - h_2\|_\infty \leq \|l\|_{L^1}\|u - \overline{u}\|_\infty.$$

By an analogous relation, obtained by interchanging the roles of u and $\overline{u}$, it follows that

$$H_d(A(u), A(\overline{u})) \leq \|l\|_{L^1}\|u - \overline{u}\|_\infty.$$

So, A is a multivalued contraction on X.

Step 3: Now, we show that the multivalued operator B is compact and upper semi-continuous on $B[0,r]$. First, we show that B is compact on $B[0,r]$. Let $u \in B[0,r]$ be arbitrary. Then, for each $h \in B(u)$, there exists $v \in S_{G,u}$ such that for each $(x,y) \in J_a \times J_b$ we have

$$h(x,y) = \int_0^x \int_0^y v(t,s)dsdt.$$

From (H5) we have

$$|h(x,y)| \leq \int_0^a \int_0^b |v(t,s)|dsdt \leq \int_0^a \int_0^b q(t,s)\psi(\|u\|_\infty)dsdt \leq \|q\|_{L^1}\psi(r).$$

Next, we show that B maps bounded sets into equicontinuous sets of X. Let $(x_1,y_1), (x_2,y_2) \in J_a \times J_b,\ x_1 < x_2, y_1 < y_2$ and $u \in B[0,r]$. For each $h \in B(u)$,

$$\begin{aligned}
|h(x_2,y_2) - h(x_1,y_2)| &\leq \int_{x_1}^{x_2} \int_{y_1}^{y_2} |v(t,s)|dsdt \\
&\leq \int_{x_1}^{x_2} \int_{y_1}^{y_2} q(t,s)\psi(\|u\|_\infty)dsdt \\
&\leq \int_{x_1}^{x_2} \int_{y_1}^{y_2} q(t,s)\psi(r)dsdt.
\end{aligned}$$

The right hand side tends to zero as $(x_2,y_2) \to (x_1,y_1)$. An application of Arzelá-Ascoli Theorem yields that the operator $B : B[0,r] \to \mathcal{P}(X)$ is compact.

Step 4: Next we prove that B has a closed graph. Let $u_n \to u_*$, $h_n \in B(u_n)$ and $h_n \to h_*$. We need to show that $h_* \in B(u_*)$. $h_n \in B(u_n)$ implies that there exists $v_n \in S_{G,u_n}$ such that for each $(x,y) \in J_a \times J_b$,

$$h_n(x,y) = \int_0^x \int_0^y v_n(t,s)dsdt.$$

We must show that there exists $h_* \in S_{G,u_*}$ such that for each $(x,y) \in J_a \times J_b$,

$$h_*(x,y) = \int_0^x \int_0^y v_*(t,s)dsdt.$$

Clearly we have

$$\|h_n - h_*\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Consider the continuous linear operator $\Gamma : L^1(J_a \times J_b, \mathbb{R}) \to C(J_a \times J_b, \mathbb{R})$ given by

$$v \mapsto (\Gamma v)(x,y) = \int_0^x \int_0^y v(t,s) ds dt.$$

From Lemma 2.4, it follows that $\Gamma \circ S_G$ is a closed graph operator. Moreover, we have

$$h_n(x,y) \in \Gamma(S_{G,u_n}).$$

Since $u_n \to u_*$, it follows from Lemma 2.4 that

$$h_*(x,y) = \int_0^x \int_0^y v_*(t,s) ds dt$$

for some $v_* \in S_{G,u_*}$.

Step 5: Now, we show that the second assertion of Theorem 2.3 is not true. Let $u \in X$ be a possible solution of $\lambda u \in A(u) + B(u)$ for some real number $\lambda > 1$ with $\|u\|_\infty = r$. Then there exist $v_1 \in S_{F,u}$ and $v_2 \in S_{G,u}$ such that for each $(x,y) \in J_a \times J_b$ we have

$$u(x,y) = \lambda^{-1}\Big[f(x) + g(y) - f(0) + \int_0^x \int_0^y (v_1(t,s) + v_2(t,s)) ds dt\Big].$$

Then by (H2), (H5) we have

$$\begin{aligned}
|u(x,y)| &\leq |f(x)| + |g(y)| + |f(0)| + \int_0^a \int_0^b |v_1(t,s)| ds dt + \int_0^a \int_0^b |v_2(t,s)| ds dt \\
&\leq |f(x)| + |g(y)| + |f(0)| + \int_0^a \int_0^b [l(t,s)|u(t,s)| + l(t,s)] ds dt \\
&\quad + \int_0^a \int_0^b q(t,s)\psi(|u(t,s)|) ds dt \\
&\leq |f(x)| + |g(y)| + |f(0)| + \int_0^a \int_0^b [l(t,s)\|u\|_\infty + l(t,s)] ds dt \\
&\quad + \int_0^a \int_0^b q(t,s)\psi(\|u\|_\infty) ds dt.
\end{aligned}$$

Taking the supremum over (x,y) we get

$$\begin{aligned}
\|u\|_\infty &\leq \|f\|_\infty + \|g\|_\infty + |f(0)| + \int_0^a \int_0^b [l(t,s)\|u\|_\infty + l(t,s)] ds dt \\
&\quad + \int_0^a \int_0^b q(t,s)\psi(\|u\|_\infty) ds\, dt.
\end{aligned}$$

Substituting $\|u\|_\infty = r$ in the above inequality yields

$$r \leq \frac{\|f\|_\infty + \|g\|_\infty + |f(0)| + \|l\|_{L^1} + \psi(r)\|q\|_{L^1}}{1 - \|l\|_{L^1}}$$

which contradicts (H6). As a result, the conclusion (ii) of Theorem 2.3 does not hold. Hence, the conclusion (i) holds and consequently the problem (1.1)-(1.2) has a solution u on $J_a \times J_b$. This completes the proof. $\square$

4. Existence of Extremal Solutions

In this section, we shall prove the existence of maximal and minimal solutions of the problem (1.1)–(1.2) under suitable monotonicity conditions on the multi-functions involved in it. We equip the space $X = C(J_a \times J_b, \mathbb{R})$ with the order relation $\leq$ defined by the cone K in X, that is,

$$K = \{u \in X : u(x,y) \geq 0, \forall (x,y) \in J_a \times J_b\}.$$

It is known that the cone K is normal in X. The details of cones and their properties may be found in Heikkila and Lakshmikantham [17]. Let $a, b \in X$ be such that $a \leq b$. Then, by an order interval $[a,b]$ we mean a set of points in X given by

$$[a,b] = \{u \in X : a \leq u \leq b\}.$$

Let $D, Q \in \mathcal{P}_{cl}(X)$. Then by $D \leq Q$ we mean $a \leq b$ for all $a \in D$ and $b \in Q$. Thus $a \leq D$ implies that $a \leq b$ for all $b \in Q$ in particular, if $D \leq D$, then, it follows that D is a singleton set.

Definition 4.1. Let X be an ordered Banach space. A mapping $T : X \to \mathcal{P}_{cl}(X)$ is called isotone increasing if $x, y \in X$ with $x < y$, then we have that $T(x) \leq T(y)$. Similarly, T is called isotone decreasing if $T(x) \geq T(y)$ whenever $x < y$.

We use the following fixed point theorem in the proof of the main existence result of this section.

Theorem 4.2 (Dhage[16]). *Let $[a,b]$ be an order interval in a Banach space and let $A, B : [a,b] \to \mathcal{P}_{cl}(X)$ be two multivalued operators satisfying*

(a) *A is multivalued contraction,*
(b) *B is completely continuous,*
(c) *A and B are isotone increasing, and*
(d) *$A(x) + B(x) \subset [a,b]$, $\forall x \in [a,b]$.*

Further if the cone K in X is normal, then the operator inclusion $x \in A(x) + B(x)$ has a least fixed point x_* and a greatest fixed point $x^* \in [a,b]$. Moreover $x_* = \lim_n x_n$ and $x^* = \lim_n y_n$, where $\{x_n\}$ and $\{y_n\}$ are the sequences in $[a,b]$ defined by

$$x_{n+1} \in A(x_n) + B(x_n),\ x_0 = a \text{ and } y_{n+1} \in A(y_n) + B(y_n),\ y_0 = b.$$

Now, we introduce the concept of lower and upper solutions of (1.1)-(1.2). It will be the basic tool in the approach that follows.

Definition 4.3. A function $\underline{u}(\cdot,\cdot) \in C(J_a \times J_b, \mathbb{R})$ is said to be a lower solution of (1.1)-(1.2) if there exist $v_1, v_2 \in L^1(J_a \times J_b, \mathbb{R})$ such that $v_1(t,s) \in F(t,s,u(t,s)), v_2(t,s) \in G(t,s,u(t,s))$ *a.e.* on $J_a \times J_b$, and

$$\underline{u}(x,y) \leq f(x) + g(y) - f(0) + \int_0^x \int_0^y (v_1(t,s) + v_2(t,s)) ds dt$$

for each $(x,y) \in J_a \times J_b$.

A function $\bar{u}(\cdot,\cdot) \in C(J_a \times J_b, \mathbb{R})$ is said to be an upper solution of (1.1)-(1.2) if there exist $v_1, v_2 \in L^1(J_a \times J_b, \mathbb{R})$ such that $v_1(t,s) \in F(t,s,u(t,s)), v_2(t,s) \in G(t,s,u(t,s))$ *a.e.* on $J_a \times J_b$, and

$$\bar{u}(x,y) \geq f(x) + g(y) - f(0) + \int_0^x \int_0^y (v_1(t,s) + v_2(t,s)) ds dt$$

for each $(x,y) \in J_a \times J_b$.

Definition 4.4. A solution u_M of the problem (1.1)-(1.2) is said to be maximal if for any other solution u to the problem (1.1)-(1.2) one has $u(x,y) \leq u_M(x,y)$ for all $(x,y) \in J_a \times J_b$. Again a solution u_m of the problem (1.1)-(1.2) is said to be minimal if $u_m(x,y) \leq u(x,y)$ for all $(x,y) \in J_a \times J_b$, where u is any solution of the problem (1.1)-(1.2) on $J_a \times J_b$.

Definition 4.5. A multivalued function $F(x,y,z)$ is called strictly monotone increasing in z almost everywhere for $(x,y) \in J_a \times J_b$ if $F(x,y,z) \leq F(x,y,\bar{z})$ a.e. $(x,y) \in J_a \times J_b$ for all $z, \bar{z} \in \mathbb{R}$ with $z < \bar{z}$. Similarly $F(x,y,z)$ is called strictly monotone decreasing in z almost every where for $(x,y) \in J_a \times J_b$ if $F(x,y,z) \geq F(x,y,\bar{z})$ a.e. $(x,y) \in J_a \times J_b$ for all $z, \bar{z} \in \mathbb{R}$ with $z < \bar{z}$.

We consider the following assumptions in the sequel.

- (H7) The multivalued maps $F(x,y,z)$ and $G(x,y,z)$ are strictly monotone increasing in z for almost each $(x,y) \in J_a \times J_b$.
- (H8) The problem (1.1)–(1.3) has a lower solution $\underline{u}$ and an upper solution $\bar{u}$ with $\underline{u} \leq \bar{u}$.

Theorem 4.6. *Assume that the hypotheses (H1)-(H5), (H7)-(H8) hold. Then the problem* (1.1)–(1.2) *has minimal and maximal solutions on* $J_a \times J_b$.

Proof. It can be shown, as in the proof of Theorem 3.2, that A and B define the multi-valued operators $A : [\underline{u}, \bar{u}] \to \mathcal{P}_{cl,cv,bd}(X)$ and $B : [\underline{u}, \bar{u}] \to \mathcal{P}_{cp,cv}(X)$. It can be similarly shown that A and B are respectively multi-valued contraction and compact and upper semi-continuous on $[\underline{u}, \bar{u}]$. We shall show that A and B are isotone increasing on $[\underline{u}, \bar{u}]$. Let $u, v \in [\underline{u}, \bar{u}]$ be such that $u \leq v, u \neq v$. Then by (H7), we have for each $(x,y) \in J_a \times J_b$

$$\begin{aligned} A(u) &= \{h \in X : h(x,y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v_1(t,s)dsdt, v_1 \in S_{F,u}\} \\ &\leq \{h \in X : h(x,y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v_1(t,s)dsdt, v_1 \in S_{F,v}\} \\ &= A(v). \end{aligned}$$

Hence $A(u) \leq A(v)$. Similarly by (H7), we have for each $(x,y) \in J_a \times J_b$

$$\begin{aligned} B(u) &= \{h \in X : h(x,y) = \int_0^x \int_0^y v_2(t,s)dsdt, v_2 \in S_{G,u}\} \\ &\leq \{h \in X : h(x,y) = \int_0^x \int_0^y v_2(t,s)dsdt, v_2 \in S_{G,v}\} \\ &= B(v). \end{aligned}$$

Hence $B(u) \leq B(v)$. Thus, A and B are isotone increasing on $[\underline{u}, \bar{u}]$.

Finally, let $u \in [\underline{u}, \bar{u}]$ be any element. Then by (H8),

$$\underline{u} \leq A(\underline{u}) + B(\underline{u}) \leq A(u) + B(u) \leq A(\bar{u}) + B(\bar{u}) \leq \bar{u},$$

which shows that $A(u) + B(u) \in [\underline{u}, \bar{u}]$ for all $u \in [\underline{u}, \bar{u}]$. Thus, the multivalued operators A and B satisfy all the conditions of Theorem 4.2 to yield that the problem (1.1)–(1.2) has maximal and minimal solutions on $J_a \times J_b$. This completes the proof. □

5. Nonlocal Hyperbolic problem

In this section, we indicate some generalizations of the problem (1.1)-(1.2). By using the same methods as in Theorems 3.2 and 4.2 (with obvious modifications), we can prove existence results for the problem (1.3)-(1.5) under the following additional assumptions:

(H9) There exist two nonnegative constants d_1 and d_2 such that
$$|Q(u)-Q(\overline{u})| \le d_1\|u-\overline{u}\|_\infty \quad \text{for all } u,\overline{u} \in C(J_a\times J_b,\mathbb{R}),$$
$$|K(u)-K(\overline{u})| \le d_2\|u-\overline{u}\|_\infty \quad \text{for all } u,\overline{u} \in C(J_a\times J_b,\mathbb{R}).$$

(H10) There exists a real number $r>0$ such that
$$r > \frac{\|f\|_\infty+\|g\|_\infty+|Q(0)|+|K(0)|+|f(0)|+\|l\|_{L^1}+\psi(r)\|q\|_{L^1}}{1-d_1-d_2-\|l\|_{L^1}}.$$

(H11) The functions $Q, K : C(J_a\times J_b,\mathbb{R}) \to \mathbb{R}$ are continuous and nonincreasing.

(H12) The problem (1.3)–(1.5) has a lower solution $\underline{u}$ and an upper solution $\bar{u}$ with $\underline{u} \le \bar{u}$.

Theorem 5.1. *Assume that hypotheses (H1)-(H5), (H9)-(H10) hold. If*
$$\|l\|_{L^1}+d_1+d_2<1,$$
then the perturbed nonlocal problem (1.3)-(1.5) *has at least one solution on* $J_a\times J_b$.

Proof. Consider the operator $\bar{N}$ defined by
$$\begin{aligned}\bar{N}(u) = \{h\in X : h(x,y) &= f(x)+g(y)-Q(u)-K(u)-f(0)\\ &+\int_0^x\int_0^y (v_1(t,s)+v_2(t,s))dsdt,\ v_1\in S_{F,u}, v_2\in S_{G,u}\}.\end{aligned}$$
We can show as in Theorem 3.2 that $\bar{N}$ satisfies the conditions of Theorem 2.3. The details of the proof are left to the reader. □

Theorem 5.2. *Assume that hypotheses (H1)-(H5), (H7), (H11)-(H12) hold. Then the perturbed nonlocal problem* (1.3)-(1.5) *has minimal and maximal solutions on* $J_a\times J_b$.

The proof of the above theorem is left to the reader.

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