SOLUTION MATCHING FOR A THREE-POINT BOUNDARY-VALUE PROBLEM ON A TIME SCALE

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Abstract. Let $T$ be a time scale such that $t_1, t_2, t_3 \in T$. We show the existence of a unique solution for the three-point boundary value problem

$$y^\Delta\Delta\Delta (t) = f(t, y(t), y^\Delta(t), y^\Delta\Delta(t)), \quad t \in [t_1, t_3] \cap T,$$

$$y(t_1) = y_1, \quad y(t_2) = y_2, \quad y(t_3) = y_3.$$ We do this by matching a solution to the first equation satisfying a two-point boundary conditions on $[t_1, t_2] \cap T$ with a solution satisfying a two-point boundary conditions on $[t_2, t_3] \cap T$.

1. Introduction

Bailey, Shampine and Waltman [2] were the first to use solution matching techniques to obtain solutions of two-point boundary value problems for the second order equation $y'' = f(x, y, y')$ by matching solutions of initial value problems. Since then, many authors have used this technique on three-point boundary value problems on an interval $[a, c]$ for an $n$th order differential equation by piecing together solutions of two-point boundary value problems on $[a, b]$, where $b \in (a, c)$ is fixed, with solutions of two-point boundary value problems on $[b, c]$; see for example, Barr and Sherman [3], Das and Lalli [6], Henderson [7, 8], Henderson and Tananton [9], Lakshmikantham and Murty [12], Moorti and Garner [13], and Rao, Murty and Rao [14].

All the above cited works considered boundary value problems for differential equations. In this work, we will use the solution matching technique to obtain a solution to a three-point boundary value problem for a $\Delta$-differential equation on a time scale. The theory of time scales was introduced by Stephan Hilger, [10], as a means of unifying theories of differential equations and difference equations. Three excellent sources about dynamic systems on time scales are the books by Bohner and Peterson [4], Bohner and Peterson [5], and Kaymakcalan et. al., [11]. The definitions below can be found in [4].

A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t < \sup \mathbb{T}$ and $r > \inf \mathbb{T}$, we define the forward jump operator, $\sigma$, and the backward jump operator, $\rho$,
respectively, by
\[
\sigma(t) = \inf \{ \tau \in T : \tau > t \} \in T, \\
\rho(r) = \sup \{ \tau \in T : \tau < r \} \in T.
\]

If \( \sigma(t) > t, \) \( t \) is said to be right scattered, and if \( \sigma(t) = t, \) \( t \) is said to be right dense.

If \( \rho(t) < t, \) \( t \) is said to be left scattered, and if \( \rho(t) = t, \) \( t \) is said to be left dense.

If \( T \) has a left-scattered maximum at \( M, \) then we define \( T^\kappa = T \setminus \{ M \}. \) Otherwise we define \( T^\kappa = T. \) If \( T \) has a right-scattered minimum at \( m, \) then we define \( T_\kappa = T \setminus \{ m \}. \) Otherwise we define \( T_\kappa = T. \)

We say that the function \( x \) has a generalized zero (g.z.) at \( t \) if \( x(t) = 0 \) or if \( x(\sigma(t)) \cdot x(t) < 0. \) In the latter case, we would say the generalized zero is in the real interval \(( t, \sigma(t) ).\)

For \( x : T \to \mathbb{R} \) and \( t \in T, \) (assume \( t \) is not left scattered if \( t = \sup T), \) we define the delta derivative of \( x(t), \) \( x^\Delta(t), \) to be the number (when it exists), with the property that, for each \( \varepsilon > 0, \) there is a neighborhood, \( U, \) of \( t \) such that
\[
|x(\sigma(t)) - x(s) - x^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|,
\]
for all \( s \in U. \)

For \( x : T \to \mathbb{R} \) and \( t \in T, \) (assume \( t \) is not right scattered if \( t = \inf T), \) we define the nabla derivative of \( x(t), \) \( x^\nabla(t), \) to be the number (when it exists), with the property that, for each \( \varepsilon > 0, \) there is a neighborhood, \( U, \) of \( t \) such that
\[
|x(\rho(t)) - x(s) - x^\nabla(t)(\rho(t) - s)| \leq \varepsilon|\rho(t) - s|,
\]
for all \( s \in U. \)

**Remarks:** If \( T = \mathbb{R}, \) then \( x^\Delta(t) = x^\nabla(t) = x'(t). \) If \( T = \mathbb{Z}, \) then \( x^\Delta(t) = x(t + 1) - x(t) \) is the forward difference operator while \( x^\nabla(t) = x(t) - x(t - 1) \) is the backward difference operator.

Let \( T \) be a time scale such that \( t_1, t_2, t_3 \in T. \) We consider the existence of solutions of the three-point boundary value problem
\[
y^\Delta\Delta(t) = f(t, y(t), y^\Delta(t), y^\Delta\Delta(t)), \quad t \in (t_1, t_3) \cap T, \quad (1.1) \\
y(t_1) = y_1, \quad y(t_2) = y_2, \quad y(t_3) = y_3. \quad (1.2)
\]

We obtain solutions by matching a solution of \( (1.1) \) satisfying two-point boundary conditions on \([t_1, t_2] \cap T\) to a solution of \( (1.1) \) satisfying two-point boundary conditions on \([t_2, t_3] \cap T. \) In particular, we will give sufficient conditions such that if \( y_1(t) \) is the solution of \( (1.1) \) satisfying the boundary conditions \( y(t_1) = y_1, y(t_2) = y_2, y^\Delta(t_2) = m, \) \( j = 1 \) or \( 2 \) and \( y_2(t) = y_2, y^\Delta(t_2) = m, y(t_3) = y_3, \) (using the same \( j \)), then the solution of \( (1.1), (1.2) \) is
\[
y(t) = \begin{cases} 
  y_1(t), & t \in [t_1, t_2] \cap T, \\
  y_2(t), & t \in [t_2, t_3] \cap T.
\end{cases}
\]

We will assume that \( f : T \times \mathbb{R}^3 \to \mathbb{R} \) is continuous and that solutions of initial value problems for \( (1.1) \) exist and are unique on \([t_1, t_3] \cap T. \) Moreover, we require that \( t_2 \in T \) is dense and fixed throughout. In addition to these hypotheses, we suppose that there exists a function \( g : T \times \mathbb{R}^3 \to \mathbb{R} \) such that:

(A) For each \( v_3, u_3 \in \mathbb{R} \) the function \( f \) satisfies
\[
f(t, v_1, v_2, v_3) - f(t, u_1, u_2, u_3) > g(t, v_1 - u_1, v_2 - u_2, v_3 - u_3)
\]
when \( t \in (t_1, t_2] \cap \mathbb{T} \), \( u_1 - v_1 \geq 0 \), and \( u_2 - v_2 < 0 \), or when \( t \in [t_2, t_3) \cap \mathbb{T} \), \( u_1 - v_1 \leq 0 \), and \( u_2 - v_2 < 0 \)

(B) There exists \( \varepsilon_1 > 0 \) such that, for each \( 0 < \varepsilon < \varepsilon_1 \), the initial value problem

\[
y^{\Delta\Delta\Delta}(t) = g(t, y(t), y^\Delta(t), y^{\Delta\Delta}(t)), \quad t \in [t_1, t_3] \cap \mathbb{T},
y(t_2) = 0, \quad y^{\Delta\Delta}(t_2) = 0, \quad y^\Delta(t_2) = \varepsilon,
\]

has a solution \( z \) such that \( z^\Delta \) does not change sign on \([t_1, t_3] \cap \mathbb{T}\)

(C) There exists \( \varepsilon_2 > 0 \) such that, for each \( 0 < \varepsilon < \varepsilon_2 \), the initial value problem

\[
y^{\Delta\Delta\Delta}(t) = g(t, y(t), y^\Delta(t), y^{\Delta\Delta}(t)), \quad t \in [t_1, t_3] \cap \mathbb{T},
y(t_2) = 0, \quad y^{\Delta\Delta}(t_2) = 0, \quad y^\Delta(t_2) = \varepsilon(-\varepsilon)
\]

has a solution \( z \) on \([t_2, t_3] \cap \mathbb{T}, ([t_1, t_2] \cap \mathbb{T})\), such that \( z^{\Delta\Delta} \) does not change sign on \([t_2, t_3] \cap \mathbb{T}, ([t_1, t_2] \cap \mathbb{T})\)

(D) For each \( w \in \mathbb{R} \), the function \( g \) satisfies \( g(t, v_1, v_2, w) \geq g(t, u_1, u_2, w) \) when \( t \in (t_1, t_2] \cap \mathbb{T} \), \( u_1 - v_1 \geq 0 \) and \( v_2 > u_2 \geq 0 \), or when \( t \in [t_2, t_3) \cap \mathbb{T} \), \( u_1 - v_1 \leq 0 \) and \( v_2 > u_2 \geq 0 \)

We will need also the following two theorems due to Atici and Guseinov, (Theorems 2.5 and 2.6 in [1] pg. 79).

**Theorem 1.1.** If \( f : \mathbb{T} \to \mathbb{C} \) is \( \Delta \)-differentiable on \( \mathbb{T}^c \) and if \( f^\Delta \) is continuous on \( \mathbb{T}^c \), then \( f \) is \( \nabla \)-differentiable on \( \mathbb{T}_\kappa \) and

\[
f^\nabla(t) = f^\Delta(\rho(t))
\]

for all \( t \in \mathbb{T}_\kappa \).

**Theorem 1.2.** If \( f : \mathbb{T} \to \mathbb{C} \) is \( \nabla \)-differentiable on \( \mathbb{T}_\kappa \) and if \( f^\nabla \) is continuous on \( \mathbb{T}_\kappa \), then \( f \) is \( \Delta \)-differentiable on \( \mathbb{T}^c \) and

\[
f^\Delta(t) = f^\nabla(\sigma(t))
\]

for all \( t \in \mathbb{T}^c \).

2. **Existence and Uniqueness of Solutions**

Consider the boundary conditions,

\[
y(t_1) = y_1, \quad y(t_2) = y_2, \quad y^\Delta(t_2) = m \tag{2.1}
\]

for \( j = 1, 2 \), and

\[
y(t_2) = y_2, \quad y^\Delta(t_2) = m, \quad y(t_3) = y_3 \tag{2.2}
\]

for \( j = 1, 2 \), where \( y_1, y_2, y_3, m \in \mathbb{R} \). In this section, the solution of \((1.1), (2.1), (j = 1, 2)\) is matched with the solution of \((1.1), (2.2), (j = 1, 2)\) to obtain a unique solution of \((1.1), (2.2), (j = 1, 2)\). Our first theorem states that solutions of \((1.1), (2.1), (j = 1, 2), (1.1), (2.2), j = 1, 2\) are unique.

**Theorem 2.1.** Let \( y_1, y_2, y_3 \in \mathbb{R} \), and assume that conditions (A) through (D) are satisfied. Then, given \( m \in \mathbb{R} \), each of the boundary value problems \((1.1), (2.1), (j = 1, 2), (1.1), (2.2), j = 1, 2\), has at most one solution.
Proof. We will consider only the proof for (1.1), (2.1) with $j = 1$; the arguments for the other cases is similar.

Let us assume that there are distinct solutions $\alpha$ and $\beta$ of (1.1), (2.1) (with $j = 1$). Define $w \equiv \alpha - \beta$. Then $w(t_1) = w(t_2) = w(t_3) = 0$. By uniqueness of solutions of initial value problems for (1.1) we know that $w^{\Delta \Delta}(t_2) \neq 0$. Without loss of generality, we let $w^{\Delta \Delta}(t_2) < 0$.

Since $w(t_1) = 0$ and since $t_2$ is dense, there exists an $r_1 \in (t_1, t_2) \cap T$ such that $w^{\Delta \Delta}(t)$ has a g.z. at $r_1$, $w^{\Delta}(t) > 0$ on $[r_1, t_2) \cap T$, $w(t) < 0$ on $(r_1, t_2) \cap T$, and $w^{\Delta \Delta}(t) < 0$ on $(r_1, t_2) \cap T$. From the definition of a generalized zero, we have either $w^{\Delta \Delta}(r_1) = 0$ or $w^{\Delta \Delta}(r_1) \cdot w^{\Delta \Delta}(\sigma(r_1)) < 0$. If $r_1$ is right dense, then $w^{\Delta \Delta}(r_1) = 0$. If $r_1$ is right scattered and $w^{\Delta \Delta}(r_1) \neq 0$, then $w^{\Delta \Delta}(r_1) \cdot w^{\Delta \Delta}(\sigma(r_1)) < 0$. Since $w^{\Delta \Delta}(t) < 0$ on $(r_1, t_2] \cap T$, $w^{\Delta \Delta}(r_1) > 0$. Thus $w^{\Delta \Delta}(r_1) \geq 0$.

Now let $0 < \varepsilon < \frac{1}{2} \min\{z_2, -w^{\Delta \Delta}(t_2)\}$ and let $z_\varepsilon$ satisfy the criteria of hypothesis (C) relative to the interval $[t_1, t_2] \cap T$; that is

\[ z^{\Delta \Delta \Delta}(t) = g(t, z(t), z^{\Delta}(t), z^{\Delta \Delta}(t)), \quad t \in [t_1, t_2] \cap T, \]
\[ z^{\Delta \Delta}(t_2) = 0, \quad z^{\Delta \Delta}(t_2) = -\varepsilon \]

and $z^{\Delta \Delta \Delta}$ does not change sign in $[t_1, t_2] \cap T$.

Set $Z \equiv w - z_\varepsilon$. Then $Z(t_2) = Z^{\Delta}(t_2) = 0$, and $Z^{\Delta \Delta}(t_2) < 0$. Moreover, $Z^{\Delta \Delta \Delta}(r_1) = w^{\Delta \Delta \Delta}(r_1) - z^{\Delta \Delta \Delta}(r_1) > 0$, and $Z^{\Delta \Delta}(t_2) < 0$ imply that there exists an $r_2 \in (r_1, t_2) \cap T$ such that $Z^{\Delta \Delta \Delta}$ has a g.z. at $r_2$ and $Z^{\Delta \Delta}(t) < 0$ on $(r_2, t_2) \cap T$. As above, since $Z^{\Delta \Delta \Delta}$ has a g.z. at $r_2$, $Z^{\Delta \Delta \Delta}(r_2) \geq 0$. Also, $Z(t) > 0$ and $Z(t) < 0$ on $[r_2, t_2) \cap T$.

When $\sigma(r_2) > r_2$,

\[ Z^{\Delta \Delta \Delta}(r_2) = \frac{Z^{\Delta \Delta \Delta}(\sigma(r_2)) - Z^{\Delta \Delta \Delta}(r_2)}{\sigma(r_2) - r_2} < 0. \]

When $\sigma(r_2) = r_2$,

\[ Z^{\Delta \Delta \Delta}(r_2) = \lim_{t \to r_2^+} \frac{Z^{\Delta \Delta \Delta}(t)}{t - r_2} < 0. \]

Regardless of whether $r_2$ is right dense or right scattered we have, from the definition of the delta derivative, that $Z^{\Delta \Delta \Delta}(r_2) < 0$.

From conditions (A) and (D) we have

\[ Z^{\Delta \Delta \Delta}(r_2) = w^{\Delta \Delta \Delta}(r_2) - z^{\Delta \Delta \Delta}(r_2) \]
\[ > g(r_2, w(r_2), w^{\Delta}(r_2), w^{\Delta \Delta}(r_2)) - g(r_2, z_\varepsilon(r_2), z^{\Delta}(r_2), z^{\Delta \Delta}(r_2)) \]
\[ \geq 0. \]

That is, $Z^{\Delta \Delta \Delta}(r_2) > 0$, which is a contradiction. Our assumption must be wrong and consequently (1.1), (2.1) has at most one solution.

\[ \square \]

Theorem 2.2. Assume that hypotheses (A) through (D) are satisfied. Then (1.1), (1.2) has at most one solution.

Proof. Assume that there exist two distinct solutions $\alpha$ and $\beta$ of (1.1), (1.2). Define $w = \alpha - \beta$. Then $w(t_1) = w(t_2) = w(t_3) = 0$. From Theorem 2.1, $w^{\Delta}(t_2) \neq 0$ and $w^{\Delta \Delta}(t_2) \neq 0$. Without loss of generality let $w^{\Delta}(t_2) = \alpha^{\Delta}(t_2) - \beta^{\Delta}(t_2) > 0$. By Theorem 1.2 we have $w^{\Delta}(t_2) > w^{\Delta}(t_2) > 0$. Then there exist points $r_1 \in (t_1, t_2) \cap T$ and $r_2 \in (t_2, t_3) \cap T$ such that $w^{\Delta}$ has a g.z. at $r_1$ and $r_2$ and $w^{\Delta}(t) > 0$ on $(r_1, r_2) \cap T$. 

Let $\varepsilon = \frac{1}{2} \min \{ \varepsilon_1, w^\Delta(t_2) \}$ and let $z_\varepsilon$ be the solution of the initial value problem $z^\Delta(t) = g(t, z(t), z^\Delta(t)), t \in [t_1, t_3] \cap \mathbb{T}$, $z_\varepsilon(t_2) = 0, z^\Delta(t_2) = \varepsilon, z_\varepsilon(t_2) = 0$. By condition (B), $Z^\Delta$ does not change sign on $[t_1, t_3] \cap \mathbb{T}$.

Define $Z \equiv w - z_\varepsilon$. Then $Z(t_2) = 0, Z^\Delta(t_2) > 0$, and $Z^\Delta(t_2) = w^\Delta(t_2) \neq 0$. There are two cases to consider.

**Case 1:** $Z^\Delta(t_2) < 0$. Recall that $w^\Delta$ has a g.z. at $r_1$. If $r_1$ is right dense, then $w^\Delta(r_1) = 0$. If $r_1$ is right scattered, then either $w^\Delta(r_1) = 0$ or $w^\Delta(\sigma(r_1)) \cdot w^\Delta(r_1) < 0$. In the latter case since $w^\Delta(t) > 0$ on $(r_1, r_2) \cap \mathbb{T}$, we have $w^\Delta(r_1) < 0$. Regardless of whether $r_1$ is right dense or right scattered we have $Z^\Delta(r_1) = w^\Delta(r_1) - z^\Delta_\varepsilon(r_1) \leq 0$.

Since $Z^\Delta(r_1) < 0$ and $Z^\Delta(t_2) < 0$, there exists an $r_3 \in (r_1, t_2) \cap \mathbb{T}$ such that $Z^\Delta$ has a g.z. at $r_3$ and $Z^\Delta(t) < 0$ on $(r_3, t_2] \cap \mathbb{T}$.

On the one hand, if $\sigma(r_3) > r_3$, then

$$Z^\Delta(t) = Z^\Delta(\sigma(r_3)) - Z^\Delta(r_3) < 0.$$ If $\sigma(r_3) = r_3$, then

$$Z^\Delta(t) = \lim_{t \to r_3^+} \frac{Z^\Delta(t)}{t - r_3} < 0.$$ Regardless of whether $r_3$ is right dense or right scattered we have, from the definition of the delta derivative, that $Z^\Delta(t) < 0$.

On the other hand, from conditions (A) and (D) we have

$$Z^\Delta(t) = w^\Delta(r_3) - z^\Delta_\varepsilon(r_3)$$

$$> g(r_4, w(r_4), w^\Delta(t), w^\Delta(r_3)) - g(r_3, z_\varepsilon(r_3), z^\Delta_\varepsilon(r_3))$$

$$\geq 0.$$ That is, conditions (A) and (D) imply that $Z^\Delta(t) > 0$ which is a contradiction. Consequently, $Z^\Delta(t_2) \neq 0$.

**Case 2:** $Z^\Delta(t_2) > 0$. Again, we know that $w^\Delta$ has a g.z. at $r_2$. If $\sigma(r_2) = r_2$, then $w^\Delta(r_2) = 0$. If $\sigma(r_2) > r_2$, then either $w^\Delta(r_2) = 0$ or $w^\Delta(r_2) > 0$ and $w^\Delta(\sigma(r_2)) < 0$ or $w^\Delta(r_2) < 0$ and $w^\Delta(\rho(r_2)) > 0$. Consequently, either $Z^\Delta(t_2) < 0$ or $Z^\Delta(\sigma(r_2)) < 0$.

Since $Z^\Delta(r_2) > 0$, (where $r^* = r_2$ or $r^* = \sigma(r_2)$), and since $Z^\Delta(t_2) > 0$, there exists $r_4 \in (t_2, r^*)$ such that $Z^\Delta$ has a g.z. at $r_4$, $Z^\Delta(t) > 0$ on $[t_2, r_4) \cap \mathbb{T}$, and $Z^\Delta$ does not have a g.z. in $[t_2, r_4) \cap \mathbb{T}$.

We now obtain a contradiction. On the one hand, we can use the definition of the $\Delta$-derivative to calculate $Z^\Delta(t_4)$. If $\rho(r_4) = r_4$, then by Theorem 1.1 we have

$$Z^\Delta(t_4) = Z^\Delta_\varepsilon(t_4) = \lim_{t \to r_4^-} \frac{Z^\Delta(t) - 0}{t - r_4} < 0.$$ If $\rho(r_4) < r_4$, then either $\sigma(r_4) = r_4$ or $\sigma(r_4) > r_4$. If $\sigma(r_4) = r_4$, then

$$Z^\Delta(t_4) = \lim_{t \to r_4^+} \frac{Z^\Delta(t)}{t - r_4} < 0.$$ If $\sigma(r_4) > r_4$, then

$$Z^\Delta(t_4) = \frac{Z^\Delta(\sigma(r_4)) - Z^\Delta(r_4)}{\sigma(r_4) - r_4} < 0.$$ In any case, we have, by definition of the $\Delta$-derivative, that $Z^\Delta(t_4) < 0$. 


On the other hand, we have from conditions (A) and (D),
\[
Z^\Delta\Delta\Delta(r_4) = w^\Delta\Delta\Delta(r_4) - z^\Delta\Delta\Delta(r_4)
\]
\[
> g(r_4, w(r_4), w^\Delta(r_4), w^\Delta\Delta(r_4)) - g(r_4, z^\Delta(r_4), z^\Delta\Delta(r_4), z^\Delta\Delta\Delta(r_4))
\]
\[
\geq 0.
\]
Conditions (A) and (D) imply \(Z^\Delta\Delta\Delta(r_4) > 0\) which is a contradiction. Thus \(Z^\Delta\Delta\Delta(t_2) \neq 0\).

Since \(Z^\Delta\Delta\Delta(t_2) \neq 0\) and \(Z^\Delta\Delta\Delta(t_2) < 0\) and \(Z^\Delta\Delta\Delta(t_2) > 0\) lead to contradictions, our original assumption must be false. As such, the boundary value problem (1.1), (1.2) has at most one solution and the theorem is proved.

Our original assumption must be false. As such, the boundary value problem (1.1), (1.2) has at most one solution and the theorem is proved.

\[\square\]

Now given \(m \in \mathbb{R}\), let \(\alpha(x, m), \beta(x, m), u(x, m)\) and \(v(x, m)\) denote the solutions, when they exist, of the boundary value problems for (1.1), (1.2), \(j = 1, 2\), respectively.

**Theorem 2.3.** Suppose that (A) through (D) are satisfied and that, for each \(m \in \mathbb{R}\), there exist solutions of (1.1), (2.1) and (1.1), (2.2), \(j = 1, 2\). Then \(w^\Delta(t_2, m)\) and \(\alpha^\Delta\Delta(t_2, m)\) are strictly increasing functions of \(m\) whose range is \(\mathbb{R}\), and \(v^\Delta(t_2, m)\) and \(\beta^\Delta\Delta(t_2, m)\) are strictly decreasing functions of \(m\) with ranges all of \(\mathbb{R}\).

**Proof.** The “strictness” of the conclusion arises from Theorem 2.2. We will prove the theorem with respect to the solution \(\alpha(t, m)\). Let \(m_1 > m_2\) and let \(w(t) \equiv \alpha(t, m_1) - \alpha(t, m_2)\). Then when \(w(t_1) = w(t_2) = 0, w^\Delta(t_2) > 0\), and \(w^\Delta\Delta(t_2) \neq 0\).

Assume that \(w^\Delta\Delta(t_2) < 0\). Then there exists an \(r_1 \in (t_1, t_2) \cap \mathbb{T}\) such that \(w^\Delta\Delta\) has a g.z. at \(r_1\) and \(w^\Delta(t) > 0\) on \((r_1, t_2) \cap \mathbb{T}\). By continuity, there exists an \(r_2 \in (r_1, t_2) \cap \mathbb{T}\) such that \(w^\Delta\Delta\) has a g.z. at \(r_2\) and \(w^\Delta\Delta(t) < 0\) on \((r_2, t_2) \cap \mathbb{T}\). Note that \(w(t) < 0\) on \([r_2, t_2) \cap \mathbb{T}\).

Let \(0 < \varepsilon < \min\{\varepsilon_2, -w^\Delta\Delta(t_2)\}\) and let \(z_\varepsilon\) be the solution of the initial value problem satisfying conditions of (C), and set \(Z \equiv w - z_\varepsilon\). Then \(Z(t_2) = 0, Z^\Delta(t_2) = w^\Delta(t_2) > 0\), and \(Z^\Delta\Delta(t_2) < 0\). Furthermore \(Z^\Delta\Delta\Delta(r_2) > 0\). Thus there exist \(r_3 \in (r_2, t_2) \cap \mathbb{T}\) such that \(Z^\Delta\Delta\Delta(r_3) = 0\) and \(Z^\Delta\Delta\Delta(t) < 0\) on \((r_3, t_2)\). Then \(Z(t) > 0\) and \(Z(T) < 0\) on \([r_3, t_2)\). As in the proofs of Theorems 2.1 and 2.2 we can then argue that \(Z^\Delta\Delta\Delta(t_3) < 0\) and \(Z^\Delta\Delta\Delta(t_3) > 0\), which is again a contradiction. Thus \(w^\Delta\Delta\Delta(t_2) > 0\) and consequently, \(\alpha^\Delta\Delta(t_2, m)\) is strictly increasing as a function of \(m\).

We now show that \(\{\alpha^\Delta\Delta(t_2, m) : m \in \mathbb{R}\} = \mathbb{R}\). Let \(k \in \mathbb{R}\) and consider the solution \(u(x, k)\) of (1.1), (2.1) (with \(j = 2\)) with \(u\) as specified above. Consider also the solution \(\alpha(x, u^\Delta(t_2, k))\) of (1.1), (2.1) (with \(j = 1\)). Then \(\alpha(x, u^\Delta(t_2, k))\) and \(u(x, k)\) are solutions of (1.1), (2.1). Hence, by Theorem 2.1 \(\alpha(x, u^\Delta(t_2, k)) \equiv u(x, k)\). Therefore, \(\alpha^\Delta\Delta(t_2, u^\Delta(t_2, k)) = k\) and so \(\{\alpha^\Delta\Delta(t_2, m) : m \in \mathbb{R}\} = \mathbb{R}\). The other three parts are established in a similar manner and the proof is complete. \(\square\)

**Theorem 2.4.** Assume the hypothesis of Theorem 2.3. Then (1.1), (1.2) has a unique solution.

**Proof.** By Theorem 2.3, there exists a unique \(m_0\) such that \(w^\Delta(t_2, m_0) = w^\Delta(t_2, m_0)\). Also \(u^\Delta\Delta(t_2, m_0) = m_0 = v^\Delta\Delta(t_2, m_0)\). Then,

\[
y(t) = \begin{cases} u(t, m_0) = y_1(t), & t_1 \leq t \leq t_2, \\ v(t, m_0) = y_2(t), & t_2 \leq t \leq t_3, \end{cases}
\]

is a solution of (1.1), (1.2). By Theorem 2.2 \(y(t)\) is the unique solution. \(\square\)
References


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