Multiple positive solutions for a class of quasilinear elliptic boundary-value problems *

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Abstract

Using variational arguments we prove some nonexistence and multiplicity results for positive solutions of a class of elliptic boundary-value problems involving the $p$-Laplacian and a parameter.

1 Introduction

In a recent paper, Maya and Shivaji [4] studied the existence, multiplicity, and non-existence of positive classical solutions of the semilinear elliptic boundary-value problem

$$
-\Delta u = \lambda f(u) \quad \text{in } \Omega,
$$

$$
\quad u = 0 \quad \text{on } \partial \Omega \quad (1.1)
$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\lambda > 0$ is a parameter, and $f$ is a $C^1$ function such that

$$
f(0) = 0, \quad \lim_{t \to \infty} \frac{f(t)}{t} = 0. \quad (1.2)
$$

Assuming

$(f_1)$ \quad $f'(0) < 0,$

$(f_2)$ \quad $\exists \beta > 0$ such that $f(t) < 0$ for $0 < t < \beta$ and $f(t) > 0$ for $t > \beta,$

$(f_3)$ \quad $f$ is eventually increasing,

they showed using sub-super solutions arguments and recent results from semipositone problems that there are $\underline{\lambda}$ and $\overline{\lambda}$ such that (1.1) has no positive solution for $\lambda < \underline{\lambda}$ and at least two positive solutions for $\lambda \geq \overline{\lambda}.$

In the present paper we consider the corresponding quasilinear problem

$$
-\Delta_p u = \lambda f(x, u) \quad \text{in } \Omega,
$$

$$
\quad u = 0 \quad \text{on } \partial \Omega \quad (1.3)
$$

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where $\Delta_p u = \text{div} (|\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian, $1 < p < \infty$, $\lambda > 0$, and $f$ is a Carathéodory function on $\Omega \times [0, \infty)$ satisfying

$$f(x, 0) = 0, \quad |f(x, t)| \leq Ct^{p-1} \quad \text{(1.4)}$$

for some constant $C > 0$. Note that when $p = 2$ and $f$ is $C^1$ and satisfies (1.2), the existence of the limits $\lim_{t \to 0} f(t)/t = f'(0)$ and $\lim_{t \to \infty} f(t)/t$ imply (1.4). Using variational methods, we shall prove the following theorems.

**Theorem 1.1.** There is a $\lambda$ such that (1.3) has no positive solution for $\lambda < \lambda$.

**Theorem 1.2.** Set $F(x, t) = \int_0^t f(x, s) ds$, and assume

(F$_1$) $\exists \delta > 0$ such that $F(x, t) \leq 0$ for $0 \leq t \leq \delta$,

(F$_2$) $\exists t_0 > 0$ such that $F(x, t_0) > 0$,

(F$_3$) $\lim_{t \to \infty} \frac{F(x, t)}{t^p} \leq 0$ uniformly in $x$.

Then there is a $\lambda$ such that (1.3) has at least two positive solutions $u_1 > u_2$ for $\lambda \geq \lambda$.

Note that we have substantially relaxed the assumptions in [4] and therefore our results seem to be new even in the semilinear case $p = 2$. More specifically, we have let $f$ depend on $x$ and dropped the assumption of differentiability in $t$, and replaced (f$_1$), (f$_2$), and (f$_3$) with the much weaker assumptions (F$_1$) and (F$_2$) on the primitive $F$. We emphasize that (F$_1$) follows from (f$_1$), while (f$_2$) and (f$_3$) together imply (F$_2$), and that we make no monotonicity assumptions. The limit in (F$_3$) equals 0 in the $p$-sublinear case

$$\lim_{t \to \infty} \frac{f(x, t)}{t^{p-1}} = 0 \text{ uniformly in } x, \quad \text{(1.5)}$$

in particular, in the special case considered in [4].

## 2 Proofs of Theorems 1.1 and 1.2

Recall that the first Dirichlet eigenvalue of $-\Delta_p$ is positive and is given by

$$\lambda_1 = \min_{u \neq 0} \frac{\int_{\Omega} |\nabla u|^p}{\int_{\Omega} |u|^p} \quad \text{(2.1)}$$

(see Lindqvist [3]). If (1.3) has a positive solution $u$, multiplying (1.3) by $u$, integrating by parts, and using (1.4) gives

$$\int_{\Omega} |\nabla u|^p = \lambda \int_{\Omega} f(x, u) u \leq C \lambda \int_{\Omega} u^p, \quad \text{(2.2)}$$

and hence $\lambda \geq \lambda_1/C$ by (2.1), proving Theorem 1.1.
We will prove Theorem 1.2 using critical point theory. Set \( f(x,t) = 0 \) for \( t < 0 \), and consider the \( C^1 \) functional
\[
\Phi_\lambda(u) = \int_\Omega |\nabla u|^p - \lambda p F(x,u), \quad u \in W^{1,p}_0(\Omega). \tag{2.3}
\]
If \( u \) is a critical point of \( \Phi_\lambda \), denoting by \( u^- \) the negative part of \( u \),
\[
0 = (\Phi'_\lambda(u), u^-) = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla u^+ - \lambda f(x,u)u^- = \|u^-\|^p \tag{2.4}
\]
shows that \( u \geq 0 \). Furthermore, \( u \in L^\infty(\Omega) \cap C^1(\Omega) \) by Anane [1] and di Benedetto [2], so it follows from the Harnack inequality (Theorem 1.1 of Trudinger [6]) that either \( u > 0 \) or \( u \equiv 0 \). Thus, nontrivial critical points of \( \Phi_\lambda \) are positive solutions of (1.3).

By (F3) and (1.4), there is a constant \( C_\lambda > 0 \) such that
\[
\lambda p F(x,t) \leq \frac{\lambda_1}{2} |t|^p + C_\lambda \tag{2.5}
\]
and hence
\[
\Phi_\lambda(u) \geq \int_\Omega |\nabla u|^p - \frac{\lambda_1}{2} |u|^p - C_\lambda \geq \frac{1}{2} \|u\|^p - C_\lambda \mu(\Omega) \tag{2.6}
\]
where \( \mu \) denotes the Lebesgue measure in \( \mathbb{R}^n \), so \( \Phi_\lambda \) is bounded from below and coercive. This yields a global minimizer \( u_1 \) since \( \Phi_\lambda \) is weakly lower semicontinuous.

**Lemma 2.1.** There is a \( \overline{\lambda} \) such that \( \inf \Phi_\lambda < 0 \), and hence \( u_1 \neq 0 \), for \( \lambda \geq \overline{\lambda} \).

**Proof.** Taking a sufficiently large compact subset \( \Omega' \) of \( \Omega \) by Anane [1] and di Benedetto [2], so it follows from the Harnack inequality (Theorem 1.1 of Trudinger [6]) that either \( u > 0 \) or \( u \equiv 0 \). Thus, nontrivial critical points of \( \Phi_\lambda \) are positive solutions of (1.3).

By (F3) and (1.4), there is a constant \( C_\lambda > 0 \) such that
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Now fix \( \lambda \geq \overline{\lambda} \), let
\[
\tilde{f}(x,t) = \begin{cases} f(x,t), & t \leq u_1(x), \\ f(x,u_1(x)), & t > u_1(x), \end{cases} \quad \text{and} \quad \tilde{F}(x,t) = \int_0^t \tilde{f}(x,s) ds. \tag{2.8}
\]
Then consider
\[
\tilde{\Phi}_\lambda(u) = \int_\Omega |\nabla u|^p - \lambda p \tilde{F}(x,u). \tag{2.9}
\]
If $u$ is a critical point of $\tilde{\Phi}_\lambda$, then $u \geq 0$ as before, and
\[
0 = (\tilde{\Phi}'_\lambda(u) - \Phi'_\lambda(u_1), (u - u_1)^+) \\
= \int_{\Omega} (|\nabla u|^{p-2}\nabla u - |\nabla u_1|^{p-2}\nabla u_1) \cdot \nabla (u - u_1) \\
- \lambda (\tilde{f}(x, u) - f(x, u_1))(u - u_1)^+ \\
= \int_{u > u_1} ((|\nabla u|^{p-2}\nabla u - |\nabla u_1|^{p-2}\nabla u_1) \cdot (\nabla u - \nabla u_1) \\
\geq \int_{u > u_1} (|\nabla u|^{p-1} - |\nabla u_1|^{p-1}) (|\nabla u| - |\nabla u_1|) \geq 0.
\]
implies that $u \leq u_1$, so $u$ is a solution of (1.3) in the order interval $[0, u_1]$. We will obtain a critical point $u_2$ with $\Phi_\lambda(u_2) > 0$ via the mountain-pass lemma, which would complete the proof since $\tilde{\Phi}_\lambda(0) = 0 > \tilde{\Phi}_\lambda(u_1)$.

**Lemma 2.2.** The origin is a strict local minimizer of $\tilde{\Phi}_\lambda$.

*Proof.* Setting $\Omega_u = \{ x \in \Omega : u(x) > \min \{ u_1(x), \delta \} \}$, by (2.8) and (F1), $\tilde{F}(x, u(x)) \leq 0$ on $\Omega \setminus \Omega_u$, so
\[
\tilde{\Phi}_\lambda(u) \geq \|u\|^p - \lambda p \int_{\Omega_u} \tilde{F}(x, u).
\]
By (1.4), Hölder’s inequality, and Sobolev imbedding,
\[
\int_{\Omega_u} \tilde{F}(x, u) \leq C \int_{\Omega_u} u^p \leq C \mu(\Omega_u)^{1-\frac{q}{p}} \|u\|^p
\]
where $q = np/(n-p)$ if $p < n$ and $q > p$ if $p \geq n$, so it suffices to show that $\mu(\Omega_u) \to 0$ as $\|u\| \to 0$.

Given $\varepsilon > 0$, take a compact subset $\Omega_\varepsilon$ of $\Omega$ such that $\mu(\Omega \setminus \Omega_\varepsilon) < \varepsilon$ and let $\Omega_{u,\varepsilon} = \Omega_u \cap \Omega_\varepsilon$. Then
\[
\|u\|^p \geq \int_{\Omega_{u,\varepsilon}} u^p \geq c^p \mu(\Omega_{u,\varepsilon})
\]
where $c = \min \{ \min u_1(\Omega_\varepsilon), \delta \} > 0$, so $\mu(\Omega_{u,\varepsilon}) \to 0$. But, since $\Omega_u \subset \Omega_{u,\varepsilon} \cup (\Omega \setminus \Omega_\varepsilon)$,
\[
\mu(\Omega_u) < \mu(\Omega_{u,\varepsilon}) + \varepsilon,
\]
and $\varepsilon$ is arbitrary. \qed

An argument similar to the one we used for $\Phi_\lambda$ shows that $\tilde{\Phi}_\lambda$ is also coercive, so every Palais-Smale sequence of $\tilde{\Phi}_\lambda$ is bounded and hence contains a convergent subsequence as usual. Now the mountain-pass lemma gives a critical point $u_2$ of $\tilde{\Phi}_\lambda$ at the level
\[
c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} \tilde{\Phi}_\lambda(u) > 0
\]
where $\Gamma = \{ \gamma \in C([0,1], W_0^{1,p}(\Omega)) : \gamma(0) = 0, \gamma(1) = u_1 \}$ is the class of paths joining the origin to $u_1$ (see, e.g., Rabinowitz [5]).
References


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