AN EMBEDDING NORM AND THE LINDQVIST TRIGONOMETRIC FUNCTIONS

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Abstract. We shall calculate the operator norm \( \|T\|_p \) of the Hardy operator
\[ Tf = \int_0^x f, \]
where \( 1 \leq p \leq \infty \). This operator is related to the Sobolev embedding operator from \( W^{1,p}(0,1)/C \) into \( W^p(0,1)/C \). For \( 1 < p < \infty \), the extremal, whose norm gives the operator norm \( \|T\|_p \), is expressed in terms of the function \( \sin_p \) which is a generalization of the usual sine function and was introduced by Lindqvist [6].

1. Introduction

Evans, Harris and Saitō [3] give the following result: Let \( W^{1,p}(0,1) \) be the complex first order Sobolev space given by
\[ W^{1,p}(0,1) = \{ f \mid \int_0^1 (|f|^p + |f'|^p) < \infty \}, \]
where \( 1 < p < \infty \). Then
\[ \sup \frac{\|f\|_{p,s}}{\|f\|_p} = \frac{\|T\|_p}{2}, \tag{1.1} \]
where the supremum is over all non-zero functions in \( W^{1,p}(0,1) \), \( \|\cdot\|_p \) is the norm of \( L^p(0,1) \), \( \|\cdot\|_{p,s} \) is given by
\[ \|f\|_{p,s} = \inf_{C \in C} \|f - C\|_p, \]
and \( \|T\|_p \) is the operator norm of the Hardy operator
\[ T : L^p(0,1) \ni f \mapsto Tf(x) = \int_0^x f \in L^p(0,1). \]
The left-hand side of (1.1) is the norm of the embedding from the factor space \( W^{1,p}(0,1)/C \) into the factor space \( L^p(0,1)/C \). Since the Poincaré inequality holds on \( (0,1) \), the norm \( \|f\|_p, f \in W^{1,p}(0,1) \), is one of the equivalent norms of the space \( W^{1,p}(0,1)/C \). For more details on the embedding operator we refer to Evans and Harris [4], [5], in addition to [3].

In this note we calculate the norm \( \|T\|_p = p^{1/q} q^{1/p} \sin(\pi/p)/\pi \), \( 1 < p < \infty \), where \( 1/p + 1/q = 1 \), and \( \|T\|_1 = \|T\|_\infty = 1 \). Our main arguments depend on classical and elementary calculus of variations. We will also calculate all extremals,
i.e., functions \( f \) for which \( \|Tf\|_p = \|f\|_p \). In particular, if \( 1 < p < \infty \) these are expressed in terms of ‘\( \sin_p \)’ and ‘\( \cos_p \)’ functions which has been introduced in Lindqvist [6]. These are natural generalizations of the usual trigonometric functions when the Euclidean norm in \( \mathbb{R}^2 \) is replaced by a \( p \)-norm.

After finishing our work, we found two recent works, Edmunds and Lang [2] and Drabek and Manásevich [1], which, among others, produced quite a similar results in real \( L^p \) spaces on intervals using the results on the \( p \)-Laplacian.

In Section 2 we shall give a short account on \( \sin_p \) and \( \cos_p \) functions. In Section 3 we shall first deal with the cases that \( p = 1 \) and \( p = \infty \). Then we shall study the case that \( 1 < p < \infty \). After showing that we have only to find the nonnegative extremals (Lemmas 3 and 4), we shall show that the extremal is expressed in terms of the \( \sin_p \) function and the norm \( \|T\|_p \) is computed using some properties of the \( \sin_p \) and \( \cos_p \) functions (Theorem 3.1). We shall discuss the operator

\[
T : L^p(0, 1) \ni f \mapsto Tf(x) = \int_0^x f \in L^p(0, 1).
\]

in Section 4.

2. \( \sin_p \), \( \cos_p \) AND \( \tan_p \) FUNCTIONS

Suppose \( 1 < p < \infty \) and consider the function \( x \mapsto \int_0^x \frac{dt}{(1 - t^p)^{1/p}}, 0 \leq x \leq 1 \). This is strictly increasing, and for \( p = 2 \) we obtain \( \arcsin x \). By analogy we denote the inverse for general \( p \) by \( \sin_p x \). It is defined on the interval \([0, \pi_p/2]\) where \( \pi_p = 2 \int_0^1 \frac{dt}{(1 - t^p)^{1/p}} \), it is strictly increasing on this interval, \( \sin_p 0 = 0 \) and \( \sin_p(\pi_p/2) = 1 \).

We may extend the definition to the interval \([0, \pi_p]\) by setting \( \sin_p x = \sin_p(\pi_p - x) \) for \( \pi_p/2 \leq x \leq \pi_p \), and to \([-\pi_p, \pi_p]\) by extending it as an odd function. Finally, we may define it on all of \( \mathbb{R} \) by extending it as a periodic function of period \( 2\pi_p \).

Next we define \( \cos_p x = \frac{d}{dx} \sin_p x \), which is an even, \( 2\pi_p \)-periodic function, odd around \( \pi_p/2 \). In \([0, \pi_p/2]\), setting \( y = \sin_p x \), we have

\[
\cos_p x = \frac{dy}{dx} \sin_p x = (1 - y^p)^{1/p} = (1 - (\sin_p x)^p)^{1/p},
\]

so that \( \cos_p \) is strictly decreasing on this interval, \( \cos_p 0 = 1 \) and \( \cos_p(\pi_p/2) = 0 \).

Furthermore

\[
|\cos_p x|^p + |\sin_p x|^p = 1,
\]

first in \([0, \pi_p/2]\), but then by symmetry and periodicity in all of \( \mathbb{R} \). One may also introduce an analogue of the tangent function as \( \tan_p x = \sin_p x / \cos_p x \). We then obtain

\[
\frac{d}{dx} \cos_p x = -|\tan_p x|^p \sin_p x.
\]

We also get

\[
\frac{d}{dx} \tan_p x = 1/|\cos_p x|^p = 1 + |\tan_p x|^p,
\]

so that the inverse of the restriction of \( \tan_p x \) to the interval \( (-\pi_p/2, \pi_p/2) \) has derivative \( 1/(1 + |y|^p) \), \( y \in \mathbb{R} \). The constant \( \pi_p \) is easily calculated. In fact, by a change of variable \( t = s^{1/p} \) we obtain

\[
\pi_p = \frac{2}{p} \int_0^1 (1 - s)^{-1/p}s^{1/p-1} \, ds = \frac{2}{p} B(1 - 1/p, 1/p) = 2 \frac{\pi/p}{\sin(\pi/p)},
\]

(2.1)

where \( B \) is the classical beta function. If \( q \) is the conjugate exponent to \( p \), so that \( 1/p + 1/q = 1 \), this means in particular that \( q \pi_p = p \pi_q \).

The cases \( p = 1 \) and \( p = \infty \) are somewhat degenerate, especially for \( p = 1 \). For \( p = \infty \) one gets \( \pi_\infty = 2 \), and \( \sin_p \) becomes a triangular wave, with \( \sin_p x = x \) on \([-1, 1]\), and \( \cos_p \) the corresponding square wave. For \( p = 1 \) one gets \( \pi_1 = \infty \), and \( \sin_p x \) is the odd extension of \( 1 - e^{-x} \), \( x \geq 0 \), whereas \( \cos_p x = e^{-|x|} \).
3. An operator norm

Consider the linear operator \( Tf(x) = \int_{0}^{x} f \) on \( L^p(0,1) \), \( 1 \leq p \leq \infty \). We shall determine the norm \( \|T\|_p \) and all extremals, i.e., all functions \( f \in L^p(0,1) \) for which \( \|Tf\|_p = \|T\|_p \|f\|_p \). In particular, we shall show that if \( q \) is the conjugate exponent to \( p \), then \( \|T\|_p = \|T\|_q \). This may be proved by duality if \( 1 < p < \infty \), but will follow from our explicit calculations below.

**Theorem 3.1.** For \( 1 < p < \infty \), \( \|T\|_p = 2p^{1/q}q^{1/p}/(p\pi_p) = p^{1/q}q^{1/p} \sin(\pi/p)/\pi \), where \( q \) is the dual exponent to \( p \). The corresponding extremals are all multiples of \( \cos_p(x/2) \), where \( \pi_p \) is given by (2.1). Furthermore, \( \|T\|_1 = \|T\|_\infty = 1 \), and the extremals for \( p = \infty \) are all constants, whereas no extremals exist in \( L^1(0,1) \) for the case \( p = 1 \). If one extends \( T \) to the space of finite Borel measures on \([0,1] \), however, the extremals are all multiples of the Dirac measure at 0.

In particular \( \|T\|_p = \|T\|_q \) since \( p\pi_p = q\pi_q \). The statement of the theorem indicates that it will be advantageous, for \( p = 1 \), to extend \( T \) to operate on finite Borel measures on \([0,1] \), normed by total variation, thus considering \( L^1(0,1) \) as the subset of absolutely continuous Borel measures on \([0,1] \). We will do this in the sequel without further comment. The proof of the theorem is almost immediate in the cases \( p = 1 \) and \( p = \infty \).

**Lemma 3.2.** For \( p = 1 \) extremals exist, they are precisely the non-zero multiples of the Dirac measure at 0, and \( \|T\|_1 = 1 \).

**Proof.** If \( \mu \) is a finite Borel measure on \([0,1] \) and we define \( T\mu(x) = \int_{[0,x]} d\mu \), we obtain
\[
\|T\mu\|_1 = \int_{0}^{1} \left| \int_{[0,x]} d\mu \right| dx \leq \int_{0}^{1} \int_{[0,x]} |d\mu| dx = \int_{[0,1]} (1-t)|d\mu(t)| \leq \int_{[0,1]} |d\mu| = \|\mu\|_1, \quad (3.1)
\]
where the middle equality follows from Fubini’s theorem. Thus \( \|T\|_1 \leq 1 \). However, we obtain equality throughout in (3.1) precisely if \( \int_{[0,1]} t|d\mu(t)| = 0 \), i.e., the support of \( \mu \) consists of the point 0. Thus, if \( \mu \) is a multiple of the Dirac measure at 0. \( \square \)

The proof in the case \( p = \infty \) is even simpler.

**Lemma 3.3.** For \( p = \infty \) extremals exist, they are precisely the functions which are a.e. constant, and \( \|T\|_\infty = 1 \).

**Proof.** If \( f \in L^\infty(0,1) \) we obtain
\[
\|Tf\|_\infty = \sup_{0 \leq x \leq 1} \left| \int_{0}^{x} f \right| \leq \int_{0}^{1} |f| \leq \|f\|_\infty,
\]
and we have equality throughout if and only if \( f \) is a.e. a multiple of \( \|f\|_\infty \) (see also the proof of Lemma 3.4). \( \square \)

The case when \( 1 < p < \infty \) is less trivial, and the first step in the proof is to show that we may restrict ourselves to positive functions when looking for extremals.

**Lemma 3.4.** Any extremal for \( 1 \leq p \leq \infty \) is a multiple of a non-negative extremal. Therefore, the norm \( \|T\|_p \) when \( T \) is defined on the complex \( L^p \) space is the same as the one when \( T \) is defined on the real \( L^p \) space.
\textbf{Proof.} We already know this if \( p = 1 \) or \( p = \infty \), so we now assume that \( 1 < p < \infty \). Suppose \( f \) is an extremal and put \( g = |f| \). Since \( \|f\|_p = \|g\|_p \) and \( |f^0| f \leq f^0 g \) it follows that if \( f \) is an extremal, then so is \( g \). Furthermore, it follows that \( \|Tf\|_p = \|Tg\|_p \). Now \( |f^0| f \leq f^0 g \) for all \( x \in [0,1] \), so in order that \( \|Tf\|_p = \|Tg\|_p \) we must have equality throughout \([0,1]\). In particular, we must have
\[
\int_0^1 f = \int_0^1 g.
\]
i.e., we must have equality in the triangle inequality. This proves the lemma, since it requires that \( f \) and \( g \) are a.e. proportional. We remind the reader why this is so.

If we choose \( \theta \in \mathbb{R} \) so that \( e^{i\theta} f^0 \) is positive, and we have equality in the triangle inequality \( |f^0| f \leq f^0 |f| \), then we have \( \int_0^1 |e^{i\theta} f| = \int_0^1 |f| \) while at the same time \( \text{Re}(e^{i\theta} f) = |e^{i\theta} f| = |f| \). Thus \( |f| = e^{i\theta} f \) a.e.

The next step in the proof of Theorem 3.1 is to show the existence of extremals.

\textbf{Lemma 3.5.} If \( 1 < p < \infty \) there exists a function \( f \in L^p(0,1) \) with non-zero norm such that \( \|Tf\|_p = \|T\|_p \|f\|_p \).

\textbf{Proof.} Let \( f_1, f_2, \ldots \) be a sequence of unit vectors in \( L^p(0,1) \) such that
\[
\|Tf_n\|_p \to \|T\|_p \quad \text{as} \quad n \to \infty.
\]
We may assume that \( f_n \to f \in L^p(0,1) \) weakly in \( L^p(0,1) \). For each fixed \( x \in [0,1] \) the characteristic function of \([0,x]\) is in \( L^q(0,1) \), \( q = p/(p-1) \), so we obtain
\[
F_n(x) = \int_0^x f_n \to \int_0^x f = F(x) \quad \text{as} \quad n \to \infty.
\]
Since \( |F_n(x)| \leq \int_0^1 |f_n| \leq \|f_n\|_p = 1 \) by Hölder’s inequality, we obtain by dominated convergence that \( \|F\|_p = \lim \|F_n\|_p = \|F\|_p \), and since clearly \( \|T\|_p > 0 \) we can not have \( F = 0 \), so that \( \|f\|_p > 0 \). Since \( \|T\|_p = \|F\|_p \leq \|T\|_p \|f\|_p \) it actually follows that \( \|f\|_p = 1 \). The lemma is proved.

We are now ready to prove Theorem 3.1. We will do this by applying standard methods of the calculus of variations.

\textbf{Proof of Theorem 3.1.} We need only consider the case \( 1 < p < \infty \). Suppose \( f \geq 0 \) is an extremal with non-zero norm and put \( F(x) = Tf(x) = f^0 \) \( f \geq 0 \). Then setting \( \lambda = \|F\|_p^p / \|F\|_p^p \) we have \( \|T\|_p = \lambda^{1/p} \). If \( \varphi \in C^1(0,1) \) with \( \varphi(0) = 0 \), then \( \|F + \varepsilon \varphi\|_p / \|F + \varepsilon \varphi\|_p^p \) has a maximum for \( \varepsilon = 0 \). Differentiating we get, for \( \varepsilon = 0 \), that
\[
\int_0^1 F^{p-1} \varphi - \lambda \int_0^1 (F')^{p-1} \varphi' = 0,
\]
so integrating by parts we obtain
\[
\int_0^1 (\lambda(F')^{p-1} - \int_x^1 F^{p-1}) \varphi' = 0 
\]
for all \( \varphi \in C^1(0,1) \) with \( \varphi(0) = 0 \), so that, by du Bois Reymond’s lemma, \( \lambda(F')^{p-1} - \int_x^1 F^{p-1} \) is constant. It follows that \( (F')^{p-1} \) is continuously differentiable, and differentiating we obtain
\[
F^{p-1} + \lambda((F')^{p-1})' = 0.
\]
Thus, integrating by parts in (3.2) we obtain \((F'(1))^{p-1} \psi(1) = 0\), so that \(F'(1) = f(1) = 0\). Multiplying (3.3) by \(F'\) and integrating we obtain

\[ F^p + \lambda(p - 1)(F')^p = C, \tag{3.4} \]

where \(C > 0\) is a constant and we should note that

\[ (p - 1)p^{-1} \frac{d}{dx} \left( (F'(p-1)p/(p-1)) \right) = ((F')^{p-1})F'. \]

Here, after multiplying \(F\) by an appropriate constant, we may assume that \(C = 1\) in (3.4). Although (3.4) is then satisfied in any interval where \(F = 1\), this will not satisfy (3.2). In any point where \(F \neq 1\) we then obtain

\[ \frac{F'}{(1 - F^p)^{1/p}} = a, \]

where \(a = (\lambda(p - 1))^{-1/p}\). Integrating again we obtain \(F(x) = \sin_p(ax)\), since \(F(0) = 0\). Thus \(f(x) = F'(x) = a \cos_p(ax)\). From \(f(1) = 0\) it follows that \(a\) is a zero of \(\cos_p\), and since we have \(f \geq 0\), \(a\) is the first positive zero of \(\cos_p\). Thus \(a = \pi p/2\).

We also have \(\lambda = a^{-p}/(p - 1)\) so that \(\|T\|_p = (p - 1)^{-1/p}/a = 2(p - 1)^{-1/p}/\pi p\). Now, if \(q\) is the conjugate exponent to \(p\) we have \(p(p - 1)^{-1/p} = p^{1/q}q^{1/p}\) and we are done, in view of the formula (2.1).

\[ \square \]

4. Generalizations

One may of course also consider the operator

\[ T : L^p(0, 1) \ni f \mapsto Tf(x) = \int_0^x f \in L^q(0, 1), \]

where now \(p\) and \(q\) are unrelated exponents in \([1, \infty]\). Even in this case it is possible to calculate the norm \(\|T\|_{p,q}\). We introduce the constant \(\pi_{p,q} = 2 \int_0^1 \frac{dt}{(1 - t^p)^{1/p}}\), which is finite unless \(p = 1\), \(q < \infty\), and then the function \(\sin_{p,q}\), first on the interval \([0, \pi_{p,q}/2]\) as the inverse of the strictly increasing function

\[ [0, 1) \ni x \mapsto \int_0^x \frac{dt}{(1 - t^p)^{1/p}}, \]

and then suitably extended to an odd function, which is \(2\pi_{p,q}\)-periodic and even around \(\pi_{p,q}/2\) if \(\pi_{p,q}\) is finite. We next define \(\cos_{p,q} x = \frac{d}{dx} \sin_{p,q} x\) and may the easily deduce that

\[ |\cos_{p,q} x|^p + |\sin_{p,q} x|^q = 1, \]

except if \(p = \infty\), \(q < \infty\). Drabek and Manásevich [1] also introduced \(\sin_{p,q}\) and \(\cos_{p,q}\) functions. They are quite similar to ours but not the same.

**Proposition 4.1.** For \(p, q \in (1, \infty)\) we have \(\pi_{p,q} = \frac{2}{q} B(1/p', 1/q)\), where \(p'\) is the dual exponent of \(p\) and \(B\) is the classical beta function. Furthermore,

- \(\pi_{p,\infty} = 2\), \(1 \leq p \leq \infty\),
- \(\pi_{\infty,q} = 2\), \(1 \leq q \leq \infty\),
- \(\pi_{p,1} = 2p'\), \(1 \leq p \leq \infty\),
- \(\pi_{1,q} = \infty\), \(1 \leq q < \infty\).

We may now carry out the analysis of the operator \(T\) in much the same way as in Section 3, with the following conclusion.
Theorem 4.2. For \( p, q \in (1, \infty) \) we have
\[
\|T\|_{p,q} = \left( p' + q \right)^{1/p'} \left( q' \right)^{1/q'} B(1/p', 1/q),
\]
where \( p' \) is the dual exponent of \( p \) and \( B \) is the classical beta function. Extremals are all non-zero multiples of \( \cos p, q(\pi p,q x/2) \). Furthermore,
- \( \|T\|_{p,\infty} = 1, \ 1 \leq p \leq \infty \). Extremals are all constants \( \neq 0 \). In the case \( p = 1 \) any non-zero multiple of a non-zero positive measure is an extremal.
- \( \|T\|_{\infty,q} = (1 + q)^{-1/q}, \ 1 \leq q < \infty \). Extremals are all constants \( \neq 0 \).
- \( \|T\|_{p,1} = (1 + p')^{-1/p'}, \ 1 < p \leq \infty \). Extremals are all non-zero multiples of \( (1 - x)^{1/(p-1)} \).
- \( \|T\|_{1,q} = 1, \ 1 \leq q < \infty \). Extremals are all non-zero multiples of the Dirac measure at the origin.

It will be seen that as a function of \((1/p, 1/q)\) the norm \( \|T\|_{p,q} \) is continuous in the unit square \( 0 \leq 1/p \leq 1, \ 0 \leq 1/q \leq 1 \). Furthermore, it is clear that \( \|T\|_{p,q} = \|T\|_{q',p'} \) for all \( p, q \in [1, \infty] \), a fact that could presumably also be proved by duality.

References


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