Existence of solutions for a variational unilateral system *

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Abstract

In this work the authors study the existence of weak solutions of the nonlinear unilateral mixed problem associated to the inequalities

\[ u_{tt} - M(|\nabla u|^2)\Delta u + \theta \geq f, \]
\[ \theta_t - \Delta \theta + u_t \geq g, \]

where \( f, g, M \) are given real-valued functions with \( M \) positive.

1 Introduction

Let \( \Omega \) be a bounded and open set of \( \mathbb{R}^n \), with smooth boundary \( \Gamma = \partial \Omega \), and let \( T \) be a positive real number. Let \( Q = \Omega \times [0,T] \) be the cylinder with lateral boundary \( \Sigma = \Gamma \times [0,T] \).

We study the variational nonlinear system

\[ u_{tt} - M(|\nabla u|^2)\Delta u + \theta \geq f \quad \text{in} \quad Q, \quad (1.1) \]
\[ \theta_t - \Delta \theta + u_t \geq g \quad \text{in} \quad Q, \quad (1.2) \]
\[ u = \theta = 0 \quad \text{in} \quad \Sigma, \quad (1.3) \]
\[ u(0) = u_0, \quad u'(0) = u_1, \quad \theta(0) = \theta_0. \quad (1.4) \]

The above system with \( M(s) = m_0 + m_1 s \) (\( m_0 \) and \( m_1 \) positive constants) and \( \theta = 0 \) is a nonlinear perturbation of the canonical Kirchhoff model

\[ u_{tt} - (m_0 + m_1 \int_{\Omega} |\nabla u|^2 dx) \Delta u = f. \quad (1.5) \]

This model describes small vibrations of a stretched string when only the transverse component of the tension is considered, see for example, Arosio & Spagnolo [1], Pohozaev [12].

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Several authors have studied (1.5). For $\Omega$ bounded, we can cite: D’Ancona & Spagnolo [5], Medeiros & Milla Miranda [9], Hosoya & Yamada [7], Lions [8], Medeiros [10], and Matos [9]. For $\Omega$ unbounded, we can cite Bisiguin [2], Clark & Lima [4], and Matos [9]. The system (1.1)–(1.4) was studied also in the case when (1.1) and (1.2) are equations, see for example [3].

In the present work we show the existence of a weak solution for the variational nonlinear system (1.1)–(1.4), under appropriate assumptions on $M$, $f$ and $g$. We employ Galerkin’s approximation method and the penalization method used by Frota & Lar’kin [6].

2 Notation and main result

We represent the Sobolev space of order $m$ on $\Omega$ by

$$W^{m,p}(\Omega) = \{ u \in L^p(\Omega); D^\alpha u \in L^p(\Omega), \forall |\alpha| \leq m \}$$

and its associated norm by

$$\|u\|_{m,p} = \left( \sum_{|\alpha| \leq m} |D^\alpha u|_{L^p(\Omega)}^p \right)^{1/p}, \quad u \in W^{m,p}(\Omega), \quad 1 \leq p < \infty.$$

When $p = 2$, we have the usual Sobolev space $H^m(\Omega)$. Let $D(\Omega)$ be the space of the test functions on $\Omega$, and let $W^{m,p}_0(\Omega)$ be the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$. When $p = 2$, we have $W^{2,2}_0(\Omega) = H^m_0(\Omega)$. The dual space of $W^{m,p}_0(\Omega)$ is denoted by $W^{-m,p'}(\Omega)$, with $p'$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. For the rest of this paper we use the symbol $(\cdot, \cdot)$ to indicate the inner product in $L^2(\Omega)$, and $((\cdot, \cdot))$ to indicate the inner product in $H^1_0(\Omega)$.

Let $K = \{ \psi \in W^{2,4}_0(\Omega); |\Delta \psi| \leq 1 \text{ and } \psi \geq 0 \text{ a.e. in } \Omega \}$. Then we have the following proposition whose proof can be found in [6]

**Proposition 2.1** The set $K$ is a closed and connected in $W^{2,4}_0(\Omega)$.

**Definition** Let $V$ be a Banach space and $V'$ its dual. An operator $\beta$ from $V$ to $V'$ is called hemicontinuous if the function

$$\lambda \rightarrow (\beta(\lambda + \lambda v),w)$$

is continuous for all $\lambda \in \mathbb{R}$. The operator $\beta$ is called monotone if

$$(\beta(u) - \beta(v), u - v) \geq 0, \quad \forall u, v \in V.$$

We consider the penalization operator $\beta: W^{2,4}_0(\Omega) \rightarrow W^{-2,4/3}(\Omega)$ such that $\beta(\zeta) = \beta_1(\zeta) + \beta_2(\zeta)$, $\zeta \in W^{2,4}_0(\Omega)$, where $\beta_1(\zeta)$ and $\beta_2(\zeta)$ are defined by

$$\langle \beta_1(\zeta), v \rangle = - \int_\Omega z^{-}(x)v(x)dx,$$

$$\langle \beta_2(\zeta), v \rangle = - \int_\Omega (1 - |\Delta \zeta(x)|^2)^{-\frac{1}{2}} \Delta \zeta(x) \Delta v(x)dx$$

for all $v$ in $W^{2,4}_0(\Omega)$. 
Proposition 2.2 The operator $\beta$ defined above satisfies the following coditions:

i) $\beta$ is monotone and hemicontinuous

ii) $\beta$ is bounded; this is, $\beta(S)$ is bounded in $W^{2,4/3}(\Omega)$ for all bounded set $S$ in $W^{2,4}_{0}(\Omega)$.

iii) $\beta(u) = 0$ if only if $u$ belongs to $K$.

The proof of this proposition can be found in [6].

In this article, we assume the following hypotheses:

A1) $M \in C^1[0, \infty)$, $M(s) \geq 0$ for $s \geq 0$, and $\int_0^\infty M(s)ds = \infty$

A2) $f, g$ belong to $H^1_0(0, T; L^2(\Omega)).$

The main result of this paper is stated as follows.

Theorem 2.3 Assume A1) and A2). For $u_0 \in H^1_0(\Omega) \cap H^2(\Omega)$, $u_1, \theta_0$ in the interior of $K$, there exist functions $u, \theta: Q \to \mathbb{R}$ such that

\begin{align*}
&u \in L^\infty(0, T; H^1_0(\Omega) \cap H^2(\Omega)) \\
u' \in L^1(0, T; W^{2,4}_{0}(\Omega)) \text{ and } u'(t) \in K \text{ a.e. in } [0, T] \\
u'' \in L^\infty(0, T; L^2(\Omega)) \\
&\theta \in L^\infty(0, T; H^1_0(\Omega)) \text{ and } \theta(t) \in K \text{ a.e. in } [0, T].
\end{align*}

Also

\begin{align*}
(u''(t) - M(\|u(t)\|)^2)\Delta u(t) + \theta(t) - f(t), v - u'(t) \geq 0, \forall v \in K \text{ a.e. in } [0, T] \quad (2.5) \\
(\theta'(t) - \Delta \theta(t) + u'(t) - g(t), v - \theta(t)) \geq 0 \forall v \in K \text{ a.e. in } [0, T] \quad (2.6) \\
u(0) = u_0, ~ u'(0) = u_1, ~ \theta(0) = \theta_0. \quad (2.7)
\end{align*}

To obtain the solution $\{u, \theta\}$ of problem (2.1)–(2.4) in Theorem 2.3, we consider the following associated penalized problem. For $0 < \varepsilon < 1$, consider

\begin{align*}
u''_\varepsilon(t) - M(\|u_\varepsilon(t)\|^2)\Delta u_\varepsilon(t) + \theta_\varepsilon(t) + \frac{1}{\varepsilon}\beta(u'_\varepsilon(t)) = f(t) \text{ in } Q \quad (2.8) \\
\theta'_\varepsilon(t) - \Delta \theta_\varepsilon(t) + u'_\varepsilon + \frac{1}{\varepsilon}\beta(\theta_\varepsilon(t)) = g(t) \text{ in } Q \quad (2.9) \\
u_\varepsilon(0) = u_{0\varepsilon}, \ u'_\varepsilon(0) = u_{1\varepsilon}, \ \theta_\varepsilon(0) = \theta_{0\varepsilon} \text{ in } \Omega \quad (2.10)
\end{align*}

Here $\beta$ is a penalization operator, $M$, $f$, and $g$ are as above. The solution $\{u_\varepsilon, \theta_\varepsilon\}$ of the penalized problem (2.8)–(2.10) are guaranteed by the following theorem.
Theorem 2.4 Suppose the hypotheses of the Theorem 2.3 hold, and for $0 < \varepsilon < 1$, then there exist functions $\{u_\varepsilon, \theta_\varepsilon\}$ such that

\[
\begin{align*}
  u_\varepsilon, \theta_\varepsilon &\in L^\infty(0,T;H^1_0(\Omega) \cap H^2(\Omega)) \quad (2.11) \\
  u'_\varepsilon &\in L^4(0,T;W^{2,4}_0(\Omega)) \quad (2.12) \\
  u''_\varepsilon &\in L^\infty(0,T;L^2(\Omega)) \quad (2.13) \\
  \theta_\varepsilon &\in L^4(0,T;W^{2,4}_0(\Omega)) \quad (2.14)
\end{align*}
\]

\[
(u''_\varepsilon(t), v) + M(||u_\varepsilon(t)||^2)((u_\varepsilon(t), v)) + (\theta_\varepsilon(t), v) + \frac{1}{\varepsilon}<\beta(u'_\varepsilon(t)), v>
\]

\[
= (f(t), v) \text{ a.e. in } [0, T] \text{ for all } v \in W^{2,4}_0(\Omega),
\]

\[
(\theta'_\varepsilon(t), v) + ((\theta_\varepsilon(t), v)) + (u'_\varepsilon(t), v) + \frac{1}{\varepsilon}<\beta(\theta_\varepsilon(t)), v>
\]

\[
= (g(t), v) \text{ a.e. in } [0, T] \text{ for all } v \in W^{2,4}_0(\Omega),
\]

\[
u_\varepsilon(0) = u_{0\varepsilon}, u'_\varepsilon(0) = u_{1\varepsilon}, \theta_\varepsilon(0) = \theta_{0\varepsilon}.
\]

**Proof** We will use Galerkin’s method and a compactness argument.

**First step** (Approximated system) Let $w_1, \ldots, w_m$ be an orthonormal basis of $W^{2,4}_0(\Omega)$ consisting of eigenfunctions of the Laplacian operator. Let $V_m = [w_1, \ldots, w_m]$ the subspace of $W^{2,4}_0(\Omega)$, generated by the first $m$ vectors $w_j$. We look for a pair of functions $u_{\varepsilon m}(t) = \sum_{j=1}^{m} g_{jm}(t)w_j$, $\theta_{\varepsilon m}(t) = \sum_{j=1}^{m} h_{jm}(t)w_j$ in $V_m$

with $g_{jm} \in C^2([0,T])$ and $h_{jm} \in C^1([0,T])$, for all $j = 1, \ldots, m$. Which are solutions of the following system of ordinary differential equations

\[
(u''_{\varepsilon m}(t), w_j) + M(||u_{\varepsilon m}(t)||^2)((u_{\varepsilon m}(t), w_j)) + (\theta_{\varepsilon m}(t), w_j) + \frac{1}{\varepsilon}<\beta(u'_{\varepsilon m}(t)), w_j> = (f(t), w_j),
\]

\[
(\theta'_{\varepsilon m}(t), w_j) + ((\theta_{\varepsilon m}(t), w_j)) + (u'_{\varepsilon m}(t), w_j) + \frac{1}{\varepsilon}<\beta(\theta_{\varepsilon m}(t)), w_j> = (g(t), w_j),
\]

for $j = 1, \ldots, m$, with the initial conditions: $u_{\varepsilon m}(0) = u_{0\varepsilon m}$, $u'_{\varepsilon m}(0) = u_{1\varepsilon m}$, $\theta_{\varepsilon m}(0) = \theta_{0\varepsilon m}$, where

\[
u_{0\varepsilon m} = \sum_{j=1}^{m} (u_{0\varepsilon}, w_j)w_j \rightarrow u_0 \text{ strongly in } H^1_0(\Omega) \cap H^2(\Omega),
\]

\[
u_{1\varepsilon m} = \sum_{j=1}^{m} (u_{1\varepsilon}, w_j)w_j \rightarrow u_1 \text{ strongly in } H^1_0(\Omega),
\]

\[
\theta_{0\varepsilon m} = \sum_{j=1}^{m} (\theta_{0\varepsilon}, w_j)w_j \rightarrow \theta_0 \text{ strongly in } W^{2,4}_0(\Omega).
\]
The system (2.18)–(2.20) contains 2m unknowns functions \(g_{jm}(t), h_{jm}(t)\); \(j = 1, 2, \ldots, m\). By Caratheodory’s Theorem it follows that (2.18)–(2.20) has a local solution \(\{u_{xm}(t), \theta_{xm}(t)\}\) on \([0, t_m]\). In order to extend these local solutions to the interval \([0, T]\) and to take the limit in \(m\), we must obtain some a priori estimates.

**Estimate (i)** Note that finite linear combinations of the \(w_j\) are dense in \(W_0^{2, 4}(\Omega)\), then we can take \(w \in W_0^{2, 4}(\Omega)\) in (2.18) and (2.19) instead of \(w_j\).

Taking \(w = 2u'_{xm}(t)\) in (2.18) and \(w = 2\theta_{xm}(t)\) in (2.19) we obtain

\[
\frac{d}{dt} |u'_{xm}(t)|^2 + \frac{d}{dt} \overline{M}(\|u_{xm}(t)\|^2) + \frac{2}{\varepsilon} \langle \beta(u'_{xm}(t)), u'_{xm}(t) \rangle = 2(f(t), u'_{xm}(t)) - 2(\theta_{xm}(t), u'_{xm}(t)),
\]

\[
\frac{d}{dt} |\theta_{xm}(t)|^2 + \|\theta_{xm}(t)\|^2 + \frac{2}{\varepsilon} \langle \beta(\theta_{xm}(t)), \theta_{xm}(t) \rangle = -2(u'_{xm}(t), \theta_{xm}(t)) + 2(g(t), \theta_{xm}(t)),
\]

where \(\overline{M}(\lambda) = \int_0^\lambda M(s)ds\). Adding (2.21) and (2.22), and integrating from 0 to \(t \leq t_m\) we have

\[
|u'_{xm}(t)|^2 + |\theta_{xm}(t)|^2 + \int_0^t |u_{xm}(s)|^2 ds + \int_0^t |\theta_{xm}(s)|^2 ds + \int_0^t |f(t)|^2 ds + 3 \int_0^t |u'_{xm}(s)|^2 ds + 3 \int_0^t |\theta_{xm}(s)|^2 ds + \int_0^T |g(t)|^2 dt + |\theta_{xm}|^2 + |u_{xm}|^2.
\]

From (2.20) and hypothesis (A2) there exists a positive constant \(C\), independently of \(\varepsilon > 0\) and \(m\) such that

\[
|u'_{xm}(t)|^2 + |\theta_{xm}(t)|^2 + \int_0^t |u_{xm}(s)|^2 ds + \int_0^t |\theta_{xm}(s)|^2 ds + \frac{2}{\varepsilon} \left[ \int_0^t \langle \beta(u'_{xm}(s)), u'_{xm}(s) \rangle ds + \int_0^t \langle \beta(\theta_{xm}(s)), \theta_{xm}(s) \rangle ds \right] \leq C + 3 \int_0^t |u'_{xm}(s)|^2 ds + 3 \int_0^t |\theta_{xm}(s)|^2 ds.
\]

Next we analyze the sign of the term \(\int_0^t \langle \beta(u'_{xm}(s)), u'_{xm}(s) \rangle ds\). Note that \(-u'_{xm}(t) \leq u'_{xm}(t)^-\). Then, by the definition of \(\beta\), we have

\[
\langle \beta(u'_{xm}(t)), u'_{xm}(t) \rangle = \langle \beta_1(u'_{xm}(t)), u'_{xm}(t) \rangle + \langle \beta_2(u'_{xm}(t)), u'_{xm}(t) \rangle = - \int_\Omega (u'_{xm}(x, t) - u'_{xm}(x, t) dx + \int_\Omega (1 - |\Delta u'_{xm}(t)|^2) - (\Delta u'_{xm}(t))^2 dx \geq 0.
\]
Similarly, we have,
\[ \langle \beta(\theta_{\varepsilon m}(t)), \theta_{\varepsilon m}(t) \rangle \geq 0. \]

Because \( M(s) \geq 0 \) for all \( s \), from (2.24) and Gronwall’s inequality it follows that
\[ |u_{\varepsilon m}'(t)|^2 + |\theta_{\varepsilon m}(t)|^2 \leq C_1, \quad \forall \varepsilon, m, \forall t \in [0, t_m]. \]

Returning to (2.24), we obtain
\[
|u_{\varepsilon m}'(t)|^2 + |\theta_{\varepsilon m}(t)|^2 + \int_0^t |u_{\varepsilon m}(s)|^2 M(s) ds + \int_0^t |\theta_{\varepsilon m}(s)|^2 ds + \frac{2}{\varepsilon} \int_0^t \langle \beta(u_{\varepsilon m}'(s)), u_{\varepsilon m}'(s) \rangle ds + \frac{2}{\varepsilon} \int_0^t \langle \beta(\theta_{\varepsilon m}(s)), \theta_{\varepsilon m}(s) \rangle ds \leq C + 3C_1 T. \tag{2.25}
\]

Since \( \int_0^\infty M(s) ds = \infty \), by (2.25) we can find \( C_1 \) such that
\[ |u_{\varepsilon m}(t)|^2 \leq C_1, \quad \forall \varepsilon, m, \forall t \in [0, t_m]. \]

Thus there exists, other constant \( C = C(T) \) independently of \( \varepsilon, m \) and \( t \in [0, t_m] \) such that
\[
|u_{\varepsilon m}'(t)|^2 + |\theta_{\varepsilon m}(t)|^2 + \int_0^t |u_{\varepsilon m}(s)|^2 ds + \frac{2}{\varepsilon} \int_0^t \langle \beta(u_{\varepsilon m}'(s)), u_{\varepsilon m}'(s) \rangle ds + \frac{2}{\varepsilon} \int_0^t \langle \beta(\theta_{\varepsilon m}(s)), \theta_{\varepsilon m}(s) \rangle ds \leq C \tag{2.26}
\]

**Estimate (ii)** We will obtain a bound for \( |u_{\varepsilon m}''(0)| \). For this, we note that \( u_1 \) being in the interior of \( K \) and \( u_{1\varepsilon m} \to u_1 \) imply that \( u_{1\varepsilon m} \) is in the interior of \( K \), for \( m \) large. Therefore, \( |\Delta u_{1\varepsilon m}| \leq 1 \) and \( u_{1\varepsilon m} \geq 0 \) a.e. in \( \Omega \). Also we have \( (u_{1\varepsilon m})^2 \to 0 \) and \( (1 - |\Delta u_{1\varepsilon m}|)^2 \to 0 \) a.e. in \( \Omega \). Thus
\[ \langle \beta(u_{1\varepsilon m}), u_{1\varepsilon m}''(0) \rangle = 0 \tag{2.27} \]

Taking \( t = 0 \) and \( v = u_{1\varepsilon m}''(0) \) in (2.14), and observing (2.27), we obtain
\[ |u_{\varepsilon m}''(0)|^2 + M(\|u_{0\varepsilon m}\|^2)(u_{0\varepsilon m}, u_{1\varepsilon m}''(0)) + \langle \theta_{\varepsilon m}, u_{\varepsilon m}''(0) \rangle = (f(0), u_{\varepsilon m}''(0)) \]
which implies
\[ |u_{\varepsilon m}''(0)|^2 \leq |f(0)||u_{\varepsilon m}''(0)| + M(\|u_{0\varepsilon m}\|^2)|\Delta u_{0\varepsilon m}||u_{\varepsilon m}''(0)| + |\theta_{0\varepsilon m}||u_{\varepsilon m}''(0)|. \]

From \( u_{0\varepsilon m} \to u_0 \) in \( H^1_0(\Omega) \cap H^2(\Omega), \theta_{0\varepsilon m} \to \theta_0 \) in \( H^1_0(\Omega) \), \( M \in C^1[0, \infty) \), and \( f \in H^1(0, T; L^2(\Omega)) \), we obtain
\[ |u_{\varepsilon m}''(0)| \leq C, \tag{2.28} \]
with \( C \) independent of \( \varepsilon, m \), and \( t \in [0, T] \).

**Estimate (iii)** We obtain estimates for \( |\Delta u_{\varepsilon m}'(t)|, |\Delta \theta_{\varepsilon m}''(t)|, \int_0^t |u_{\varepsilon m}'(s)|^2 ds \), and \( \int_0^t |\theta_{\varepsilon m}(s)|^2 ds \). For this, we need the following lemma whose proof can be found in [6].
Lemma 2.5 Let $h : \Omega \to \mathbb{R}$ be an arbitrary function. Then

$$h^4 - 1 \leq 2(1 - h^2) - h^2.$$ 

By this lemma, we have

$$(\Delta u'_{\varepsilon m})^4 - 1 \leq 2[1 - (\Delta u'_{\varepsilon m})^2] - (\Delta u'_{\varepsilon m})^2.$$ 

Therefore,

$$\|\Delta u'_{\varepsilon m}\|_{L^4(\Omega)}^4 = \int_0^T \int_\Omega |\Delta u'_{\varepsilon m}(x, t)|^4 dx \, dt$$

$$\leq 2 \int_0^T \int_\Omega (1 - \Delta |u'_{\varepsilon m}(x, t)|^2) (\Delta u'_{\varepsilon m}(x, t))^2 dx \, dt + \text{meas}(Q)$$

$$= 2 \int_0^T \langle \beta(\Delta u'_{\varepsilon m}(t)), u'_{\varepsilon m}(t) \rangle dx \, dt + \text{meas}(Q).$$

Using (2.26), we obtain

$$\|\Delta u'_{\varepsilon m}\|_{L^4(\Omega)}^4 \leq C_{\varepsilon} + \text{meas}(Q) < C + \text{meas}(Q) \tag{2.29}$$

with $C$ independent of $\varepsilon, m$ and $t \in [0, T]$. Analogously, using the Lemma 2.5 with $h = \Delta \theta_{\varepsilon m}$ and (2.26), we obtain

$$\|\Delta \theta_{\varepsilon m}\|_{L^4(\Omega)}^4 \leq C + \text{meas}(Q) \tag{2.30}$$

On the other hand, from (2.18) and (2.19), we obtain

$$\frac{1}{\varepsilon} \langle \beta(u'_{\varepsilon m}(t)), v \rangle + \frac{1}{\varepsilon} \langle \beta(\theta_{\varepsilon m}(t)), v \rangle \leq C(\|f(t)\| + \|g(t)\|) v + 2 \|\beta\|v + C(\|u_{\varepsilon m}(t)\|\|v\| + \|\theta_{\varepsilon m}(t)\|\|v\|) \leq$$

$$\frac{1}{\varepsilon} \|\beta(u'_{\varepsilon m}(t))\| v + \|\beta(\theta_{\varepsilon m}(t))\| v + 2 \|\beta\|v + C(\|u_{\varepsilon m}(t)\|\|v\| + \|\theta_{\varepsilon m}(t)\|\|v\|) \leq$$

$$\frac{1}{\varepsilon} \|\beta(u'_{\varepsilon m}(t))\| v + \|\beta(\theta_{\varepsilon m}(t))\| v + 2 \|\beta\|v + C(\|u_{\varepsilon m}(t)\|\|v\| + \|\theta_{\varepsilon m}(t)\|\|v\|) \leq$$

Since $f, g \in C^0([0, T]; L^2(\Omega))$, from the inequality above we obtain

$$\frac{1}{\varepsilon} \|\beta(u'_{\varepsilon m}(t))\| v \leq C_1 \|v\| \quad \forall v \in W_0^{2,4}(\Omega), \tag{2.31}$$

$$\frac{1}{\varepsilon} \|\beta(\theta_{\varepsilon m}(t))\| v \leq C_1 \|v\| \quad \forall v \in W_0^{2,4}(\Omega), \tag{2.32}$$

independent of $\varepsilon, m$ and $t \in [0, T]$; this is,

$$\|\beta(u'_{\varepsilon m})\|_{L^\infty(0,T;W^{2,4/3}(\Omega))} \leq C_1, \tag{2.33}$$

$$\|\beta(\theta_{\varepsilon m})\|_{L^\infty(0,T;W^{2,4/3}(\Omega))} \leq C_1. \tag{2.34}$$
To estimate $|\Delta u_{\varepsilon m}(t)|$, we note that

$$
|\Delta u_{\varepsilon m}(t)|^2 = |\Delta u_{\varepsilon m}(t)|^2 + \int_0^t \frac{d}{ds}|\Delta u_{\varepsilon m}(s)|^2 ds
$$

$$
= |\Delta u_{\varepsilon m}(t)|^2 + 2C \int_0^t |\Delta u_{\varepsilon m}(s)||\Delta u_{\varepsilon m}(s)| ds
$$

$$
\leq |\Delta u_{\varepsilon m}(t)|^2 + C \int_0^t (|\Delta u_{\varepsilon m}(s)|^2 + \|\Delta u_{\varepsilon m}(s)\|^2) ds,
$$

where $C$ is the constant of the embedding from $H_0^1(\Omega)$ into $L^2(\Omega)$. From (2.20), (2.29) and Gronwall’s inequality, we obtain

$$
|\Delta u_{\varepsilon m}(t)|^2 < C, \tag{2.35}
$$

where $C$ is a constant independent of $\varepsilon, m$ and $t \in [0, T]$. 

Next, we obtain an estimate for $\int_0^T |\Delta u_{\varepsilon m}(s)|^3 ds$. Let $C$ represent various positive constants of the embedding in the sequence

$$W^{2,1}_0(\Omega) \hookrightarrow H^1_0(\Omega) \hookrightarrow H^1_0(\Omega).$$

Observing that $W^{2,1}_0(\Omega) \subseteq C|\Delta w|$ we obtain

$$
\int_0^t ||u_{\varepsilon m}(s)||^3 ds \leq C \int_0^t ||u_{\varepsilon m}(s)||^{4/3}_H ds \leq C \int_0^t |\Delta u_{\varepsilon m}(s)|^3 ds, \tag{2.36}
$$

independently of $\varepsilon$ and $m$. It follows from Hölder’s inequality that

$$
\int_0^t |\Delta u_{\varepsilon m}(s)|^3 ds \leq (\int_0^T 1^{1/4} ds)^{1/4} (\int_0^T ||u_{\varepsilon m}(s)||^4 ds)^{3/4}
$$

and substituting in (2.36) and observing (2.29), we obtain

$$
\int_0^t ||u_{\varepsilon m}(s)||^3 ds \leq C, \tag{2.37}
$$

independent of $\varepsilon, m$ and $t \in [0, T]$.

**Estimate (iv)** We will obtain the estimative for $|u_{\varepsilon m}''(t)|$. Let us consider the functions

$$
\Psi_h(t) = \frac{1}{h} [u_{\varepsilon m}(t + h) - u_{\varepsilon m}(t)],
$$

$$
M_h(t) = \frac{1}{h} [M(||u_{\varepsilon m}(t + h)||^2) - M(||u_{\varepsilon m}(t)||^2)],
$$

$$
f_h(t) = \frac{1}{h} [f(t + h) - f(t)].
$$

Setting $w = 2\Psi_h(t)$ in (1.14), we obtain

$$
2(u''_{\varepsilon m}(t), \Psi_h'(t)) + 2M(||u_{\varepsilon m}(t)||^2)(u_{\varepsilon m}(t), \Psi_h'(t)) + \frac{2}{\varepsilon} \langle \beta(u_{\varepsilon m}(t)), \Psi_h'(t) \rangle = 2(f(t), \Psi_h'(t)). \tag{2.38}
$$
Substituting \( t \) by \( t + h \in [0,T] \) in (2.18) and taking \( w = 2\Psi'_h(t) \), we set
\[
2(u''_{\epsilon m}(t + h), \Psi'_h(t)) + 2M(\|u_{\epsilon m}(t + h)\|^2)((u_{\epsilon m}(t + h), \Psi'_h(t))) + \frac{2}{\varepsilon} \langle \beta'(u'_{\epsilon m}(t + h)), \Psi'_h(t) \rangle = 2(f(t + h), \Psi'_h(t)).
\] (2.39)

Now, from (2.38) and (2.39) it follows, for \( h \neq 0 \), that
\[
2\left(\frac{u''_{\epsilon m}(t + h) - u''_{\epsilon m}(t)}{h}, \Psi'_h(t)\right) + \frac{2}{h} M(\|u_{\epsilon m}(t + h)\|^2)((u_{\epsilon m}(t + h), \Psi'_h(t))) - \frac{2}{h} M(\|u_{\epsilon m}(t)\|^2)((u_{\epsilon m}(t), \Psi'_h(t))) + \frac{2}{h\varepsilon} \langle \beta(u'_{\epsilon m}(t + h)) - \beta(u'_{\epsilon m}(t)), \Psi'_h(t) \rangle = 2\left(\frac{f(t + h) - f(t)}{h}, \Psi'_h(t)\right),
\]
which implies
\[
\frac{d}{dt} |\Psi'_h(t)|^2 + \frac{2}{h} M(\|u_{\epsilon m}(t + h)\|^2)((u_{\epsilon m}(t + h), \Psi'_h(t))) - \frac{2}{h} M(\|u_{\epsilon m}(t)\|^2)((u_{\epsilon m}(t), \Psi'_h(t))) + \frac{2}{h\varepsilon} \langle \beta(u'_{\epsilon m}(t + h)) - \beta(u'_{\epsilon m}(t)), \Psi'_h(t) \rangle = 2\left(\frac{f(t + h) - f(t)}{h}, \Psi'_h(t)\right).
\] (2.40)

Nothing that
\[
\frac{2}{h} M(\|u_{\epsilon m}(t + h)\|^2)((u_{\epsilon m}(t + h), \Psi'_h(t))) - \frac{2}{h} M(\|u_{\epsilon m}(t)\|^2)((u_{\epsilon m}(t), \Psi'_h(t))) = 2M(\|u_{\epsilon m}(t + h)\|^2)((\Psi_h(t), \Psi'_h(t))) + \frac{2M(\|u_{\epsilon m}(t + h)\|^2)}{h}\left(\frac{u_{\epsilon m}(t + h) - u_{\epsilon m}(t)}{h}, \Psi'_h(t)\right) - \frac{2M(\|u_{\epsilon m}(t)\|^2)}{h}\left(\frac{u_{\epsilon m}(t + h) - u_{\epsilon m}(t)}{h}, \Psi'_h(t)\right) = M(\|u_{\epsilon m}(t + h)\|^2) \frac{d}{dt} (\|\Psi'_h(t)\|^2) + 2M_h(t)(\|u_{\epsilon m}(t)\|^2)(\Psi'_h(t)).
\]

From (2.40) it follows that
\[
\frac{d}{dt} |\Psi'_h(t)|^2 + M(\|u_{\epsilon m}(t + h)\|^2) \frac{d}{dt} (\|\Psi'_h(t)\|^2) + \frac{2}{h\varepsilon} \langle \beta(u'_{\epsilon m}(t + h)) - \beta(u'_{\epsilon m}(t)), u'_{\epsilon m}(t + h) - u'_{\epsilon m}(t) \rangle = -2M_h(t)((u_{\epsilon m}(t), \Psi'_h(t))) + 2(f_h(t), \Psi'_h(t)).
\]

By the monotonicity of the operator \( \beta \), we obtain
\[
\frac{d}{dt} |\Psi'_h(t)|^2 + M(\|u_{\epsilon m}(t + h)\|^2) \frac{d}{dt} (\|\Psi'_h(t)\|^2) \leq 2|M_h(t)(\Delta u_{\epsilon m}(t), \Psi'_h(t))] + 2\|(f_h(t), \Psi'_h(t))\|.
\] (2.41)
On the other hand, using integration by parts, we get

\[ |\Psi_0'(0)|^2 + 2 \int_0^t |M_h(s)(\Delta u_{\varepsilon m}(s), \Psi_h(s))|ds + 2 \int_0^t |(f_h(s), \Psi_h(s))|ds. \]

Taking the limit as \( h \to 0 \), it follows

\[ |u_{\varepsilon m}'(t)|^2 + \int_0^t M(||u_{\varepsilon m}(s)||^2) \frac{du_{\varepsilon m}(s)}{ds}^2ds \leq \]

\[ |u_{\varepsilon m}''(0)|^2 + 2 \int_0^t [M'(||u_{\varepsilon m}(s)||^2)||u_{\varepsilon m}'(s)||\Delta u_{\varepsilon m}(s), u_{\varepsilon m}''(s)]ds + \quad (2.42) \]

\[ 2 \int_0^t |(f'(s), u_{\varepsilon m}(s))|ds. \]

Using Assumption (A2) and (2.28), we obtain, from (2.42),

\[ |u_{\varepsilon m}''(t)|^2 + \int_0^t M(||u_{\varepsilon m}(s)||^2) \frac{du_{\varepsilon m}(s)}{ds}^2ds \leq \]

\[ C + 4 \int_0^t M'(||u_{\varepsilon m}(s)||^2)||u_{\varepsilon m}'(s)||\Delta u_{\varepsilon m}(s)||u_{\varepsilon m}''(s)||ds + \quad (2.43) \]

\[ \int_0^t |u_{\varepsilon m}''(s)|^2ds. \]

From (2.26), (2.35) and (2.37) it follows that there exists a positive constant \( C \) such that

\[ ||u_{\varepsilon m}(t)||^2 + ||\Delta u_{\varepsilon m}(t)||^2 + \int_0^t ||u_{\varepsilon m}'(s)||^2ds \leq C, \quad \forall \varepsilon, m, t. \quad (2.44) \]

Since \( M \in C^1([0, \infty)), \) we also obtain from (2.44),

\[ M'(||u_{\varepsilon m}(s)||^2) \leq C, \quad \forall \varepsilon, m, t. \quad (2.45) \]

On the other hand, using integration by parts, we get

\[ \int_0^t M(||u_{\varepsilon m}(s)||^2) \frac{d}{ds}||u_{\varepsilon m}'(s)||^2ds = \]

\[ M(||u_{\varepsilon m}(s)||^2)||u_{\varepsilon m}'(s)||^2 - M(||u_{0\varepsilon m}(s)||^2)||u_{1\varepsilon m}(s)||^2 - \]

\[ \int_0^t M'(||u_{\varepsilon m}(s)||^2) \frac{d}{ds}||u_{\varepsilon m}'(s)||^2||u_{\varepsilon m}(s)||^2ds. \]

Estimates (2.37), (2.44), and (2.45) together imply

\[ - \int M'(||u_{\varepsilon m}(s)||^2) \frac{d}{ds}||u_{\varepsilon m}(s)||^2||u_{\varepsilon m}'(s)||^2ds \geq -C \int_0^t ||u_{\varepsilon m}'(s)||^3 \geq -C, \]
independently of \( \varepsilon, m, \) and \( t \). Therefore,

\[
\int_0^t M(\|u_{\varepsilon m}(s)\|^2) \frac{d}{ds}\|u_{\varepsilon m}(s)\|^2 ds \geq M(\|u_{\varepsilon m}(t)\|^2)\|u'_{\varepsilon m}(t)\|^2 - C, \tag{2.46}
\]

independently of \( \varepsilon, m \) and \( t \). Here, \( C \) denote various positive constants. Making use of inequalities (2.44)–(2.46) in (2.43) we obtain

\[
|u''_{\varepsilon m}(t)|^2 + M(\|u_{\varepsilon m}(t)\|^2)\|u'_{\varepsilon m}(s)\|^2 \leq C + C \int_0^t |u''_{\varepsilon m}(s)|^2 ds, \tag{2.47}
\]

independently of \( \varepsilon, m, \) and \( t \). From (2.47) and using Gronwall’s inequality, we have

\[
|u''_{\varepsilon m}(t)|^2 \leq C, \tag{2.48}
\]

independently of \( \varepsilon, m, \) and \( t \).

**Passage to the limit** By estimates (2.26) and (2.35) we obtain

\[
(u_{\varepsilon m}) \text{ is bounded in } L^\infty(0,T;H^1_0(\Omega) \cap H^2(\Omega)),
\]

\[
(u'_{\varepsilon m}) \text{ is bounded in } L^\infty(0,T;L^2(\Omega)),
\]

\[
(\theta_{\varepsilon m}) \text{ is bounded in } L^\infty(0,T;L^2(\Omega)).
\]

Therefore, we can get subsequences, if necessary, denoted by \((u_{\varepsilon m})\) and \((\theta_{\varepsilon m})\), such that

\[
u_{\varepsilon m} \to u_\varepsilon \text{ weak star in } L^\infty(0,T;H^1_0(\Omega) \cap H^2(\Omega)), \tag{2.49}
\]

\[
u'_{\varepsilon m} \to u'_\varepsilon \text{ weak star in } L^\infty(0,T;L^2(\Omega)), \tag{2.50}
\]

\[
\theta_{\varepsilon m} \to \theta_\varepsilon \text{ weak star in } L^\infty(0,T;L^2(\Omega)). \tag{2.51}
\]

Similarly by (2.48), we obtain

\[
u''_{\varepsilon m} \to u''_\varepsilon \text{ weak star in } L^\infty(0,T;L^2(\Omega)). \tag{2.52}
\]

Also, by (2.33) and (2.34), there exist functions \( X_\varepsilon, \phi_\varepsilon \in L^{4/3}(0,T;W^{2,4/3}(\Omega)) \) such that

\[
\beta(u'_{\varepsilon m}) \to X_\varepsilon \text{ in } L^{4/3}(0,T;W^{2,4/3}(\Omega)), \tag{2.53}
\]

\[
\beta(\theta_{\varepsilon m}) \to \phi_\varepsilon \text{ in } L^{4/3}(0,T;W^{2,4/3}(\Omega)). \tag{2.54}
\]

It follows from the embedding \( W^{2,4}_0(\Omega) \) into \( L^4(\Omega) \) and of (2.29) that

\[
\|u'_{\varepsilon m}\|_{L^4(0,T;W^{2,4}_0(\Omega))}^4 \leq C \|\Delta u'_{\varepsilon m}\|_{L^2(\Omega)}^4 \leq K.
\]

Therefore, there exists a subsequence of \((u_{\varepsilon m})\) such that

\[
u'_{\varepsilon m} \to u'_\varepsilon \text{ weak star in } L^4(0,T;W^{2,4}_0(\Omega)). \tag{2.55}
\]
Analogously, by (2.30) we obtain
\[ \theta_{\varepsilon m} \to \theta_{\varepsilon} \text{ weak star in } L^4(0, T; W^{2, 4}_0(\Omega)). \] (2.56)

Being the embedding from \( H^1_0(\Omega) \cap H^2(\Omega) \) into \( H^1_0(\Omega) \) compact, we can set a subsequence, again denoted by \((u_{\varepsilon m})\), such that:
\[ u_{\varepsilon m} \to u_{\varepsilon} \text{ strong in } L^2(0, T; H^1_0(\Omega)). \] (2.57)

By assumption (A1) we obtain
\[ M(\|u_{\varepsilon m}(t)\|^2) \to M(\|u_{\varepsilon}(t)\|^2). \] (2.58)

From the compactness of the embedding \( H^1_0(\Omega) \hookrightarrow L^2(\Omega) \) we obtain
\[ u'_{\varepsilon m} \to u'_{\varepsilon} \text{ strong in } L^2(0, T; L^2(\Omega)). \] (2.59)

Then taking limit in the system (2.18)–(2.20), when \( m \to \infty \), with \( w = v\varphi(t), \ v \in W^{2, 4}_0(\Omega), \varphi(t) \in D(0, T) \) instead of \( w_j \), and using the fact that \( \beta \) is monotone and hemicontinuous operator, we obtain that \( \{u_{\varepsilon}, \theta_{\varepsilon}\} \) is a weak solution of the system (2.18)–(2.20).

The initial conditions (2.19) can be obtained by observing the convergence above and the definition of weak solution; this is,
\[ u'_{\varepsilon}(0) = \lim_{m \to \infty} u_{0\varepsilon m} \to \lim_{m \to \infty} \sum_{j=1}^{m} (u_{0\varepsilon}, w_j) w_j = u_0, \]
\[ u'_\varepsilon(0) = \lim_{m \to \infty} u_{1\varepsilon m} \to \lim_{m \to \infty} \sum_{j=1}^{m} (u_{1\varepsilon}, w_j) w_j = u_1, \]
\[ \phi_{\varepsilon}(0) = \lim_{m \to \infty} \theta_{0\varepsilon m} \to \lim_{m \to \infty} \sum_{j=1}^{m} (\theta_{0\varepsilon}, w_j) w_j = \theta_0. \]

This concludes the proof of Theorem 2.4

3 Main Result

In this section, we will prove the Theorem 2.3. By Theorem 2.4, there exists functions \( u_{\varepsilon}, \theta_{\varepsilon}: \mathbb{Q} \to \mathbb{R} \) such that
\[ u_{\varepsilon} \in L^\infty(0, T; H^1_0(\Omega) \cap H^2(\Omega)), \]
\[ u'_{\varepsilon}, \theta_{\varepsilon} \in L^4(0, T; W^{2, 4}_0(\Omega)), \]
\[ u''_{\varepsilon} \in L^\infty(0, T; L^2(\Omega)), \]
\[ \theta_{\varepsilon} \in L^\infty(0, T; L^2(\Omega)), \]
We observe that

\[ (u'_\varepsilon(t), w) + M[||u_\varepsilon(t)||^2][(u_\varepsilon(t), w)] + \frac{1}{\varepsilon} \langle \beta(u'_\varepsilon(t), w) \rangle = (g(t), w), \]

\[ (\theta_\varepsilon(t), w) + ((\theta_\varepsilon(t), w)) + (u'_\varepsilon(t), w) + \frac{1}{\varepsilon} \langle \beta(\theta_\varepsilon(t), w) \rangle = (g(t), w), \]

a.e. in \([0, T]\), for all \(w \in W^{2,4}_0(\Omega)\). \(u_\varepsilon(0) = u_0; u'_\varepsilon(0) = u_1, \) and \(\theta_\varepsilon(0) = \theta_0.\)

Being the estimates (2.26), (2.29), (2.30), (2.33), (2.34), (2.32) and (2.44) independently of \(\varepsilon, \) \(m\) and \(t\) we obtain by Uniform Boundedness Theorem that there exists a positive constant \(C\) such that

\[ |u'_\varepsilon(t)|^2 + |\theta_\varepsilon(t)|^2 + \|u_\varepsilon(t)\|^2 + \int_0^T \|\theta_\varepsilon(t)\|^2 ds + \]

\[ \frac{2}{\varepsilon} \int_0^T \langle \beta(u'_\varepsilon(s), u'_\varepsilon(s)) \rangle ds + \frac{2}{\varepsilon} \int_0^T \langle \beta(\theta_\varepsilon(s)), \theta_\varepsilon(s) \rangle ds \leq C \|\Delta u'_\varepsilon\|_{L^4(Q)}^4 \leq C, \]

and

\[ \|\Delta \theta_\varepsilon\|_{L^4(Q)}^4 \leq C, \quad \|\beta(u'_\varepsilon)\|_{L^4(0, T; W^{2,4/3}(\Omega))} \leq C, \]

\[ \|\beta(\theta_\varepsilon)\|_{L^{4/3}(0, T; W^{2,4/3}(\Omega))} \leq C, \quad |\Delta u_\varepsilon(t)|^2 \leq C, \quad |u''_\varepsilon(t)|^2 \leq C. \]

Consequently, we can find a subnet, which we still represent by \((u_\varepsilon), (\theta_\varepsilon)\) such that

\[ u_\varepsilon \to u \quad \text{weak star in} \quad L^\infty(0, T; H^1_0(\Omega) \cap H^2(\Omega)), \]

\[ u'_\varepsilon \to u' \quad \text{weak star in} \quad L^\infty(0, T; L^2(\Omega)), \]

\[ u''_\varepsilon \to u'' \quad \text{weak star in} \quad L^\infty(0, T; L^2(\Omega)), \]

\[ \beta(u'_\varepsilon) \to \beta(u') \quad \text{weak in} \quad L^{4/3}(0, T; W^{-2,4/3}(\Omega)), \]

\[ \beta(\theta_\varepsilon) \to \beta(\theta) \quad \text{weak in} \quad L^{4/3}(0, T; W^{-2,4/3}(\Omega)), \]

\[ u'_\varepsilon \to u' \quad \text{weak in} \quad L^4(0, T; W^{0,4}_0(\Omega)), \]

\[ \theta_\varepsilon \to \theta \quad \text{weak in} \quad L^4(0, T; W^{0,4}_0(\Omega)). \]

By the compactness theorem of Aubin-Lions [8], we obtain

\[ u_\varepsilon \to u \quad \text{strongly} \quad L^2(0, T; H^1_0(\Omega)), \]

\[ u'_\varepsilon \to u' \quad \text{strongly} \quad L^2(0, T; L^2(\Omega)). \]

We observe that

\[ (u''_\varepsilon(t), v(t)) + M[||u_\varepsilon(t)||^2][(u_\varepsilon(t), v(t))] + (\theta_\varepsilon(t), v(t)) + \]

\[ \frac{1}{\varepsilon} \langle \beta(u'_\varepsilon(t)), v(t) \rangle = (f(t), v(t)) \],

\[ (\theta'_\varepsilon(t), v(t)) + ((\theta_\varepsilon(t), v(t))) + (u'_\varepsilon(t), v(t)) + \frac{1}{\varepsilon} \langle \beta(\theta_\varepsilon(t)), v(t) \rangle = (g(t), v(t)). \]
is true for all \( v \in L^4(0, T; W^{2,4}_0(\Omega)) \).

On the other hand, being \( u'_\varepsilon, \theta_\varepsilon \in L^4(0, T; W^{2,4}_0(\Omega)) \) implies

\[
(u''_\varepsilon(t), u'_\varepsilon(t)) + M(\|u_\varepsilon(t)\|^2)((u_\varepsilon(t), u'_\varepsilon(t))) + (\theta_\varepsilon(t), u'_\varepsilon(t)) + \frac{1}{\varepsilon} \langle \beta(u'_\varepsilon(t)), u'_\varepsilon(t) \rangle = (f(t), u'_\varepsilon(t)),
\]

\[
(\theta'_\varepsilon(t), \theta_\varepsilon(t)) + ((\theta_\varepsilon(t), \theta_\varepsilon(t))) + (\theta'_\varepsilon(t), \theta_\varepsilon(t)) + \frac{1}{\varepsilon} \langle \beta(\theta_\varepsilon(t)), \theta_\varepsilon(t) \rangle = (g(t), \theta_\varepsilon(t)).
\]

Subtracting the equations of the system above, we obtain

\[
(u''_\varepsilon(t), v(t) - u'_\varepsilon(t)) + M(\|u_\varepsilon(t)\|^2)\big((u_\varepsilon(t), v(t) - u_\varepsilon(t))\big) + (\theta_\varepsilon(t), v(t) - u'_\varepsilon(t)) = (f(t), v(t) - u'_\varepsilon(t)),
\]

\[
(\theta'_\varepsilon(t), v(t) - \theta'_\varepsilon(t)) + ((\theta_\varepsilon(t), v(t) - \theta_\varepsilon(t))) + (\theta'_\varepsilon(t), v(t) - \theta_\varepsilon(t)) = (g(t), v(t) - \theta_\varepsilon(t)),
\]

for all \( v \in W^{2,4}_0(\Omega) \).

Let us consider \( v(t) \in K \) a.e. in \([0, T]\). Then we obtain \( \beta(v(t)) = 0 \) and being \( \beta \) a monotone operator, we have

\[
\langle \beta(u'_\varepsilon(t)) - \beta(v(t)), v(t) - u'_\varepsilon(t) \rangle \leq 0,
\]

\[
\langle \beta(\theta_\varepsilon(t)) - \beta(v(t)), v(t) - \theta_\varepsilon(t) \rangle \leq 0.
\]

Therefore,

\[
\int_0^T (u''_\varepsilon(t) - M(\|u_\varepsilon(t)\|^2)\Delta u_\varepsilon(t) + \theta_\varepsilon(t) - f(t), v(t) - u'_\varepsilon(t))dt \geq 0,
\]

\[
\int_0^T (\theta'_\varepsilon(t) - \Delta \theta_\varepsilon + u'_\varepsilon - g(t), v(t) - \theta_\varepsilon(t))dt \geq 0,
\]

for all \( v \in L^4(0, T; W^{2,4}_0(\Omega)) \) with \( v(t) \in K \) a.e. in \([0, T]\). Now, taking the limit in (3.6) and (3.7), when \( \varepsilon \to 0 \) and using (3.1)–(3.3) and observing that \( \Delta u_\varepsilon \to \Delta w \) weak in \( L^2(0, T; L^2(\Omega)) \) it follows that \( u, \theta \) satisfy (1.5) and (1.6) in Theorem 2.3.

To conclude the proof of the existence of a solution, we show that \( u'(t), \theta(t) \in K \) a.e. in \([0, T]\). In fact, by (2.33) and (2.34) we have

\[
\|\beta(u'_\varepsilon)\|_{L^\infty(0, T; W^{2,4}_0(\Omega))} \leq C\varepsilon,
\]

\[
\|\beta(\theta_\varepsilon)\|_{L^\infty(0, T; W^{2,4}_0(\Omega))} \leq C\varepsilon.
\]

Therefore, as \( \varepsilon \to 0 \), \( \beta(u'_\varepsilon) \to 0 \) and \( \beta(\theta_\varepsilon) \to 0 \) strong \( L^\infty(0, T; W^{2,4}_0(\Omega)) \).

On the other hand we have \( \beta(u'_\varepsilon) \to \beta(u') \) and \( \beta(\theta_\varepsilon) \to \beta(\theta) \) weak in \( L^{4/3}(0, T; W^{2,4/3}_0(\Omega)) \). Then, \( \beta(u'(t)) = \beta(\theta(t)) = 0 \) in \( L^\infty(0, T; W^{2,4/3}_0(\Omega)) \). Therefore, \( u'(t), \theta(t) \in K \) a.e. in \([0, T]\).

The initial conditions (1.7) can be verified easily. This concludes the proof of Theorem 2.3.
4 Uniqueness

For proving uniqueness of solutions in Theorem 2.3, we consider the restriction

\[ u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \quad u_0(x) \geq 0 \ \text{a.e. in} \ \Omega, \ \text{and} \ ||u_0|| > 0. \]

Consequently \ ||u(t)|| > 0, for all \ t \in [0,T]. \ In fact, if there exists \ t_0 \in [0,T] \ such that \ ||u(t)|| = 0, \ then

\[ \int_{\Omega} |u(x,t_0)|^2 dx \leq C||u(t_0)||^2 = 0, \]

where \ C \ is the constant of the embedding \ H_0^1(\Omega) \hookrightarrow L^2(\Omega). \ Therefore, \ u(x,t_0) = 0, \ a.e. \ in \ \Omega.

Since \ u'(t) \in K \ a.e. \ in \ [0,T], \ \text{we have} \ u'(t) \geq 0 \ a.e. \ in \ \Omega. \ \text{This implies that}

\[ u(x,t) \geq u(x,0) = u_0(x) \ \text{in} \ \Omega \ \text{a.e. in} \ [0,T]. \quad (4.1) \]

Being \ ||u_0|| > 0, \ there exists \ \Omega' \subset \Omega \ with \ ||\Omega'|| > 0 \ such that \ that \ u_0(x) > 0. \ By \ (3.1) \ it follows that \ u(x,t_0) > 0 \ in \ \Omega. \ This is a contradiction.

**Theorem 4.1** Under the hypotheses of Theorem 2.3, if

i) \ M(\lambda) > 0 \ for all \ \lambda > 0, \ and \ M(0) = 0.

ii) \ u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \ u_0(x) \geq 0 \ a.e. \ in \ \Omega, \ \text{and} \ ||u_0|| > 0,

Then the solution \ {u,\theta} \ of Theorem 2.3 is unique.

**Proof.** From (i) and (ii) it follows that

\[ m_0 = \min\{M(||u(t)||^2); \ t \in [0,T]\} > 0. \]

Suppose we have two pairs of solutions \ {u,\theta} \ and \ {w,\varphi} \ satisfying the conditions of Theorem 2.3. Let \ \Psi = u - w \ and \ \phi = \theta - \varphi. \ Thus, \ \Psi \ and \ \phi \ satisfy

\[ (\Psi''(t) - M(||u(t)||^2)\Delta\Psi(t) + \{M(||w(t)||^2) - M(||u(t)||^2)\}\Delta w + \phi(t), \Psi'(t)) \leq 0, \]

\[ (\phi'(t) - \Delta\phi(t) + \Psi'(t), \phi(t)) \leq 0, \]

which implies

\[ \frac{1}{2} \frac{d}{dt} \{||\Psi'(t)||^2 + ||\phi(t)||^2 + ||\phi(t)||^2\} + M(||u(t)||^2)\frac{d}{dt}||\Psi(t)||^2 + 2(\phi(t), \Psi'(t)) \leq \]

\[ \{M(||u(t)||^2) - M(||w(t)||^2)\}(\Delta w(t), \Psi'(t)). \]

Since

\[ M(||u(t)||^2)\frac{d}{dt}||\Psi(t)||^2 = \frac{d}{dt}\{M(||u(t)||^2)||\Psi(t)||^2\} - \frac{d}{dt}[M(||u(t)||^2)]||\Psi(t)||^2, \]

then
This completes the proof of uniqueness.

where

\[ C \parallel if follows that \]

which implies

\[ \| u(t) \|^2 + \| \phi(t) \|^2 + \| \Psi(t) \|^2 \leq \triangle \]

\[ M(\| u(t) \|^2) = M(\| w(t) \|^2) \triangle \Psi(t) \triangle M' \| u(t) \|^2 \triangle \Psi(t) \triangle (u'(t), u(t)) \| \Psi(t) \|^2. \]

Now, integrating this inequality form 0 to \( t < T \), we obtain

\[
\frac{1}{2} \left( \| \Psi'(t) \|^2 + |\phi(t)|^2 + \| \Psi(t) \|^2 \right) + \frac{d}{dt} \left[ M(\| u(t) \|^2) \right] \Psi(t) \right) \leq \left[ M(\| u(t) \|^2) - M(\| w(t) \|^2) \right] (\Delta w(t), \Psi(t)) + M' \| u(t) \|^2 (u'(t), u(t)) \| \Psi(t) \|^2.
\]

Note that \( \| u(t) \| \) and \( \| u'(t) \| \in L^\infty(0, T) \). Then there exists a positive constant \( C_0 \) such that

\[ \| u(t) \| \leq C_0 \quad \text{and} \quad \| u'(t) \| \leq C_0 \quad \text{a.e. in } [0, T]. \]

Since \( M \in C^1([0, \infty)), \) it follows \( |M'(\xi)| \leq C_1, \) for all \( \xi \in [0, C_0] \).

Now, by the Mean Value Theorem, for each \( s \in [0, T], \) there exists \( \xi_s \) between \( \| u(s) \|^2 \) and \( \| w(s) \|^2 \) such that

\[ |M(\| u(s) \|^2) - M(\| w(s) \|^2)| \leq C_1 \| u(s) \|^2 - \| w(s) \|^2 \leq C_2 \| u(s) - w(s) \| = C_2 \| \Psi(s) \|. \]

Observing that \( |\Delta w(s)| \leq C_3, \) from (4.2) and (4.3) we obtain that

\[
|\Psi'(t)|^2 + |\phi(t)|^2 + M(\| u(t) \|^2) \| \Psi(t) \|^2 \leq C_4 \int_0^t \{ |\Psi'(s)|^2 + \| \Psi(s) \|^2 + |\phi(s)|^2 \} ds,
\]

which implies

\[
|\Psi'(t)|^2 + |\phi(t)|^2 + \| \Psi(t) \|^2 \leq C_5 \int_0^t \{ |\Psi'(t)|^2 + \| \Psi(t) \|^2 + |\phi(t)|^2 \} ds.
\]

where \( C_5 = C_4 / \min \{1, m_0 \}. \) From the above inequality and Gronwall inequality if follows that \( |\phi(t)| = \| \Psi(t) \| = 0, \) i.e., \( \phi \) and \( \Psi \) are zero almost everywhere. This completes the proof of uniqueness.
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