Positive and monotone solutions of an m-point boundary-value problem *

Panos K. Palamides

Abstract

We study the second-order ordinary differential equation

$$y''(t) = -f(t, y(t), y'(t)), \quad 0 \leq t \leq 1,$$

subject to the multi-point boundary conditions

$$\alpha y(0) \pm \beta y'(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i).$$

We prove the existence of a positive solution (and monotone in some cases) under superlinear and/or sublinear growth rate in $f$. Our approach is based on an analysis of the corresponding vector field on the $(y, y')$ face-plane and on Kneser’s property for the solution’s funnel.

1 Introduction

Recently an increasing interest has been observed in investigating the existence of positive solutions of boundary-value problems. This interest comes from situations involving nonlinear elliptic problems in annular regions. Erbe and Tang [5] noted that, if the boundary-value problem

$$-\Delta u = F(|x|, u) \quad \text{in } R < |x| < \hat{R}$$

with

$$u = 0 \quad \text{for } |x| = R, \quad u = 0 \quad \text{for } |x| = \hat{R}; \quad \text{or}$$

$$u = 0 \quad \text{for } |x| = R, \quad \frac{\partial u}{\partial |x|} = 0 \quad \text{for } |x| = \hat{R}; \quad \text{or}$$

$$\frac{\partial u}{\partial |x|} = 0 \quad \text{for } |x| = R, \quad u = 0 \quad \text{for } |x| = \hat{R}$$

*Mathematics Subject Classifications: 34B10, 34B18, 34B15.

Key words: multipoint boundary value problems, positive monotone solution, vector field, sublinear, superlinear, Kneser’s property, solution’s funnel.

©2002 Southwest Texas State University.
is radially symmetric, then the boundary-value problem can be transformed into the scalar Sturm-Liouville problem

\[ x''(t) = -f(t, x(t)), \quad 0 \leq t \leq 1, \]  
\[ \alpha x(0) - \beta x'(0) = 0, \quad \gamma x(1) + \delta x'(1) = 0. \]  

(1.1) (1.2)

where \( \alpha, \beta, \gamma, \delta \) are positive constants.

By a positive solution of (1.1)-(1.2), we mean a function \( x(t) \) which is positive for \( 0 < t < 1 \) and satisfies the differential equation (1.1) with the boundary conditions (1.2).

Erbe and Wang [6] using Green’s functions and the Krasnoselskii’s fixed point theorem on cones proved the existence of a positive solution of (1.1)-(1.2), under the following assumptions:

(B1) The function \( f \) is continuous and positive on \([0, 1] \times [0, \infty)\) and

\[ f_0 := \lim_{y \to 0^+} \max_{0 \leq t \leq 1} \frac{f(t, y)}{y} = 0, \quad f_\infty := \lim_{y \to +\infty} \min_{0 \leq t \leq 1} \frac{f(t, y)}{y} = +\infty \]  

(1.3)

i.e. \( f \) is superlinear at both ends points \( x = 0 \) and \( x = \infty \); or

\[ f_0 := \lim_{y \to 0^+} \min_{0 \leq t \leq 1} \frac{f(t, y)}{y} = +\infty, \quad f_\infty := \lim_{y \to +\infty} \max_{0 \leq t \leq 1} \frac{f(t, y)}{y} = 0. \]  

(1.4)

i.e. \( f \) is sublinear at both \( x = 0 \) and \( x = \infty \).

(B2) \( \rho := \beta \gamma + \alpha \gamma + \alpha \delta > 0. \)

Also nonlinear boundary constraints have been studied, among others by Thompson [22] and by the author of this paper and Jackson [9]. There are common ingredients in these papers: an (assumed) Nagumo-type growth condition on the nonlinearity \( f \) or/and the presence of upper and lower solutions.

The multi-point boundary-value problem for second-order ordinary differential equations was initiated by Ilin and Moiseev [10, 11]. Gupta [14] studied the three-point boundary-value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multipoint boundary-value problems have been studied by several authors. Most of them used the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder, coincidence degree theory or a fixed-point theorem on cones. We refer the reader to [1, 8, 13, 20] for some recent results of nonlinear multipoint boundary-value problems.

Let \( \alpha_i \geq 0 \) for \( i = 1, \ldots, m - 2 \) and let \( \xi_i \) satisfy \( 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1 \). Ma [21] applied a fixed-point theorem on cones to prove the existence of a positive solution of

\[ u'' + a(t)f(u) = 0 \]
\[ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i) \]

under superlinearity or sublinearity assumptions on \( f \). He also assumed the following
(Γ1) \( a \in C([0,1], [0, \infty)) \), \( f \in C([0, \infty), [0, \infty)) \), and there exists \( t_0 \in [\xi_{m-2}, 1] \) such that \( a(t_0) > 0 \).

(Γ2) For \( i = 1, \ldots, m-2 \), \( a_i \geq 0 \) and \( \sum_{i=1}^{m-2} a_i \xi_i < 1 \).

Recently, Gupta [16] obtained existence results for the boundary-value problem

\[
y''(t) = f(t, y(t), y'(t)) + e(t), \quad 0 \leq t \leq 1
\]

\[
y(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} a_i y(\xi_i),
\]

by using the Leray-Schauder continuation theorem, under smallness assumptions of the form

\[
|f(t, y, y')| \leq p(t)|y| + q(t)|y'| + r(t) \quad \text{and} \quad C_1\|p(t)\| + C_2\|q(t)\| \leq 1,
\]

with \( p(t) \), \( q(t) \), \( r(t) \) and \( e(t) \) in \( L^1(0,1) \) and \( C_1 \) and \( C_2 \) constants.

In this paper, we consider the problem of existence of positive solutions for the \( m \)-point boundary-value problem

\[
y''(t) = -f(t, y(t), y'(t)), \quad 0 \leq t \leq 1
\]

\[
\alpha y(0) - \beta y'(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} a_i y(\xi_i).
\]

We assume \( \alpha > 0 \), \( \beta > 0 \), the function \( f \) is continuous, and

\[
f(t, y, y') \geq 0, \quad \text{for all} \ t \in [0,1], \ y \geq 0 \ y' \in \mathbb{R}.
\]

The presence of the third variable \( y' \) in the function \( f(t, y, y') \) causes some considerable difficulties, especially, in the case where an approach relies on a fixed point theorem on cones and the growth rate of \( f(t, y, y') \) is sublinear or superlinear. We overcome this predicament, by extending below the concept-assumptions (1.3) and (1.4) as follows:

Suppose that for any \( M > 0 \),

\[
\begin{align*}
f_{0,0} &:= \lim_{(y,y') \to (0,0)} \max_{0 \leq t \leq 1} \frac{f(t, y, y')}{y} = 0, \\
f_{+\infty} &:= \lim_{y \to +\infty} \min_{0 \leq t \leq 1} \frac{f(t, y, y')}{y} = +\infty, \quad \text{for} \ |y'| \leq M.
\end{align*}
\]

i.e. \( f \) is jointly superlinear at the end point \( (0,0) \) and uniformly superlinear at \( +\infty \). Similarly

\[
\begin{align*}
f_{0} &:= \lim_{y \to 0+} \min_{0 \leq t \leq 1} \frac{f(t, y, y')}{y} = +\infty, \quad \text{for} \ |y'| \leq M, \\
f_{+\infty, +\infty} &:= \lim_{(y,y') \to (+\infty, +\infty)} \max_{0 \leq t \leq 1} \frac{f(t, y, y')}{y} = 0.
\end{align*}
\]
Positive and monotone solutions

i.e. \( f \) is jointly sublinear at \((+\infty, +\infty)\) and uniformly sublinear at 0.

Furthermore there exist \( \bar{l} \in (0, \infty] \), such that for every \( \bar{M} > 0 \)
\[
\lim_{y' \to -\infty} \max_{0 \leq t \leq 1} \frac{f(t, y, y')}{y'} = -\bar{l}, \quad \text{for } y \in [0, \bar{M}]
\] (1.10)
i.e. \( f(t, y) \) is linear or superlinear at \(-\infty\) and for every \( \bar{\eta} > 0 \)
\[
\lim_{y' \to 0} \min_{0 \leq t \leq 1} \frac{f(t, y, y')}{y'} = 0, \quad \text{for } y \in (0, \bar{\eta}).
\] (1.11)
i.e. \( f(t, y) \) is superlinear at 0.

Remark 1.1 Note that the differential equation (1.5) defines a vector field
whose properties will be crucial for our study. More specifically, we look at the
\((y, y')\) face semi-plane \((y > 0)\). From the sign condition on \( f \) (see assumption
(1.7)), we immediately see that \( y'' < 0 \). Thus any trajectory \((y(t), y'(t))\), \( t \geq 0 \),
emanating from the semi-line
\[
E_0 := \{(y, y') : \alpha y - \beta y' = 0, \ y > 0\}
\]
“trends” in a natural way, (when \( y'(t) > 0 \)) toward the positive \( y\)-semi-axis
and then (when \( y'(t) < 0 \)) trends toward the negative \( y'\)-semi-axis. Lastly, by
setting a certain growth rate on \( f \) (say superlinearity) we can control the vector
field, so that some trajectory satisfies the given boundary condition
\[
y(1) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i)
\]
at the time \( t = 1 \). These properties will be referred as “The nature of the vector
field” throughout the rest of paper.

So the technique presented here is different to that given in the above men-
tioned papers [16, 6, 3, 13, 5], but it is closely related with those in [9, 21].
Actually, we rely on the above “nature of the vector field” and on the simple
shooting method. Finally, for completeness we refer to the well-known Kneser’s
theorem (see for example Copel’s text-book [2]).

Theorem 1.2 Consider the system
\[
x'' = f(t, x, x'), \quad (t, x, x') \in \Omega := [\alpha, \beta] \times \mathbb{R}^{2n},
\] (1.12)
with the function \( f \) continuous. Let \( \hat{E}_0 \) be a continuum (compact and connected)
set in \( \Omega_0 := \{(t, x, x') \in \Omega : t = \alpha\} \) and let \( \mathcal{X}(\hat{E}_0) \) be the family of all solutions
of (1.12) emanating from \( \hat{E}_0 \). If any solution \( x \in \mathcal{X}(\hat{E}_0) \) is defined on the
interval \([\alpha, \tau]\), then the set (cross-section at the point \( \tau \))
\[
\mathcal{X}(\tau; \hat{E}_0) := \{(x(\tau), x'(\tau)) : x \in \mathcal{X}(\hat{E}_0)\}
\]
is a continuum in \( \mathbb{R}^{2n} \).
Now consider (1.5)-(1.6) with the following notation.

\[ \sigma := \sum_{i=1}^{m-2} \alpha_i \xi_i < 1, \quad \sigma^* := \sum_{i=1}^{m-2} \alpha_i \xi_i + \frac{\beta}{\alpha} \left\{ \sum_{i=1}^{m-2} \alpha_i \frac{\xi_i^2}{\xi_i^2} - 1 \right\} < 1, \]

\[ K_0 := \max \left\{ \frac{2\alpha}{\beta}, 2\left[ \frac{\alpha + \beta}{\beta} - \frac{\sigma}{\xi_{m-2}} \right] \right\}, \]

\[ \mu_0 := \min \left\{ (1 - m^*)^\frac{\alpha}{\beta}, 2 \left[ \frac{\varepsilon(\alpha + \beta)}{\beta} - 1 \right] \right\} \]

where \( \beta/(\alpha + \beta) < \varepsilon < 1 \) and \( \sigma^* < m^* < 1 \).

So by (1.10), for any \( \bar{K} \in (0, \bar{k}) \) there exists \( H > 0 \) such that

\[ \min_{0 \leq t \leq 1} f(t, y, y') > -\bar{K} y', \quad 0 \leq y \leq H (1 + \frac{\alpha}{\beta}) \quad \text{and} \quad y' < -H. \quad (1.13) \]

By the superlinearity of \( f(t, y, y') \) at \( y = +\infty \) (see condition (1.8)), for any \( K^* > K_0 \) there exists \( H^* > H \) such that

\[ \min_{0 \leq t \leq 1} f(t, y, y') > K^* y, \quad y \geq H^* \quad \text{and} \quad -2H \leq y' \leq \frac{\alpha}{\beta} H. \quad (1.14) \]

Similarly by the superlinearity of \( f(t, y, y') \) at \( (0, 0) \), for any \( 0 < \mu^* < \mu_0 \) there is an \( \eta^* > 0 \) such that

\[ 0 < y \leq \eta^* \text{ and } 0 < y' \leq \frac{\alpha}{\alpha + \beta} \eta^* \Rightarrow \max_{0 \leq t \leq 1} f(t, y, y') \leq \mu^* y. \quad (1.15) \]

Also consider the rectangle

\[ D := \left[ 0, (1 + \frac{\alpha}{\beta}) H \right] \times \left[ -2H, \frac{\alpha}{\beta} H \right] \]

and define a bounded continuous modification \( F \) of \( f \) such that

\[ F(t, y, y') = f(t, y, y'), \quad (t, y, y') \in [0, 1] \times D. \]

2. An \( m \)-point boundary-value Problem

We consider now the boundary-value problem

\[ y'' + F(t, y, y') = 0. \]

\[ \alpha y(0) - \beta y'(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) \quad (2.1) \]

Theorem 2.1 Assume that (1.7) holds and

\[ \sigma^* = \sum_{i=1}^{m-2} \alpha_i \xi_i + \frac{\beta}{\alpha} \left\{ \sum_{i=1}^{m-2} \alpha_i \frac{\xi_i}{\xi_i} - 1 \right\} < 1. \quad (2.2) \]

Then the boundary-value problem (1.5)-(1.6) has a positive solution provided that:
• The function $f$ is superlinear (see (1.8)) along with (1.10), or
• The function $f$ is sublinear (see (1.9)), (1.11) holds and in addition,
\[ \left( \sum_{i=1}^{m-2} \alpha_i \xi_i \right) \left[ \frac{1}{2 \xi_{m-2}} + \frac{\alpha}{2 \beta} \right] > 1. \] \hfill (2.3)

Furthermore, there exists a positive number $H$ such that
\[ 0 < y(t) \leq H \quad \text{and} \quad -2H \leq y'(t) \leq \frac{\alpha}{\beta} H, \quad 0 \leq t \leq 1, \]
for any such solution.

**Proof 1)** Superlinear case. Since $f_\infty = +\infty$ and in view of (1.14), for any $K^* > K > K_0$ there exists $H^* > H > 0$ such that
\[ \min_{0 \leq t \leq 1} f(t, y(t), y'(t)) > Ky, \quad y \geq H \quad \text{and} \quad \frac{\alpha}{\beta} H \geq y' \geq -2H. \] \hfill (2.4)

Consider the function
\[ W(P) := \sum_{i=1}^{m-2} \alpha_i y(\xi_i) - y(1), \]
where $y \in X(P_1)$ is any solution of differential equation (2.1) starting at the point $P_1 := (y_1, y'_1) \in E_0$ with $y_1 = H$.

By the assumption (1.7) (i.e. the nature of the vector field, see Remark 1.1) it is obvious that $y(t) \geq y_1 = H$ and $y'(t) \leq y'_1 = \frac{\alpha}{\beta} y_1 = \frac{\alpha}{\beta} H$, for all $t$ in a sufficiently small neighborhood of $t = 0$.

Let’s suppose that there is $t^* \in (0, 1]$ such that
\[ y(t) \geq H, \quad -2H \leq y'(t) \leq \frac{\alpha}{\beta} H, \quad 0 \leq t < t^* \quad \text{and} \quad y(t^*) = H \]
or
\[ y(t) \geq H, \quad -2H \leq y'(t) \leq \frac{\alpha}{\beta} H, \quad 0 \leq t < t^* \quad \text{and} \quad y'(t^*) = -2H. \]

Consider first the case: $y(t^*) = H$. Then since $P_1 \in E_0$, by the Taylor’s formula we get $t \in [0, t^*]$ such that
\[ H = y(t^*) \leq H \left[ 1 + \frac{\alpha}{\beta} - \frac{1}{2} f(t, y(t), y'(t)) \right] \] \hfill (2.5)

and thus
\[ H \frac{2\alpha}{\beta} \geq f(t, y(t), y'(t)). \]

But since $y(t) \geq H$ and $-2H \leq y'(t) \leq \frac{\alpha}{\beta} H$, $0 \leq t < t^*$ by (2.4), we have
\[ f(t, y(t), y'(t)) \geq \min_{0 \leq r \leq 1} f(t, y(t), y'(t)) \geq Ky(t) \geq KH \]
and so we obtain $H2\alpha/\beta \geq KH$ contrary to the choice $K > \frac{2\alpha}{\beta}$. Furthermore, by (2.5),

$$H \leq y(t) < H\left[1 + \frac{\alpha}{\beta}\right], \quad 0 \leq t \leq 1. \quad (2.6)$$

We recall also (see (1.13)) that for any $\varepsilon^* \in (1, \min\{2, 1 + \bar{l}\})$ there exists $\bar{K} \in (\varepsilon^* - 1, \bar{l})$ such that

$$\min_{0 \leq t \leq 1} f(t, y, y') > -\bar{K}y', \quad 0 \leq y \leq H\left[1 + \frac{\alpha}{\beta}\right] \text{ and } y' < -H. \quad (2.7)$$

We shall prove that

$$\frac{\alpha}{\beta}H \geq y'(t) \geq -\varepsilon^*H > -2H, \quad 0 \leq t \leq 1. \quad (2.8)$$

Indeed, since $y'(t)$ is decreasing on $[0, 1]$, let’s assume that there exist $t_0, t_1 \in (0, 1)$ such that

$$y'(t_0) = -H, \quad -\varepsilon^*H < y'(t) < -H, \quad t_0 \leq t \leq t_1 \quad \text{and} \quad y'(t_1) = -\varepsilon^*H$$

Then by (2.6)-(2.8), for some $\bar{t} \in (t_0, t_1)$, we get

$$-\varepsilon^*H = y'(t_1) - y'(t_0) - f(\bar{t}, y(\bar{t}), y'(\bar{t}))$$

$$\leq -H + \bar{K}y'(t_1) \leq -H + \bar{K}y'(t_0)$$

$$= -H - \bar{K}H,$$

Thus we get another contradiction $\bar{K} \leq \varepsilon^* - 1$. On the other hand by the concavity of the solution $y \in \mathcal{X}(P_1)$ (due to the assumption (1.7)), we know that the function $y(\xi)/\xi, \; 0 < \xi \leq 1$ is decreasing and so

$$\frac{y(\xi)}{\xi} \geq \frac{y(\xi_{m-2})}{\xi_{m-2}}, \quad i = 1, 2, \ldots, m - 2. \quad (2.9)$$

Thus in view of (2.6)

$$W(P_1) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) - y(1) = \sum_{i=1}^{m-2} \alpha_i \xi_i \frac{y(\xi_i)}{\xi_i} - y(1)$$

$$\geq \left[\sum_{i=1}^{m-2} \alpha_i \xi_i\right] \frac{H}{\xi_{m-2}} - y(1) = \sigma \frac{H}{\xi_{m-2}} - y(1).$$

where we recall that $\sigma = \sum_{i=1}^{m-2} \alpha_i \xi_i < 1$. Consequently by Taylor’s formula,

$$W(P_1) \geq \frac{\sigma}{\xi_{m-2}} H - (y_1 + \frac{\alpha}{\beta} y_1 - \frac{1}{2} f(t^*, y(t^*), y'(t^*)))$$

Thus by (2.4), (2.6) and (2.8), we get

$$W(P_1) \geq \frac{\sigma}{\xi_{m-2}} H - \left[1 + \frac{\alpha}{\beta}\right]H + \frac{1}{2}K y(t^*)$$

$$\geq \frac{\sigma}{\xi_{m-2}} H - \left[1 + \frac{\alpha}{\beta}\right]H + \frac{1}{2}K H.$$
In this way we get
\[ W(P_1) \geq 0, \quad (2.10) \]
since by the choice of \( K \) at (2.4), we have
\[ K > 2\left[1 + \frac{\alpha}{\beta} - \frac{\sigma}{\xi_{m-2}}\right]. \]
Similarly by the superlinearity of \( f(t, y, y') \) at \((0, 0)\), for any \( \mu > 0 \) there is an \( \eta > 0 \) such that
\[ 0 < y \leq \eta \quad \text{and} \quad 0 \leq y' \leq \frac{2\alpha \varepsilon}{\beta} \eta \quad \text{imply} \quad \max_{0 \leq t \leq 1} f(t, y, y') < \mu y, \quad (2.11) \]
where \( \frac{\beta}{\alpha + \beta} < \varepsilon < 1 \). We choose now (see (1.15))
\[ \mu^* \leq \mu < \mu_0 = \min \left\{ \left(1 - m^*\right) \frac{\varepsilon \alpha}{\beta}, \ 2\left[\frac{\varepsilon(\alpha + \beta)}{\beta} - 1\right]\right\} \quad (2.12) \]
and then clearly \( \eta \geq \eta^* \).

Let now \( y \in X(P_0) \) be a solution of differential equation (2.1) starting at the point \( P_0 := (y_0, y_0') \in E_0 \) with \( y_0 = \varepsilon \eta \). We shall show that
\[ \varepsilon \eta \leq y(t) \leq \eta \quad \text{and} \quad m^* \frac{\alpha \varepsilon}{\beta} \eta \leq y'(t) \leq \frac{\alpha \varepsilon}{\beta} \eta, \quad 0 \leq t \leq 1, \quad (2.13) \]
where we recall that
\[ \sigma^* = \sum_{i=1}^{m-2} \alpha_i \xi_i + \frac{\beta}{\alpha} \left\{ \sum_{i=1}^{m-2} \alpha_i \xi_i \xi_1 - 1 \right\} < m^* < 1. \]
Indeed, if there is a least \( t^* \in (0, 1] \) such that \( m^* \frac{\alpha \varepsilon}{\beta} \eta = y'(t^*) \), and
\[ \varepsilon \eta \leq y(t) \leq \eta \quad \text{and} \quad m^* \frac{\alpha \varepsilon}{\beta} \eta \leq y'(t) \leq \frac{\alpha \varepsilon}{\beta} \eta, \quad 0 \leq t < t^*, \]
then again by Taylor’s formula,
\[ m^* \frac{\alpha \varepsilon}{\beta} \eta = y'(t^*) = y_0 \frac{\alpha}{\beta} - \int_0^{t^*} f(t, y(t), y'(t)) \geq y_0 \frac{\alpha}{\beta} - \mu y(t) \geq \varepsilon \eta \frac{\alpha}{\beta} - \mu \eta, \]
and hence we obtain the contradiction \( \mu \geq \left(1 - m^*\right) \frac{\varepsilon \alpha}{\beta} \), due to the choice of \( \mu \) at (2.12). Similarly we may prove the first inequality of (2.13).

Consider once again the function
\[ W(P) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) - y(1) \]
and then by (2.9),

$$W(P_0) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) - y(1) = \sum_{i=1}^{m-2} \alpha_i \frac{y(\xi_i)}{\xi_i} - y(1)$$

$$\leq \left[ \sum_{i=1}^{m-2} \alpha_i \frac{y(\xi_1)}{\xi_1} - y(1) \right].$$

(2.14)

Now in view of (2.13),

$$\frac{y(\xi_1)}{\xi_1} = \frac{1}{\xi_1} \left( y(0) + \int_0^{\xi_1} y'(s)ds \right) \leq \frac{\varepsilon \eta}{\xi_1} + \frac{\alpha \varepsilon}{\beta} \eta$$

and

$$y(1) = y(0) + \int_0^1 y'(s)ds \geq \varepsilon \eta + m^* \frac{\alpha \varepsilon}{\beta} \eta.$$

Consequently by (2.14),

$$W(P_0) \leq \left[ \sum_{i=1}^{m-2} \alpha_i \xi_i \right] \left( \frac{\alpha \varepsilon}{\beta} \eta + \frac{\varepsilon \eta}{\xi_1} \right) - m^* \frac{\alpha \varepsilon}{\beta} \eta - \varepsilon \eta$$

$$= \left( \sum_{i=1}^{m-2} \alpha_i \xi_i - m^* \right) \frac{\alpha \varepsilon}{\beta} \eta + \left\{ \sum_{i=1}^{m-2} \alpha_i \xi_i - 1 \right\} \eta \leq 0$$

(2.15)

due to the choice of $m^ > \sum_{i=1}^{m-2} \alpha_i \xi_i + \frac{\beta}{\alpha} \left( \sum_{i=1}^{m-2} \alpha_i \xi_i - 1 \right)$.

It is now clear that the function $W = W(P), P \in [P_0, P_1]$ is continuous and thus by the Kneser’s property (see Theorem 1.2), (2.10) and (2.15), we get a point $P \in [P_0, P_1]$ (we chose the last one to the “left” of $P_1$) such that $W(P) = 0$. This fact clearly means that there is a solution $y \in \mathcal{X}(P)$ of equation (2.1), such that

$$W(P) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) - y(1) = 0.$$

It remains to be proved that the so obtaining solution $y = y(t)$ is actually a bounded function. Indeed, by the choice of $P$, the continuity of $y(t)$ with respect initial values, (2.10) and (2.15), it follows that

$$y(t) > 0, \quad 0 \leq t \leq 1,$$

because if

$$y(t) > 0, \quad 0 \leq t < 1 \quad \text{and} \quad y(1) = 0,$$

then $W(P) > 0$. Moreover by the nature of the vector field (see Remark 1.1), there is $t_p \in (0, 1)$ such that the so obtaining solution $y \in \mathcal{X}(P)$ is strictly increasing on $[0, t_p]$, strictly decreasing on $[t_p, 1]$ and further is strictly positive on $[0, 1]$. Also it holds $y(t) \leq H, \quad 0 \leq t \leq 1$, i.e.

$$0 < y(t) \leq H, \quad 0 \leq t \leq 1.$$  

(2.16)
Indeed, let’s assume that there exist \( t_0, t_1 \in [0, 1] \) such that
\[
y(t) \leq H, \quad 0 \leq t < t_0, \quad y(t_0) = H, \quad y(t) \geq H, \quad y'(t) \geq 0, \quad t_0 \leq t \leq t_1.
\]
Then we have \( 0 < y'(t_0) < \frac{\alpha}{\beta}y(t_0) \leq \frac{\alpha}{\beta}H \) and further by (2.4), for some \( \bar{t} \in (t_0, t_1) \)
\[
H \leq y(t_1) = y(t_0) + (t_1 - t_0)y'(t_0) - \frac{1}{2}f(\bar{t}, y(\bar{t}), y'(\bar{t}))
\leq H\left[1 + \frac{\alpha}{\beta}\right] - \frac{K}{2}y(\bar{t})
\leq H\left[1 + \frac{\alpha}{\beta}\right] - \frac{K}{2}H.
\]
Thus we get the contradiction \( K < \frac{2\alpha}{\beta} \). Also by assumption (1.10), we may show (exactly as at (2.8)) that the above solution \( y \in X(P) \) implies further the inequalities
\[
\frac{\alpha}{\beta}H \geq y'(t) \geq -\varepsilon^* H \geq 2H, \quad 0 \leq t \leq 1.
\]
(2.17)
and hence by (2.16) and the definition of the modification \( F \), the obtaining solution of (2.1) is actually a solution of the original equation (1.5).

2) \textit{Sublinear case.} We choose \( \varepsilon_0 > \frac{\alpha + \beta}{\beta} \) and recall that
\[
\sum_{i=1}^{m-2} \alpha_i y(\xi_i) + \frac{\beta}{\alpha} \left\{ \sum_{i=1}^{m-2} \alpha_i \frac{\xi_i}{\xi_1} - 1 \right\} < m^* < 1.
\]
Since \( f_{+\infty, +\infty} = 0 \), for \( \mu = \min \left\{ (1 - m^*) \frac{\alpha}{m^* \beta}, \frac{2}{\varepsilon_0^2} \varepsilon_0 - \frac{\alpha + \beta}{\beta} \right\} \), there exists \( H > 0 \) such that
\[
\max_{0 \leq t \leq 1} f(t, y, y') < \mu y, \quad y \geq H, \quad \text{and} \quad \frac{\alpha}{\beta}H \geq y' \geq m^* \frac{\alpha}{\beta}H.
\]
(2.18)
Let’s consider a point \( P_0 := (y_0, y_0') \in E_0 \) with \( y_0 = H \). We will prove first that for any solution \( y \in X(P_0) \),
\[
H \leq y(t) \leq \varepsilon_0 H \quad \text{and} \quad \frac{m^* \alpha}{\beta}H \leq y'(t) \leq \frac{\alpha}{\beta}H, \quad 0 \leq t \leq 1.
\]
(2.19)
Let us suppose that this is not the case. Then by the assumption (1.7), there is \( t^* \in [0, 1] \) such that
\[
H \leq y(t) \leq \varepsilon_0 H, \quad \frac{m^* \alpha}{\beta}H \leq y'(t) \leq \frac{\alpha}{\beta}H, \quad 0 < t < t^*,
\]
and \( y(t^*) = \varepsilon_0 H \) or \( y'(t^*) = \frac{m^* \alpha}{\beta}H \).
(2.20)
Assume that $y(t^*) = \varepsilon_0 H$. Then by the Taylor’s formula, (2.18) and (2.20) we obtain $t \in [0, t^*]$ such that

$$\varepsilon_0 H = y(t^*) = y_0 [1 + \frac{\alpha}{\beta}] - \frac{1}{2} f(t, y(t), y'(t))$$

$$< H [1 + \frac{\alpha}{\beta}] + \frac{1}{2} \mu y(t) \leq H [1 + \frac{\alpha}{\beta}] + \frac{1}{2} \mu \varepsilon_0 H$$

and hence it contradicts

$$\mu < \frac{2}{\varepsilon_0} [\varepsilon_0 - \frac{\alpha + \beta}{\beta}].$$

Let’s suppose now that $y'(t^*) = m^* \frac{\alpha}{\beta} H$. Then again by (2.18) and (2.20), we obtain

$$m^* \frac{\alpha}{\beta} H = y'(t^*) = y'_0 - f(t, y(t), y'(t)) \geq \frac{\alpha}{\beta} H - \mu y(t) \geq \frac{\alpha}{\beta} H - \mu \varepsilon_0 H,$$

which contradicts $\mu < (1 - m^*) \frac{\alpha}{\varepsilon_0 \beta}$.

Consider the function $W(P)$. Then

$$W(P) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) = \sum_{i=1}^{m-2} \alpha_i \frac{y(\xi_i)}{\xi_i} - y(1)$$

$$\leq \left[ \sum_{i=1}^{m-2} \alpha_i \frac{y(\xi_i)}{\xi_i} - y(1) \right]$$

and so by the second inequality of (2.19) (see also (2.15)), we get

$$W(P) \leq \left[ \sum_{i=1}^{m-2} \alpha_i \xi_i \right] \left( \frac{\alpha}{\beta} H + \frac{H}{\xi_1} \right) - m^* \frac{\alpha}{\beta} H - H$$

$$= \left( \sum_{i=1}^{m-2} \alpha_i \xi_i - m^* \right) \frac{\alpha}{\beta} H + \left( \sum_{i=1}^{m-2} \alpha_i \frac{\xi_i}{\xi_1} - 1 \right) H \leq 0$$

(2.21)

due to the fact that $m^* > \sum_{i=1}^{m-2} \alpha_i \xi_i + \frac{\alpha}{\beta} \left( \sum_{i=1}^{m-2} \alpha_i \xi_i - 1 \right)$.

On the other hand, since $f_0 = +\infty$, for any $K > \max \left\{ \frac{2(\alpha - \beta)}{\beta}, \frac{2\alpha}{\beta} \right\}$ there exist $\eta \in (0, H)$ such that

$$\min_{0 \leq t \leq 1} f(t, y, y') > K y, \quad 0 < y \leq \eta \quad \text{and} \quad -\eta \leq y' \leq \frac{\alpha \eta}{\beta}.$$

(2.22)

Consider a point $P_1 := (y_1, y'_1) \in E_0$ with $y_1 = \frac{\eta}{2}$ and any $y \in X(P_1)$. As above, by Taylor’s formula, (2.22) and the choice $K > \max \left\{ \frac{2(\alpha - \beta)}{\beta}, \frac{2\alpha}{\beta} \right\}$ we can easily prove that

$$\frac{\eta}{2} \leq y(t) \leq \eta, \quad 0 \leq t \leq 1.$$  

(2.23)
We choose now $\varepsilon_0^* \in (1, 2)$ and then by Assumption (1.11), there exist $\bar{\eta}_0 \in (0, \eta)$ and

$$0 < K* < \min \left\{ \frac{\varepsilon_0^* - 1}{\varepsilon_0^*}, \min\{1, \frac{2\beta}{\alpha}\}, \left[ \frac{1}{2} \frac{\sigma}{\xi_m - 2} \right]^{-1} \right\}$$

such that

$$\max_{0 \leq t \leq 1} f(t, y, y') < K*|y'|, \quad \frac{\eta}{2} \leq y \leq \eta, \quad \text{and} \quad -2\bar{\eta}_0 \leq y' < \frac{\alpha \eta}{2\beta}.$$  (2.25)

Besides (2.23) we shall prove that

$$\frac{\alpha \eta}{\beta} \geq y'(t) \geq -\bar{\eta}_0 > -\eta, \quad 0 \leq t \leq 1.$$  (2.26)

Indeed since $y'(t)$ is decreasing on $[0, 1]$ and $\varepsilon_0^* \in (1, 2)$ is arbitrary, let’s assume that there exist $t_0, t_1 \in [0, 1]$ such that $y'(t_0) = -\bar{\eta}_0$,

$$-2\bar{\eta}_0 < -\varepsilon_0^*\bar{\eta}_0 \leq y'(t) \leq -\bar{\eta}_0, \quad t_0 \leq t < t_1, \quad \text{and} \quad y'(t_1) = -\varepsilon_0^*\bar{\eta}_0.$$

Thus by (2.23)-(2.25), we have for some $\ell \in (t_0, t_1)$

$$-\varepsilon_0^*\bar{\eta}_0 = y'(t_1) = y'(t_0) - f(\bar{\ell}, y(\bar{\ell}), y'(\bar{\ell})) \geq -\bar{\eta}_0 + K^*y'(\bar{\ell}) \geq -\bar{\eta}_0 - K^*\varepsilon_0^*\bar{\eta}_0,$$

and so, we get another contradiction $K^* \geq (\varepsilon_0^* - 1)/\varepsilon_0^*$, due to (2.24).

Now as above (see (2.9) and (2.21)), we have

$$W(P_1) \geq \left[ \sum_{i=1}^{m-2} \alpha_i \xi_i \right] \frac{y(\xi_{m-2})}{\xi_{m-2}} - y(1) = \frac{y(\xi_{m-2})}{\xi_{m-2}} - y(1).$$

Consequently by (2.23) and the Taylor’s formula,

$$W(P_1) \geq \frac{\sigma}{\xi_{m-2}} \left( y_1 + \frac{\alpha}{\beta} y_1 \xi_m - \xi_{m-2} f(\bar{\ell}, y(\bar{\ell}), y'(\bar{\ell})) \right) - \eta - \eta$$

$$\geq \frac{\sigma}{\xi_{m-2}} \frac{\eta}{2} + \frac{\sigma}{\beta} - \frac{1}{2} \sigma \xi_{m-2} f(\bar{\ell}, y(\bar{\ell}), y'(\bar{\ell})) - \eta$$

Thus by (2.23) and (2.26), we get

$$W(P_1) \geq \frac{\sigma}{\xi_{m-2}} \frac{\eta}{2} + \frac{\sigma}{\beta} \frac{\eta}{2} - \frac{\sigma \xi_{m-2}}{2} K^*|y'| - \eta$$

$$\geq \frac{\sigma}{\xi_{m-2}} \frac{\eta}{2} + \frac{\sigma}{\beta} \frac{\eta}{2} - \frac{\sigma \xi_{m-2}}{2} K^* \hat{\eta} - \eta,$$

where $\hat{\eta} := \max\{\eta, \frac{\eta \alpha \eta}{2\beta} \}$. In this way, by the assumption (2.3) and the choice of $K^*$ at (2.24), we get

$$W(P_1) \geq 0.$$

Thus as at the superlinear case, we obtain a point $P \in [P_0, P_1]$ such that $W(P) = 0$ and this clearly completes the proof.
Remark 2.2 By the choice of \( m^* \in \left( \sum_{i=1}^{m-2} \alpha_i \xi_i + \frac{\beta}{2} \left( \sum_{i=1}^{m-2} \alpha_i \xi_i - 1 \right) \right) \) and following Ma [21], we may easily show that for

\[
\sum_{i=1}^{m-2} \alpha_i \xi_i \geq 1,
\]

there is not (positive) solution \( y \in X(P) \) of the BVP (1.5)-(1.6). Indeed, if there is one, then

\[
y(1) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) = \sum_{i=1}^{m-2} \alpha_i \xi_i y(\xi_i) \geq \sum_{i=1}^{m-2} \alpha_i \xi_i \frac{y(\xi^*)}{\xi^*} > \frac{y(\xi^*)}{\xi^*},
\]

where clearly \( \xi^* = \xi_{m-2} \) and this contradicts the concavity of the solution \( y = y(t) \). Furthermore we must seek the monotone (obviously increasing) solutions of (1.5)-(1.6), only for the case \( \sum_{i=1}^{m-2} \alpha_i \geq 1 \), since otherwise we get

\[
0 = W(P) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) - y(1) < \left[ \sum_{i=1}^{m-2} \alpha_i - 1 \right] y(1) < 0.
\]

The question of existence of such a monotone solution remains open. However we can obtain a strictly decreasing solution for the boundary-value problem

\[
\begin{align*}
y''(t) &= -f(t, y(t), y'(t)), \quad 0 \leq t \leq 1, \\
\alpha y(0) + \beta y'(0) &= 0, \\
y(1) &= \sum_{i=1}^{m-2} \alpha_i y(\xi_i).
\end{align*}
\tag{2.27}
\]

where \( \alpha \geq 0 \) and \( \beta > 0 \).

Remark 2.3 Suppose that the concept of jointly sublinearity is modified to

\[
\begin{align*}
f_0 &:= \lim_{y \to 0^+} \min_{0 \leq t \leq 1} \frac{f(t, y, y')}{y} = +\infty, \quad \text{for } |y'| \leq M, \\
f_{\infty, -\infty} &:= \lim_{(y, y') \to (+\infty, -\infty)} \max_{0 \leq t \leq 1} \frac{f(t, y, y')}{y} = 0.
\end{align*}
\tag{2.28}
\]

Then, following almost the same line as above (under the obvious modifications) we may prove the next theorem.

Theorem 2.4 Assume that (1.7) holds and further

\[
\sigma^* = \sum_{i=1}^{m-2} \alpha_i \xi_i + \frac{\beta}{\alpha} \left\{ \sum_{i=1}^{m-2} \alpha_i \xi_i - 1 \right\} < 1.
\]

Then the boundary-value problem (2.27) has a positive strictly decreasing solution provided that:
• The function $f$ is superlinear (see (1.8)) along with (1.10), or
• The function $f$ is sublinear (see (2.28)), (1.11) is true and in addition,

$$\sum_{i=1}^{m-2} \alpha_i \xi_i \left( \frac{1}{\xi_{m-2}} - \frac{\alpha}{\beta} \right) > 1.$$  

Furthermore there exists a positive number $H$ such that

$$0 < y(t) \leq H \quad \text{and} \quad -2H \leq y'(t) \leq -\frac{\alpha}{\beta}H, \quad 0 \leq t \leq 1,$$

for any such solution.

**Remark 2.5** Again, as in Remark 2.2, we may show that for

$$\sum_{i=1}^{m-2} \alpha_i \xi_i \geq 1,$$

there is no (positive) solution $y \in X(P)$ of the BVP (2.27). Furthermore we must seek the possible solutions of (2.27) only for the case

$$\sum_{i=1}^{m-2} \alpha_i \leq 1,$$

since otherwise, by the monotonicity of $y(t)$, we get the contradiction

$$0 = W(P) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) \geq \left[ \sum_{i=1}^{m-2} \alpha_i - 1 \right] y(1) > 0.$$

Finally consider the boundary-value problem

$$y'' + f(t, y, y') = 0, \quad y'(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i). \quad (2.29)$$

Then following almost the same lines as above, we may prove the next theorem.

**Theorem 2.6** Assume that (1.7) holds and

$$\sum_{i=1}^{m-2} \alpha_i \frac{\xi_i}{\xi_1} < 1.$$  

Then the boundary-value problem (2.29) has a positive strictly decreasing solution provided that

• The function $f$ is superlinear (see (1.8)) along with (1.10), or
The function $f$ is sublinear (see (2.28)), (1.11) holds and in addition
\[ \sum_{i=1}^{m-2} \alpha_i \xi_i^2 > 1. \]

Furthermore there exists a positive number $H$ such that
\[ 0 < y(t) \leq H \quad \text{and} \quad -2H \leq y'(t) \leq 0, \quad 0 \leq t \leq 1, \]
for any such solution.

References


Positive and monotone solutions


Panos K. Palamides
Naval Academy of Greece, Piraeus 183 03, Greece
and
Department of Mathematics, Univ. of Ioannina,
451 10 Ioannina, Greece
e-mail:ppalam@otenet.gr