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GEOMETRIC CRITERIA FOR INVISCID 2D SURFACE QUASIGEOSTROPHIC EQUATIONS

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Abstract. It remains an open question if all classical solutions of the inviscid surface quasigeostrophic (SQG) equation are global in time or not. In this article, this issue is addressed through a geometric approach. This article contains three sections. The first section introduces the SQG equation, and presents existing results along with open problems. The second section presents local uniqueness and existence results of the SQG equations. Finally, the third section presents several geometric criteria under which the solutions of the SQG equation become regular for all time. The relation between the geometry of the level curves and the regularity of the solutions is the central focus of this part.

1. Surface quasigeostrophic equations

The surface quasigeostrophic (SQG) equations are evolution equations for scalars which are carried by a fluid flow. The 2D SQG equations are given by

$$\begin{gathered} \theta_t + u \cdot \nabla\theta + \kappa(-\Delta)^\alpha \theta = 0, \quad \nabla \cdot u = 0, \\ \theta(x,0) = \theta_0(x), \end{gathered} \tag{1.1}$$

where $\theta = \theta(x,t)$ is a scalar and $x \in \mathbb{R}^2$ or $x \in \mathbb{T}^2$, a periodic box. The quantities $\kappa \geq 0$ and $\alpha > 0$ are parameters. The vector $u = (u_2, u_2)$ is the velocity field and $u \cdot \nabla \equiv u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2}$. Since u is divergence free, there exists a stream function ψ such that

$$u = (-\partial_{x_2}\psi, \partial_{x_1}\psi).$$

The scalar θ is related to ψ through the relation

$$(-\Delta)^{1/2}\psi = \theta.$$

Using the notation $\Lambda \equiv (-\Delta)^{1/2}$ and $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$, the velocity u can be expressed in terms of θ as

$$u = \nabla^\perp \Lambda^{-1}\theta = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta),$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are the usual Riesz transforms. The fractional Laplacian $(-\Delta)^\alpha$ is defined through the Fourier transform as

$$\widehat{\Lambda^\alpha f}(k) = |k|^\alpha \widehat{f}(k). \tag{1.2}$$

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Using (1.2) the velocity u in the Fourier space can be expressed in terms of θ as

$$\widehat{u} = \frac{i(-k_2\widehat{\theta}, k_1\widehat{\theta})}{|k|}. \tag{1.3}$$

Depending upon the values of κ and α, we classify the equation (1.1) in the following categories.

When $\kappa = 0$, equation (1.1) is called inviscid SQG.
When $\kappa > 0$, equation (1.1) is called dissipative SQG.

The dissipative SQG is further divided into the following categories.

When $\alpha > 1/2$, equation (1.1) is called subcritical SQG.
When $\alpha = 1/2$, equation (1.1) is called critical SQG.
When $\alpha < 1/2$, equation (1.1) is called supercritical SQG.

The SQG equations model the buoyancy, or potential temperature on the 2D horizontal boundaries. These equations are derived through the quasigeostrophic approximations for non homogeneous fluid flow [4]. The original 3D quasigeostrophic equations were first derived by Charney in the 1940s. They have been used in describing the major features of large-scale fluid motion in atmosphere and oceans [15, 17, 18]. On the 2D horizontal boundaries, the 3D quasigeostrophic equations with uniform potential vorticity reduce to the SQG equations (1.1). The inviscid SQG was derived by Blumen in 1978 [1], by Constantin, Majda and Tabak in 1994 [4] and by Pierrehumbert in 1995 [19].

The global regularity problem for the subcritical case was solved by Constantin and Wu in [9]. If $\alpha > \frac{1}{2}$, the dissipative term $(-\Delta)^{\alpha}\theta$ controls the nonlinear term. In this case one can use energy type estimates to show that the solutions are global in time [7]. If $\alpha < \frac{1}{2}$, the dissipative term is not sufficient to control the non linear term. This phenomenon makes the study of the long time behavior of the solution much more difficult. So far for the long time behavior of the solutions in this case, we have to rely upon the numerical results [4, 5]. Therefore, at a formal level, we can see that $\alpha = \frac{1}{2}$ is a critical index. The critical case was solved by Kiselev, Nazarov and Volverg [16] for the periodic cases, and by Caffarelli and Vasseur [2] for the whole space. A more direct proof is given by Constantin and Vicol [6]. The supercritical case was studied and partial results were obtained by Constantin and Wu in ([10, 11]).

The study of these equations is important because of several practical reasons. The inviscid SQG is useful in modeling the frontogenesis, the formation of strong fronts between masses of hot and cold air [4, 18]. And also, due to its similarities with the 3D Euler equations in many aspects [4], the study of inviscid SQG can be very useful in the study of possible finite time singularities of 3D Euler and 3D Navier-Stokes Equations. Moreover, the dissipative SQG with $\kappa > 0$ and $\alpha = \frac{1}{2}$ is used in the studies of strongly rotating fluids [3].

The main mathematical issue concerning the SQG equation (1.1) is: Given $\theta(x, 0) = \theta_0(x)$, which is smooth and becomes 0 at infinity, does the equation (1.1) have a global classical solution? The global regularity issues for the super critical case and the inviscid case are still open. This article focuses only on the inviscid case.

2. Local well-Posedness and Regularity Criteria

In this section we consider the inviscid SQG equation, namely equation (1.1) with $\kappa = 0$, given by

$$\begin{gathered} \theta_t + u \cdot \nabla\theta = 0, \quad \nabla \cdot u = 0, \\ \theta(x,0) = \theta_0(x) \end{gathered} \tag{2.1}$$

The following theorem guarantees the global existence of weak solutions of equations (2.1).

Theorem 2.1. *For a given $\theta_0 \in L^2\left(\mathbb{R}^2\right)$ or $L^2(\mathbb{T}^2)$, there exist a global weak solution of* (2.1)

$$\int_0^T \int [\theta(x,t)\phi_t(x,t) + \theta(x,t)\ u\cdot\nabla\phi]\,dx\ dt + \int \theta_0(x)\phi(x,0)\ dx = 0$$

for smooth test function $\phi(x,t)$ and $T > 0$, where $\phi(x,T) = 0$.

This theorem was proved in the Ph.D. thesis of Resnick under the supervision of Constantin [20].

We remark that whether or not weak solutions are unique is unknown.

Recently there has been significant progress in the studies of global regularity of equation (2.1) under certain conditions in the topology of level set curves by Constantin et al. [4], Cordoba et al. [12], and Hou et al. [13, 14]. However, the global regularity issue has remained open in the general case.

The local existence results have been already proved for SQG equation (2.1) by Constantin et al. [4]. More precisely, if the initial value $\theta(x,0) = \theta_0(x) \in H^s(\mathbb{R}^2)$ for some integer $s \geq 3$, then there is a smooth solution $\theta(x,t)$ of equation (2.1) that also belongs to $H^s(\mathbb{R}^2)$ for each time, t, in a sufficiently small time interval, $0 \leq t < T_\star$. Furthermore, if $T_\star$, the maximal interval of smooth existence is finite, i.e. $T_\star < \infty$, then

$$|\theta(\cdot,t)|_s \to \infty \quad \text{as } t \nearrow T_\star.$$

Theorem 2.2. *Let $\theta_0 \in H^s(\mathbb{R}^2)$ with $s > 2$ and $T_\star < \infty$. Then the interval $0 \leq t < T_\star$ is a maximal interval of H^s existence of the solution of SQG equation* (2.1) *if and only if*

$$\int_0^T |\nabla\theta|_{L^\infty}(s)ds \to \infty \quad \textit{as } T \nearrow T_\star.$$

3. Geometry of level curves and regularity

The main goal of this section is to predict or rule out the finite time singularity of the solution of the inviscid SQG equations in certain geometric configuration of the level sets. We first review a few concepts in the geometric theory on the inviscid SQG equations which will be useful in our proofs later.

3.1. **Particle trajectory.** Given the velocity field u, the particle trajectory is defined by

$$\begin{gathered} \frac{dX(a,t)}{dt} = u(X(a,t),t) \\ X(a,0) = a. \end{gathered} \tag{3.1}$$

The existence for the ODE (3.1) is guaranteed by the following theorem.

Theorem 3.1. *Assume* $u \in C(\mathbb{R}^2 \times [0,T))$ *or* $u \in C(\mathbb{T}^2 \times [0,T))$ *and for each* $t \in [0,T)$,

$$|u(x,t) - u(y,t)| \le L_t |x-y|$$

for any $x, y \in \mathbb{R}^2$ *(or* $\mathbb{T}^2$*). In other words* u *is uniformly Lipschitz. Then the ODE* (3.1) *has a unique solution*

$$X = X(a,t) \in C^1([0,T)).$$

The incompressibility of the velocity field in equation (2.1) plays a very important role in the regularity of the solution.

3.2. **Level sets and tangent vector.** Assume that θ is a classical solution of the SQG (2.1) on $[0,T)$ and u is the corresponding velocity field. Then

$$x_2 = f(x_1, t)$$

is a level curve if

$$\theta(x_1, f(x_1,t), t) = C \quad \text{(a constant)},$$

where C is independent of x_1 and t.

Lemma 3.2. *Fix* $t \in [0,T)$. *If* $x_2 = f(x_1,t)$ *is a level curve, then* $\nabla^\perp \theta$ *is tangent to this curve.*

Proof. The tangent vector to the level curve $x_2 = f(x_1,t)$ at $x = x_1$ is given by $(1, f_{x_1}(x_1,t))$. Then it suffices to show

$$\nabla\theta \cdot (1, f_{x_1}(x_1,t)) = 0.$$

But this is easy to verify that

$$\partial_{x_1}\theta(x_1, f(x_1,t), t) + \partial_{x_2}\theta(x_1, f(x_1,t),t) f_{x_1}(x_1,t) = 0.$$

Here we used the fact $\theta\,(x_1, f(x_1,t), t) = C$. □

We now develop a relationship between magnitude of the vector $\nabla^\perp\theta$ and the geometry of the level sets using the incompressibility condition. First we prove a technical lemma.

Lemma 3.3. *Let*

$$\sigma(x,t) = \frac{\nabla^\perp\theta}{|\nabla^\perp\theta|}$$

be the unit tangent vector. Assume at a fixed time $t > 0, \nabla^\perp\theta$ *is* C^1. *Let*

$$A = \{x \in \mathbb{R}^2 : \nabla^\perp\theta \neq 0\}.$$

Then at this time t, *for any* $x \in A$ *it holds*

$$\frac{\partial}{\partial s}|\nabla^\perp\theta|(x,t) = -(\nabla\cdot\sigma(x,t))|\nabla^\perp\theta|(x,t) \tag{3.2}$$

where s *is the arc length variable along the level set curve passing* x. *Furthermore, for any* y *that is on the same level set as* x (3.2) *then gives*

$$|\nabla^\perp\theta|(y,t) = |\nabla^\perp\theta|(x,t) e^{\int_x^y(-\nabla\cdot\sigma)(s,t)ds} \tag{3.3}$$

as long as the level set segment connecting x *and* y *lies in* A, *where the integration is along the level set.*

Proof. From

$$\frac{\nabla^\perp\theta}{|\nabla^\perp\theta|}=\sigma$$

we obtain $\nabla^\perp\theta=|\nabla^\perp\theta|\sigma$. Since

$$\nabla\cdot\nabla^\perp\theta=(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2})\cdot(-\frac{\partial\theta}{\partial x_2},\frac{\partial\theta}{\partial x_1})=-\frac{\partial^2\theta}{\partial x_1\partial x_2}+\frac{\partial^2\theta}{\partial x_2\partial x_1}=0,$$

we obtain

$$\begin{aligned}0=\nabla\cdot\nabla^\perp\theta&=\nabla\cdot(|\nabla^\perp\theta|\sigma)\\&=(|\nabla^\perp\theta|)(\nabla\cdot\sigma)+\nabla(|\nabla^\perp\theta)\cdot\sigma\\&=(\nabla\cdot\sigma)(|\nabla^\perp\theta|)+(\sigma\cdot\nabla)(|\nabla^\perp\theta|)\end{aligned}$$

Next, using $\sigma\cdot\nabla=\frac{\partial}{\partial s}$, we obtain

$$(\nabla\cdot\sigma)(|\nabla^\perp\theta|)+\frac{\partial}{\partial s}(|\nabla^\perp\theta|)=0.$$

This implies

$$\frac{\partial}{\partial s}(|\nabla^\perp\theta|)=-(\nabla\cdot\sigma)|\nabla^\perp\theta|.$$

Now integrating along the level set from x to y we obtain

$$\int_x^y\frac{\frac{\partial}{\partial s}(|\nabla^\perp\theta|)ds}{|\nabla^\perp\theta|}=-\int_x^y(\nabla\cdot\sigma)(x(s),t)ds,$$

which is equivalent to

$$\int_x^y\frac{\partial}{\partial s}\ln(\nabla^\perp\theta|)ds=-\int_x^y(\nabla\cdot\sigma)(x,t)ds.$$

Form this we obtain

$$\ln(|\nabla^\perp\theta|)(y,t)-\ln(|\nabla^\perp\theta|)(x,t)=-\int_x^y(\nabla\cdot\sigma)(x,t)ds.$$

This gives

$$|\nabla^\perp\theta|(y,t)=|\nabla^\perp\theta|(x,t)e^{\int_x^y(-\nabla\cdot\sigma)(x(s),t)ds}.$$

□

Next we prove the main theorem on the geometric criteria of the level curve of equations (2.1).

Theorem 3.4. *Let $\mathscr{L}_t$ be a portion of the level curve in the solution of the inviscid SQG equations* (2.1) *such that $|\nabla^\perp\theta(x(t),t)|\ge C|\nabla^\perp\theta(\cdot,t)|_{L^\infty(\mathbb{R}^2)}$. Assume that for all $t\in[0,T)$ there is another point $y(t)$ on the same level curve that contains $\mathscr{L}_t$ such that the $\sigma(x,t)=\frac{\nabla^\perp\theta}{|\nabla^\perp\theta|}$ along a point $x(t)\in\mathscr{L}_t$, and $y(t)$ is well defined. Furthermore, assume*

$$\Big|\int_{x(t)}^{y(t)}(\nabla\cdot\sigma)(x(s),t)ds\Big|\le C$$

for some absolute constant C, and

$$\int_0^T|\nabla^\perp\theta|(y(t),t)dt<\infty.$$

Then equation (2.1) *has a classical solution up to time* T. *In addition, we have*

$$e^{-C} \le \frac{|\nabla^\perp \theta(x(t),t)|}{|\nabla^\perp \theta(y(t),t)|} \le e^C.$$

Proof. From 3.3 we have

$$|\nabla^\perp \theta(x,t)| = |\nabla^\perp \theta|(y,t) e^{\int_x^y (\nabla\cdot\sigma)(s,t)} ds.$$

Integrating the above equation from 0 to T we obtain

$$\begin{aligned} \int_0^T |\nabla^\perp \theta(x(t),t)|dt &= \int_0^T |\nabla^\perp \theta(y(t),t)| e^{\int_x^y (\nabla\cdot\sigma)(s,t)ds} dt \\ &\le \int_0^T |\nabla^\perp \theta(y,t)| e^{|\int_x^y (\nabla\cdot\sigma)(s,t)ds|} dt \\ &\le e^C \int_0^T |\nabla^\perp \theta(y(t),t)|dt < \infty. \end{aligned} \tag{3.4}$$

Therefore,

$$\int_0^T |\nabla\theta|_{L^\infty}(s)ds = \int_0^T |\nabla^\perp\theta|(x(t),t)dt < \infty$$

Now by Theorem 2.2, there will be no finite time blow up until time T. $\square$

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