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GLOBAL REGULARITY CRITERIA FOR 2D MICROPOLAR EQUATIONS WITH PARTIAL DISSIPATIONS

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Abstract. This article addresses the global regularity (in time) issue of two dimensional incompressible micropolar equations with various partial dissipations. Micropolar fluids represent a class of fluids with nonsymmetric stress tensor (called polar fluids) such as fluids consisting of suspending particles, dumbbell molecules, etc. Whether or not its classical solutions of 2D micropolar equations without velocity dissipation and micro-rotational viscosity develop finite time singularities is a difficult problem, and remains open. Here, we mainly focus on two types of partial dissipation cases, and we prove the conditional global regularity.

1. Introduction

In this article we study the global existence and regularity of classical solutions to the 2D incompressible micropolar equations with various dissipation. The standard 3D incompressible micropolar equations can be written as

$$\begin{gathered}\partial_t u+(u\cdot\nabla)u+\nabla\pi=(\gamma+\kappa)\Delta u+2\kappa\nabla\times\omega,\\ \partial_t\omega+(u\cdot\nabla)\omega+4\kappa\omega=\eta\Delta\omega+\alpha\nabla\nabla\cdot\omega+2\kappa\nabla\times u,\\ \nabla\cdot u=0,\end{gathered}\tag{1.1}$$

where, for $\mathbf{x}\in\mathbb{R}^3$ and $t\geq 0$, $u=u(\mathbf{x},t),\omega=\omega(\mathbf{x},t)$ and $\pi=\pi(\mathbf{x},t)$ denote the velocity field, the micro-rotation field and the pressure, respectively, and γ denotes the kinematic viscosity, κ the micro-rotational viscosity, and α, and η the angular viscosities.

The 3D micropolar equations reduce to the 2D micropolar equations when

$$u=(u_1(x,y,t),u_2(x,y,t),0),\quad \omega=(0,0,\omega_3(x,y,t)),\quad \pi=\pi(x,y,t),$$

The 2D incompressible micropolar equations can be written as,

$$\begin{gathered}\partial_t u+(u\cdot\nabla)u+\nabla\pi=(\gamma+\kappa)\Delta u+2\kappa\nabla\times\omega,\\ \partial_t\omega+(u\cdot\nabla)\omega+4\kappa\omega=\eta\Delta\omega+2\kappa\nabla\times u,\\ \nabla\cdot u=0,\end{gathered}\tag{1.2}$$

where $u=(u_1,u_2)$, $\nabla\times\omega=(-\partial_y\omega,\partial_x\omega)$ and $\nabla\times u=\partial_x u_2-\partial_y u_1$

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Micropolar fluids represent a class of fluids with nonsymmetric stress tensor (called polar fluids) such as fluids consisting of suspending particles, dumbbell molecules, etc (see, e.g., [5, 8, 9, 10, 14]). In the absence of micro-rotational effects, this system reduces to well-known Navier Stokes equations. A generalization of the 2D micropolar equations is given by

$$\begin{gathered} \partial_t u_1 + (u\cdot\nabla)u_1 + \partial_x\pi = \mu_{11}\partial_{xx}u_1 + \mu_{12}\partial_{yy}u_1 + 2\kappa\partial_y\omega, \\ \partial_t u_2 + (u\cdot\nabla)u_2 + \partial_y\pi = \mu_{21}\partial_{xx}u_2 + \mu_{22}\partial_{yy}u_2 - 2\kappa\partial_x\omega, \\ \partial_t\omega + (u\cdot\nabla)\omega + 4\kappa\omega = \eta_1\partial_{xx}\omega + \eta_2\partial_{yy}\omega + 2\kappa\nabla\times u, \\ \nabla\cdot u = 0. \end{gathered} \tag{1.3}$$

where we have written the velocity equation in its two components. Clearly, if

$$\mu_{11} = \mu_{12} = \mu_{21} = \mu_{22} = \gamma + \kappa, \quad \eta_1 = \eta_2 = \eta$$

then (1.3) reduces to the standard 2D micropolar equations in (1.2).

We main focus on the global regularity problem on (1.3) with various dissipations. The global regularity to (1.3) with $\mu_{11} > 0$, $\mu_{12} > 0$, $\mu_{21} > 0$, $\mu_{22} > 0$, and $\eta_1 = \eta_2 = 0$ can be done easily. Global regularity of the following cases have been established.

- (I) $\mu_{11} > 0$, $\mu_{12} > 0$, $\mu_{21} > 0$, $\mu_{22} > 0$ and $\eta_1 = \eta_2 = 0$;
- (II) $\mu_{11} = \mu_{12} = \mu_{21} = \mu_{22} = 0$ and $\eta_1 = \eta_2 > 0$;
- (III) $\mu_{11} = 0$, $\mu_{12} > 0$, $\mu_{21} > 0$, $\mu_{22} = 0$, and $\eta_1 > 0$, $\eta_2 = 0$;
- (IV) $\mu_{11} > 0$, $\mu_{12} > 0$, $\mu_{21} = 0$, $\mu_{22} = 0$, and $\eta_1 > 0$, $\eta_2 = 0$;
- (V) $\mu_{11} = 0$, $\mu_{12} > 0$, $\mu_{21} = 0$, $\mu_{22} > 0$, and $\eta_1 > 0$, $\eta_2 = 0$

For (I) and (II), the global regularity was established in [7], and [6], respectively. Very recently Regmi and Wu [17] studied the global regularity of the magneto-micropolar equations with partial dissipation. The global regularity results for cases (III)–(V) are included in [17].

The global regularity to (1.3) for the case: $\mu_{11} > 0$, $\mu_{12} = 0$, $\mu_{21} > 0$, $\mu_{22} = 0$, and $\eta_1 > 0$, $\eta_2 = 0$ is very difficult. In fact the dissipation is not sufficient to control L^2-norm when we employ energy method.

In this article, we consider the global regularity to (1.3) for the following two cases:

Case 1: $\mu_{11} = 0$, $\mu_{12} > 0$, $\mu_{21} = 0$, $\mu_{22} = 0$, and $\eta_1 > 0$, $\eta_2 = 0$.
Case 2: $\mu_{11} = 0$, $\mu_{12} = 0$, $\mu_{21} > 0$, $\mu_{22} = 0$, and $\eta_1 = 0$, $\eta_2 > 0$.

More precisely, we prove the following two theorems.

Theorem 1.1. *Consider the 2D micropolar equations*

$$\begin{gathered} \partial_t u_1 + (u\cdot\nabla)u_1 + \partial_x\pi = \mu_{12}\partial_{yy}u_1 + 2\kappa\partial_y\omega, \\ \partial_t u_2 + (u\cdot\nabla)u_2 + \partial_y\pi = -2\kappa\partial_x\omega, \\ \partial_t\omega + (u\cdot\nabla)\omega + 4\kappa\omega = \eta_1\partial_{xx}\omega + 2\kappa\nabla\times u, \\ \nabla\cdot u = 0. \end{gathered} \tag{1.4}$$

Assume $(u_0,\omega_0) \in H^2(\mathbb{R}^2)$, and $\nabla\cdot u_0 = 0$. Then the system has a unique global classical solution (u,ω) satisfying, for any $T > 0$, $(u,\omega) \in L^\infty([0,T]; H^2(\mathbb{R}^2))$ provided that

$$\int_0^T \|\partial_x u\|_\infty^2 < \infty$$

Another result in this article is summarized in the following theorem.

Theorem 1.2. *Consider the 2D micropolar equations*

$$\begin{aligned} \partial_t u_1 + (u\cdot\nabla)u_1 + \partial_x \pi &= 2\kappa\partial_y\omega, \\ \partial_t u_2 + (u\cdot\nabla)u_2 + \partial_y \pi &= \mu_{21}\partial_{xx}u_2 - 2\kappa\partial_x\omega, \\ \partial_t \omega + (u\cdot\nabla)\omega + 4\kappa\omega &= \eta_2\partial_{yy}\omega + 2\kappa\nabla\times u, \\ \nabla\cdot u &= 0. \end{aligned} \tag{1.5}$$

Assume $(u_0, \omega_0) \in H^2(\mathbb{R}^2)$, and $\nabla\cdot u_0 = 0$. Then the system has a unique global classical solution (u,ω) satisfying, for any $T > 0$, $(u,\omega) \in L^\infty([0,T]; H^2(\mathbb{R}^2))$ provided that

$$\int_0^T \|\partial_y u\|_\infty^2 < \infty$$

The general approach to establish the global existence and regularity results consists of two main steps. The first step assesses the local (in time) well-posedness while the second extends the local solution into a global one by obtaining global (in time) *a priori* bounds. For the systems of equations concerned here, the local well-posedness follows from a standard approach and shall be skipped here. Our main efforts are devoted to proving the necessary global *a priori* bounds. More precisely, we show that, for any $T > 0$ and $t \le T$,

$$\|(u,\omega)(\cdot,t)\|_{H^2(\mathbb{R}^2)} \le C, \tag{1.6}$$

where C denotes a bound that depends on T and the initial data.

The rest of this paper is divided into three sections. The first section is about preliminaries. The last two sections devoted to the proof of one of the theorems stated above. To simplify the notation, we will write $\|f\|_2$ for $\|f\|_{L^2}$, $\int f$ for $\int_{\mathbb{R}^2} f\,dxdy$ and write $\frac{\partial}{\partial x}f$, $\partial_x f$ or f_x as the first partial derivative, and $\frac{\partial^2}{\partial x^2}f$ or $\partial_{xx}f$ as the second partial throughout the rest of this paper. For the simplicity we consider all non zero parameters equal 1 (although we include some of these parameter in the proof)

2. Preliminaries

In this section we state some important results which will be used later. In the proof of Theorem 1.1 and 1.2, the following anisotropic type Sobolev inequality will be frequently used. Its proof can be found in [2].

Lemma 2.1. *If $f, g, h, \partial_y g, \partial_x h \in L^2(\mathbb{R}^2)$, then*

$$\iint_{\mathbb{R}^2} |fgh|\,dx\,dy \le C\|f\|_2\|g\|_2^{1/2}\|\partial_y g\|_2^{1/2}\|h\|_2^{1/2}\|\partial_x h\|_2^{1/2}, \tag{2.1}$$

where C is a constant.

The following simple fact on the boundedness of Riesz transforms will also be used.

Lemma 2.2. *Let f be divergence-free vector field such that $\nabla f \in L^p$ for $p \in (1,\infty)$. Then there exists a pure constant $C > 0$ (independent of p) such that*

$$\|\nabla f\|_{L^p} \le \frac{C\,p^2}{p-1}\|\nabla\times f\|_{L^p}.$$

3. Proof of Theorem 1.1

As explained in the introduction, it suffices to establish the global a priori bound for the solution in H^2. For the sake of clarity, we divide this process into two subsections. The first subsection proves the global H^1-bound while the second proves the global H^2-bound.

3.1. Global L^2-bound.

Lemma 3.1. *Assume that (u_0, ω_0) satisfies the condition stated in Theorem 1.1. Let (u, ω) be the corresponding solution of* (1.4)*. Then, (u, ω) obeys the following global L^2-bound,*

$$\begin{aligned}
&\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 + \mu_{12}\int_0^t \|\partial_y u_1(\tau)\|_{L^2}^2\,d\tau \\
&+ \eta_1\int_0^t \|\partial_x \omega(\tau)\|_{L^2}^2\,d\tau + 8\kappa\int_0^t \|\omega(\tau)\|_{L^2}^2\,d\tau \\
&\le e^{32\kappa^2(\frac{1}{\mu_{12}} + \frac{1}{\eta_1})t}(\|u_0\|_2^2 + \|\omega\|_2^2)
\end{aligned}$$

for any $t \ge 0$.

Taking the L^2 inner product of (1.4) with (u, ω), integrating with respect to space variable, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_2^2 + \|\omega(t)\|_2^2\right) + \mu_{12}\|\partial_y u_1\|_2^2 + \eta_1\|\partial_x\omega\|_2^2 + 4\chi\|\omega(t)\|_2^2 \\
&= \int_{\mathbb{R}^2} 2\kappa\{(\nabla\times\omega)\cdot u + (\nabla\times u)\omega\}dx \\
&= \int_{\mathbb{R}^2} 2\kappa\{\partial_y\omega u_1 - \partial_x\omega u_2 + \partial_x u_2\omega - \partial_y u_1\omega\}dx \\
&= -\int_{\mathbb{R}^2} 4\kappa\{\partial_x\omega u_2 + \partial_y u_1\omega\}dx \\
&\le 4\kappa(\|\partial_x\omega\|_2\|u\|_2 + \|\partial_y u_1\|_2\|\omega\|_2) \\
&\le \frac{\mu_{12}}{2}\|\partial_y u_1\|_2^2 + \frac{\eta_1}{2}\|\partial_x\omega\|_2^2 + \frac{32\kappa^2}{\mu_{12}}\|\omega\|_2^2 + \frac{32\kappa^2}{\eta_1}\|u\|_2^2,
\end{aligned}$$

where we have used the following fact due to the divergence free condition

$$\int_{\mathbb{R}^2}(u\cdot\nabla)u\cdot u\,dx = \int_{\mathbb{R}^2}(u\cdot\nabla)\omega\omega\,dx = 0$$

Applying Gronwall inequality for $0 < t < \infty$,

$$\begin{aligned}
&\|u(t)\|_2^2 + \|\omega(t)\|_2^2 + \mu_{12}\int_0^t \|\partial_x u(\tau)\|_{L^2}^2\,d\tau \\
&+ \eta_1\int_0^t \|\partial_x \omega(\tau)\|_{L^2}^2\,d\tau + 8\kappa\int_0^t \|\omega(\tau)\|_{L^2}^2\,d\tau \\
&\le e^{32\kappa^2(\frac{1}{\mu_{12}} + \frac{1}{\eta_1})t}(\|u_0\|_2^2 + \|\omega\|_2^2)
\end{aligned}$$

3.2. H^1-bound.

Theorem 3.2. *Assume that (u_0, ω_0) satisfies the condition stated in Theorem 1.1. Then the solution (u, ω) corresponding to (1.4) obeys, for any $0 < t < \infty$,*

$$\begin{aligned}
&\|\nabla u(t)\|_2^2 + \|\nabla\omega(t)\|_2^2 + \mu_{12}\int_0^t \|\nabla\partial_y u_1(\tau)\|_2^2\,d\tau \\
&+ \eta_1\int_0^t \|\nabla\partial_x\omega(\tau)\|_2^2\,d\tau + 8\kappa\int_0^t \|\nabla\omega(t)\|_2^2\,d\tau \le C
\end{aligned}$$

Proof. Taking the L^2 inner product of (1.4) with $(\Delta u, \Delta\omega)$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u(t)\|_2^2 + \|\nabla\omega(t)\|_2^2\right) + \mu_{12}\|\nabla\partial_y u_1\|_2^2 + \eta_1\|\nabla\partial_x\omega(t)\|_2^2 + 4\kappa\|\nabla\omega(t)\|_2^2 \\
&= \int_{\mathbb{R}^2} (2\kappa(\nabla\times\omega)\cdot(-\Delta u) + 2\kappa(\nabla\times u)(-\Delta\omega))\,dx + \int_{\mathbb{R}^2} u\cdot\nabla\omega(-\Delta\omega)\,dx \\
&= 2I_1 + I_2,
\end{aligned}$$

where we have use the fact that

$$\int_{\mathbb{R}^2} (u\cdot\nabla)u\cdot\Delta u\,dx = \int_{\mathbb{R}^2} \nabla\pi\cdot\Delta u\,dx = 0.$$

Note that

$$\int_{\mathbb{R}^2} 2\kappa(\nabla\times\omega)\cdot(-\Delta u) = \int_{\mathbb{R}^2} 2\kappa(\nabla\times u)(-\Delta\omega))\,dx.$$

To estimate I_1 we write in component-wise

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^2} 2\kappa(\nabla\times\omega)\cdot(-\Delta u)\,dx \\
&= \int_{\mathbb{R}^2} 2\kappa(-\partial_y\omega\Delta u_1 + \partial_x\omega\Delta u_2)\,dx \\
&= \int_{\mathbb{R}^2} 2\kappa(\partial_y\omega\partial_{xx}u_1 + \partial_y\omega\partial_{yy}u_1 - \partial_x\omega\partial_{xx}u_2 - \partial_x\omega\partial_{yy}u_2)\,dx\,,
\end{aligned}$$

$$\begin{gathered}
\Big|\int_{\mathbb{R}^2} 2\kappa\partial_y\omega\partial_{xx}u_1\,dx\Big| \le \frac{\eta_1}{8}\|\nabla\partial_x\omega\|_2^2 + \frac{32\kappa^2}{\eta_1}\|\nabla u\|_2^2, \\
\Big|\int_{\mathbb{R}^2} 2\kappa\partial_y\omega\partial_{yy}u_1\,dx\Big| \le \frac{\mu_2}{2}\|\nabla\partial_y u_1\|_2^2 + \frac{16\kappa^2}{\mu_2}\|\nabla\omega\|_2^2, \\
\Big|\int_{\mathbb{R}^2} 2\kappa\partial_x\omega\partial_{xx}u_2\,dx\Big| \le \frac{\eta_1}{8}\|\nabla\partial_x\omega\|_2^2 + \frac{32\kappa^2}{\eta_1}\|\nabla u\|_2^2, \\
\Big|\int_{\mathbb{R}^2} 2\kappa\partial_x\omega\partial_{yy}u_2\,dx\Big| \le \frac{\gamma}{8}\|\nabla\partial_x\omega\|_2^2 + \frac{32\kappa^2}{\gamma}\|\nabla u\|_2^2, \\
I_2 = \int_{\mathbb{R}^2} (u\cdot\nabla)\omega(-\Delta\omega)\,dx = \int_{\mathbb{R}^2} \nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx, \\
I_2 = \int \nabla\omega\cdot\nabla u\cdot\nabla\omega = \int \partial_x u_1\omega_x^2 + 2\int u_1\omega_y\omega_{xy} + \int(\partial_x u_2 + \partial_y u_1)\omega_x\omega_y, \\
\int \partial_x u_1\omega_x^2 = -2\int u_1\omega_{xx}\omega_x, \\
\Big|\int u_1\omega_{xx}\omega_x\Big| \le \frac{1}{48}\|\omega_{xx}\|_2^2 + C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|\nabla\omega\|_2^2,
\end{gathered}$$

$$\Big|\int u_1\omega_y\omega_{xy}\Big| \le \frac{1}{48}\|\nabla\omega_x\|_2^2 + C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|\nabla\omega\|_2^2,$$
$$\Big|\int \partial_y u_1\omega_x\omega_y\Big| \le \frac{1}{48}\|\nabla\omega_x\|_2^2 + C\|\partial_y u_1\|_2^2\|\nabla\omega\|_2^2,$$
$$\Big|\int \partial_x u_2\omega_x\omega_y\Big| \le C\|\partial_x u_2\|_\infty\|\nabla\omega\|_2^2$$

Combining the estimates above, together with Gronwall's inequality, we obtain

$$\begin{aligned}&\|\nabla u(t)\|_2^2 + \|\nabla\omega(t)\|_2^2 + \mu_{12}\int_0^t \|\nabla\partial_y u_1(\tau)\|_2^2\,d\tau\\ &+ \eta_1\int_0^t \|\nabla\partial_x\omega(\tau)\|_2^2\,d\tau + 8\kappa\int_0^t \|\nabla\omega(t)\|_2^2\,d\tau \le C\end{aligned}$$

for any $t \le T$, where C depends on T and the initial H^1-norm. This completes the proof of theorem. □

3.3. **Global H^2 bound and proof of Theorem 1.1.** To estimate the H^2-norm of (u, b, ω), we consider the equations of $\Omega = \nabla\times u$, $\nabla\omega$,

$$\Omega_t + u\cdot\nabla\Omega = -\mu_{12}\partial_{yyy}u_1 + 2\kappa\Delta\omega, \tag{3.1}$$
$$\nabla\partial_t\omega + \nabla(u\cdot\nabla\omega) + 4\kappa\nabla\omega = \eta_{21}\nabla\omega_{xx} + 2\kappa\nabla\Omega \tag{3.2}$$
$$\nabla\cdot u = 0. \tag{3.3}$$

Taking the L^2 inner product of (3.1) with $\Delta\Omega$, and integrating by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\Omega\|_2^2 + \mu_2\|\Delta\partial_y u_1\|_2^2 &= \int \nabla\Omega\cdot\nabla u\cdot\nabla\Omega\,dx\,dy - 2\kappa\int \Delta\Omega\Delta\omega\,dx\,dy\\ &\equiv L_1 + L_2,\end{aligned}\tag{3.4}$$

Taking the L^2-inner product of (3.2) with $\Delta\omega$, and integrating by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta\omega\|_2^2 + \eta_1\|\Delta\omega_x\|_2^2 + 4\kappa\|\Delta\omega\|_2^2 &= -2\kappa\int \Delta\Omega\Delta\omega + \int \Delta(u\cdot\nabla\omega)\Delta\omega\\ &\equiv L_2 + L_3.\end{aligned}$$

Adding (3.4) and (3.5) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla\Omega\|_2^2 + \|\Delta\omega\|_2^2) + \mu_2\|\Delta\partial_y u_1\|_2^2 + \eta_1\|\Delta\omega_x\|_2^2 + 4\chi\|\Delta\omega\|_2^2\\ &= L_1 + 2L_2 + L_3.\end{aligned}$$

We now estimate L_1 through L_3. We further split L_1 into 4 terms.

$$\begin{aligned}L_1 &= -\int \nabla\Omega\cdot\nabla u\cdot\nabla\Omega\;\,dx\,dy\\ &= -\int \big(\partial_x u_1(\partial_x\Omega)^2 + \partial_x u_2\partial_x\Omega\partial_y\Omega + \partial_y u_1\partial_x\Omega\partial_y\Omega + \partial_y u_2(\partial_y\Omega)^2\big)\\ &= L_{11} + L_{12} + L_{13} + L_{14}.\end{aligned}$$

Since $\Omega = \nabla\times u$, we have

$$\partial_{xx}\Omega = \Delta\partial_x u_2, \quad \partial_{yy}\Omega = -\Delta\partial_y u_1, \quad \partial_{xy}\Omega = \Delta\partial_x u_2. \tag{3.5}$$

Therefore,

$$L_{11} \le \|\partial_x u_1\|_\infty\|\nabla\Omega\|_2^2, \quad L_{12} \le \|\partial_x u_2\|_\infty\|\nabla\Omega\|_2^2$$

Invoking the divergence-free condition, we note that

$$\|\nabla\partial_x u_1\|_2^2 = \|\partial_{xx}u_1\|_2^2 + \|\partial_{yy}u_2\|_2^2,$$

$$\|\nabla\partial_y u_1\|_2^2 = \|\partial_{yy}u_2\|_2^2 + \|\partial_{yy}u_1\|_2^2,$$

$$\|\nabla\partial_x u_2\|_2^2 = \|\partial_{xx}u_1\|_2^2 + \|\partial_{xx}u_2\|_2^2.$$

By Lemma 2.1,

$$\begin{aligned}
L_{13} &\le \Big|\int \partial_y u_1 \partial_x\Omega\partial_y\Omega\Big| \\
&\le C\|\partial_y u_1\|_2^{1/2}\|\partial_{xy}u_1\|_2^{1/2}\|\partial_x\Omega\|_2\|\partial_y\Omega\|_2^{1/2}\|\partial_{yy}\Omega\|_2^{1/2} \\
&\le C\|\partial_{yy}\Omega\|_2\|\partial_{xy}u_1\|_2\|\partial_y\Omega\|_2 + C\|\partial_y u_1\|_2\|\nabla\Omega\|_2^2 \\
&\le \frac{1}{48}\|\partial_{yy}\Omega\|_2^2 + C(1+\|\partial_y u_1\|_2^2 + \|\partial_{xy}u_1\|_2^2)\|\nabla\Omega\|_2^2.
\end{aligned}$$

From $\Omega = \nabla\times u$, (3.5), and lemma (2.1),

$$\begin{aligned}
L_{14} &\le \Big|2\int u_2\partial_y\Omega\partial_{yy}\Omega\Big| \\
&\le C\|\partial_{yy}\Omega\|_2\|u_2\|_2^{1/2}\|\partial_x u_2\|_2^{1/2}\|\partial_y\Omega\|_2^{1/2}\|\partial_{yy}\Omega\|_2^{1/2} \\
&\le C\|\partial_{yy}\Omega\|_2^{\frac{3}{2}}\|u_2\|_2^{1/2}\|\Omega\|_2^{1/2}\|\nabla\Omega\|_2^{1/2} \\
&\le \frac{1}{48}\|\partial_{yy}\Omega\|_2^2 + C\|u_2\|_2^2\|\Omega\|_2^2\|\nabla\Omega\|_2^2.
\end{aligned}$$

Term L_2 can be easily bounded,

$$L_2 = \int \Delta\Omega\Delta\omega = \int \Omega_{xx}\Delta\omega + \int \Omega_{yy}\Delta\omega$$

with

$$\int \Omega_{xx}\Delta\omega = -\int \Omega_x\Delta\omega_x \le \|\nabla\Omega\|_2\|\Delta\omega_x\|_2, \quad \Big|\int \Omega_{yy}\Delta\omega\Big| \le \|\Omega_{yy}\|_2\|\Delta\omega\|_2.$$

We now estimate the last term L_3.

$$L_3 = -\int \Delta(u\cdot\nabla\omega)\Delta\omega = -\int \Delta(u_1\partial_1\omega + u_2\partial_y\omega)\Delta\omega \equiv L_{31} + L_{32}.$$

We first split L_{31} and L_{32} each into two terms.

$$L_{31} = -\int \partial_{xx}(u_1\partial_x\omega + u_2\partial_y\omega)\Delta\omega = L_{311} + L_{312}.$$

$$L_{32} = -\int \partial_{yy}(u_1\partial_x\omega + u_2\partial_y\omega)\partial_{xx}\omega - \int \partial_{yy}(u_1\partial_x\omega + u_2\partial_y\omega)\partial_{yy}\omega = L_{321} + L_{322}.$$

These terms are bounded as follows.

$$
\begin{aligned}
|L_{311}| &= \big| -\int \partial_x(u_1\partial_x\omega)\Delta\omega_x\big| \\
&\le \big| -\int \partial_x u_1\partial_x\omega\Delta\omega_x\big| + \big|\int u_1\partial_{xx}\omega\Delta\omega_x\big| \\
&\le C\|\Delta\omega_x\|_2\|\partial_x u_1\|_2^{1/2}\|\partial_{xy}u_1\|_2^{1/2}\|\partial_x\omega\|_2^{1/2}\|\partial_{xx}\omega\|_2^{1/2} \\
&\quad + C\|\Delta\omega_x\|_2\|\partial_{xx}\omega\|_2^{1/2}\|\partial_{xxx}\omega\|_2^{1/2}\|u_1\|_2^{1/2}\|\partial_y u_1\|_2^{1/2} \\
&\le C\|\Delta\omega_x\|_2\|\Delta\omega\|_2^{1/2}\|\Omega\|_2^{1/2}\|\nabla\Omega\|_2^{1/2}\|\nabla\omega\|_2^{1/2} \\
&\quad + C\|\Delta\omega_x\|_2^{\frac{3}{2}}\|\nabla\omega_x\|_2^{1/2}\|u_1\|_2^{1/2}\|\Omega\|_2^{1/2} \\
&\le \frac{1}{48}\|\Delta\omega_x\|_2^2 + C\|\Omega\|_2^2\|\nabla\Omega\|_2^2 + \|\nabla\omega\|_2^2\|\Delta\omega\|_2^2 + C\|u_1\|_2^2\|\nabla\omega_x\|_2^2\|\Omega\|_2^2,
\end{aligned}
$$

$$
\big|L_{312}\big| = \big| -\int \partial_x u_2\partial_y\omega\Delta\omega_x - \int u_2\partial_{xy}\omega\Delta\omega_x\big|,
$$

$$
\big| -\int \partial_x u_2\partial_y\omega\Delta\omega_x\big| \le \|\partial_x u\|_\infty\|\partial_y\omega\|_2\|\Delta\omega_x\|_2,
$$

$$
\begin{aligned}
\big|\int u_2\partial_{xy}\omega\Delta\omega_x\big| &\le C\|\Delta\omega_x\|_2\|u_2\|_2^{1/2}\|\partial_x u_2\|_2^{1/2}\|\partial_{xy}\omega\|_2^{1/2}\|\partial_{xyy}\omega\|_2^{1/2} \\
&\le \|\Delta\omega_x\|_2^2 + C\|u_2\|_2^2\|\Omega\|_2^2\|\nabla\omega_x\|_2^2,
\end{aligned}
$$

$$
L_{321} = \int \partial_{yy}(u_1\partial_x\omega + u_2\partial_y\omega)\partial_{xx}\omega = \int \partial_{xx}(u_1\partial_x\omega + u_2\partial_y\omega)\partial_{yy}\omega.
$$

Obviously L_{321} admits the same bound as that for L_{311},

$$
|L_{321}| \le \frac{1}{48}\|\Delta\omega_x\|_2^2 + C\|\Omega\|_2^2\|\nabla\Omega\|_2^2 + \|\nabla\omega\|_2^2\|\Delta\omega\|_2^2 + C\|u_1\|_2^2\|\nabla\omega_x\|_2^2\|\Omega\|_2^2.
$$

To estimate L_{322}, we write it out explicitly and integrate by parts,

$$
\begin{aligned}
L_{322} &= \int \partial_{yy}(u_1\partial_x\omega + u_2\partial_y\omega)\partial_{yy}\omega \\
&= \int \partial_y(\partial_y u_1\partial_x\omega + u_1\partial_{xy}\omega)\partial_{yy}\omega + \int \partial_y(\partial_y u_2\partial_2\omega + u_2\partial_{yy}\omega)\partial_{yy}\omega \\
&= \int [\partial_{yy}u_1\partial_x\omega + 2\partial_y u_1\partial_{xy}\omega + u_1\partial_{xyy}\omega]\partial_{yy}\omega \\
&\quad + \int [\partial_{yy}u_2\partial_y\omega + 2\partial_y u_2\partial_{yy}\omega + u_2\partial_{yyy}\omega]\partial_{yy}\omega.
\end{aligned}
$$

The terms on the right can be bounded as follows:

$$
\begin{aligned}
&\big|\int \partial_{yy}u_1\partial_x\omega\partial_{yy}\omega\big| \\
&\le C\|\partial_x\omega\|_2\|\partial_{yy}u_1\|_2^{1/2}\|\partial_{yyy}u_1\|_1^{1/2}\|\partial_{yy}\omega\|_2^{1/2}\|\partial_{xyy}\omega\|_2^{1/2} \\
&\le C\|\partial_x\omega\|_2\|\nabla\partial_y u_1\|_2^{1/2}\|\Delta\partial_y u_1\|_2^{1/2}\|\Delta\omega\|_2^{1/2}\|\Delta\partial_x\omega\|_2^{1/2} \\
&\le \frac{1}{48}\|\Delta\partial_y u_1\|_2^2 + \frac{1}{48}\|\Delta\omega_x\|_2^2 + C\|\omega_x\|_2^2[\|\nabla\partial_y u_1\|_2^2 + \|\Delta\omega\|_2^2) \\
&\le \frac{1}{48}\|\Delta\partial_y u_1\|_2^2 + \frac{1}{48}\|\Delta\omega_x\|_2^2 + C\|\omega_x\|_2^2(\|\nabla\Omega\|_2^2 + \|\Delta\omega\|_2^2)\,.
\end{aligned}
$$

$$
\begin{aligned}
&|\int \partial_y u_1 \partial_{xy}\omega \partial_{yy}\omega| \\
&\le C\|\partial_{yy}\omega\|_2 \|\partial_{xy}\omega\|_2^{1/2} \|\partial_{xyy}\omega\|_2^{1/2} \|\partial_y u_1\|_2^{1/2} \|\partial_{xy} u_1\|_2^{1/2} \\
&\le C\|\Delta\omega\|_2 \|\nabla\omega_x\|_2^{1/2} \|\Delta\omega_x\|_2^{1/2} \|\partial_y u_1\|_2^{1/2} \|\nabla\Omega\|_2^{1/2} \\
&\le \frac{1}{48}\|\Delta\omega_x\|_2^2 + C\,\|\nabla\omega_x\|_2^2 \|\nabla\Omega\|_2^2 + C\|\partial_y u_1\|_2 \|\Delta\omega\|_2^2 .
\end{aligned}
$$

$$
\begin{aligned}
&|\int u_1 \partial_{xyy}\omega \partial_{yy}\omega| \\
&\le C\|\partial_{xyy}\omega\| \|u_1\|_2^{1/2} \|\partial_y u_1\|_2^{1/2} \|\partial_{yy}\omega\|_2^{1/2} \|\partial_{xyy}\omega\|_2^{1/2} \\
&\le C\|\Delta\omega_x\|_2^{\frac{3}{2}} \|u_1\|_2^{1/2} \|\partial_y u_1\|_2^{1/2} \|\Delta\omega\|_2^{1/2} \\
&\le \frac{1}{48}\|\Delta\omega_x\|_2^2 + C\|u_1\|_2^2 \|\partial_y u_1\|_2^2 \|\Delta\omega\|_2^2 .
\end{aligned}
$$

$$
\begin{aligned}
&|\int \partial_{yy} u_2 \partial_y\omega \partial_{yy}\omega| \\
&\le C\|\partial_{yy} u_2\|_2 \|\partial_y\omega\|_2^{1/2} \|\omega_{yy}\|_2 \|\omega_{xyy}\|_2^{1/2} \\
&\le \|\omega_y\|_2 \|\Delta\omega_x\|_2 + C\|\nabla\partial_y u_1\|_2^2 \|\Delta\omega\|_2^2 \\
&\le \frac{1}{48}\|\Delta\omega_x\|_2^2 + C\|\nabla\omega\|_2^2 + C\|\nabla\partial_y u_1\|_2^2 \|\Delta\omega\|_2^2.
\end{aligned}
$$

Integration by parts yields

$$
\begin{gathered}
\int \partial_y u_2 \partial_{yy}\omega \partial_{yy}\omega = -\int \partial_x u_1 \partial_{yy}\omega \partial_{yy}\omega = 2\int u_1 \partial_{yy}\omega \partial_{xyy}\omega , \\
\int u_2 \partial_{yyy}\omega \partial_{yy}\omega = \frac{1}{2}\int u_2 \partial_y [\partial_{yy}\omega]^2 = -\int u_1 \partial_{xyy}\omega \partial_{yy}\omega ,
\end{gathered}
$$

which can be bounded as

$$
\begin{aligned}
|\int u_1 \partial_{xyy}\omega \partial_{yy}\omega| &\le C\|\partial_{xyy}\omega\|_2 \|\partial_{yy}\omega\|_2^{1/2} \|\partial_{xyy}\omega\|_2^{1/2} \|u_1\|_2^{1/2} \|\partial_y u_1\|_2^{1/2} \\
&\le C\|\Delta\omega_x\|_2^{\frac{3}{2}} \|\Delta\omega\|_2^{1/2} \|u_1\|_2^{1/2} \|\partial_y u_1\|_2^{1/2} \\
&\le \|\Delta\omega_x\|_2^2 + C\|u_1\|_2^2 \|\partial_y u_1\|_2^2 \|\Delta\omega\|_2^2.
\end{aligned}
$$

Collecting the estimates above and applying Gronwall's inequality, we obtain the desired global H^2-bound. This completes the proof for the global H^2-bound and thus the proof of Theorem 1.1.

4. Proof of Theorem 1.2

We just prove the H^1-global bound here. The rest of the proof is similar to the proof of Theorem 1.1. The global L^2 bound can be proved easily.

Lemma 4.1. *Assume that (u_0, ω_0) satisfies the condition stated in Theorem 1.1. Let (u, ω) be the corresponding solution of (1.4). Then, (u, ω) obeys the following global L^2-bound,*

$$
\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 + \mu_{21}\int_0^t \|\partial_x u_2(\tau)\|_{L^2}^2\, d\tau
$$

$$+ \eta_2 \int_0^t \|\partial_y \omega(\tau)\|_{L^2}^2 \, d\tau + 8\kappa \int_0^t \|\omega(\tau)\|_{L^2}^2 \, d\tau \le C$$

for any $t \ge 0$.

Global H^2 bound can be obtained similar to theorem 3.2.

Theorem 4.2. *Assume that* (u_0, ω_0) *satisfies the condition stated in Theorem 1.1. Then, the corresponding solution* (u, ω) *of* (1.4) *obeys, for any* $0 < t < \infty$,

$$\begin{aligned} &\|\nabla u(t)\|_2^2 + \|\nabla \omega(t)\|_2^2 + \mu_{21} \int_0^t \|\nabla \partial_x u_2(\tau)\|_2^2 \, d\tau \\ &+ \eta_2 \int_0^t \|\nabla \partial_y \omega(\tau)\|_2^2 \, d\tau + 8\kappa \int_0^t \|\nabla \omega(t)\|_2^2 \, d\tau \le C \end{aligned}$$

Proof. Taking the L^2 inner product of (1.5) with $(\Delta u, \Delta \omega)$ yields

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\|\nabla u(t)\|_2^2 + \|\nabla \omega(t)\|_2^2 \right) + \mu_{21} \|\nabla \partial_x u_2\|_2^2 + \eta_2 \|\nabla \partial_y \omega(t)\|_2^2 + 4\kappa \|\nabla \omega(t)\|_2^2 \\ &= \int_{\mathbb{R}^2} (2\kappa(\nabla \times \omega) \cdot (-\Delta u) + 2\kappa(\nabla \times u)(-\Delta \omega)) \, dx + \int_{\mathbb{R}^2} u \cdot \nabla \omega(-\Delta \omega) \, dx \\ &= 2M_1 + M_2, \end{aligned}$$

To estimate M_1 we write in component-wise

$$M_1 = \int_{\mathbb{R}^2} 2(\partial_y \omega \partial_{xx} u_1 + \partial_y \omega \partial_{yy} u_1 - \partial_x \omega \partial_{xx} u_2 - \partial_x \omega \partial_{yy} u_2) \, dx \,,$$

$$\Big| \int_{\mathbb{R}^2} 2\kappa \partial_y \omega \partial_{xx} u_1 \, dx \Big| \le \frac{1}{8} \|\nabla \partial_y \omega\|_2^2 + \|\nabla u\|_2^2 \,,$$

$$\Big| \int_{\mathbb{R}^2} 2\kappa \partial_y \omega \partial_{yy} u_1 \, dx \Big| \le \frac{1}{8} \|\nabla \partial_y \omega\|_2^2 + \|\nabla u\|_2^2 \,,$$

$$\Big| \int_{\mathbb{R}^2} 2\kappa \partial_x \omega \partial_{xx} u_2 \, dx \Big| \le \frac{1}{8} \|\nabla \partial_x u_2\|_2^2 + \|\nabla \omega\|_2^2 \,,$$

$$\Big| \int_{\mathbb{R}^2} 2\kappa \partial_x \omega \partial_{yy} u_2 \, dx \Big| \le \frac{1}{8} \|\nabla \partial_y \omega\|_2^2 + \|\nabla u\|_2^2 \,,$$

$$M_2 = \int_{\mathbb{R}^2} \nabla \omega \cdot \nabla u \cdot \nabla \omega \, dx \,,$$

$$M_2 = \int \nabla \omega \cdot \nabla u \cdot \nabla \omega = \int \partial_x u_1 \omega_x^2 + 2 \int u_1 \omega_y \omega_{xy} + \int (\partial_x u_2 + \partial_y u_1) \omega_x \omega_y \,,$$

$$\Big| \int \partial_x u_1 \omega_x^2 \Big| \le \|\partial_y u\|_\infty \|\omega_x\|_2^2 \,,$$

$$\Big| \int u_1 \omega_y \omega_{xy} \Big| \le \|\partial_y u_1\|_\infty \|\nabla \omega_x\|_2^2 \,,$$

$$\Big| \int \partial_y u_1 \omega_x \omega_y \Big| \le \|\partial_y u_1\|_\infty \|\nabla \omega\|_2^2 \,,$$

$$\begin{aligned} \Big| \int \partial_x u_2 \omega_x \omega_y \Big| &\le \|\partial_x u_2\|_2 \|\omega_x\|_2^{1/2} \|\omega_y\|_2^{1/2} \|\nabla \omega_y\|_2 \\ &\le \frac{1}{48} \|\nabla \omega_y\|_2^2 + C \|\partial_x u_2\|_2^2 \|\nabla \omega\|_2^2 \end{aligned}$$

Combining the estimates above, together with Gronwall's inequalities, we obtain

$$\begin{aligned}&\|\nabla u(t)\|_2^2+\|\nabla\omega(t)\|_2^2+\mu_{12}\int_0^t\|\nabla\partial_y u_1(\tau)\|_2^2\,d\tau\\&+\eta_1\int_0^t\|\nabla\partial_x\omega(\tau)\|_2^2\,d\tau+8\kappa\int_0^t\|\nabla\omega(t)\|_2^2\,d\tau\le C\end{aligned}$$

for any $t\le T$, where C depends on T and the initial H^1-norm. This completes the proof of theorem. □

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