2014 Madrid Conference on Applied Mathematics in honor of Alfonso Casal,
*Electronic Journal of Differential Equations*, Conference 22 (2015), pp. 47–51.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

# A CONVERGENCE THEOREM FOR A TWO-SPECIES COMPETITION SYSTEM WITH SLOW DIFFUSION

GEORG HETZER, LOURDES TELLO

*Dedicated to Professor Alfonso Casal on his 70th birthday*

Abstract. This article concerns the effect of slow diffusion in two-species competition-diffusion problem with spatially homogeneous nearly identical reaction terms. In this case all (nonnegative) equilibria are spatially homogeneous, and the set of nontrivial equilibria is the graph of a $C^1$-curve. This article shows convergence of positive solutions to an equilibria which is determined by the initial data. The proof relies on the existence of a Lyapunov function and is adapted from [6] which dealt with linear diffusion.

## 1. Introduction

We study the asymptotic behavior of positive solutions of the two-species system

$$
\begin{gathered}
u_t - \Delta_p u = ug(u,v) \quad \text{in } (0,\infty)\times\Omega, \\
v_t - d\Delta_q v = rvg(u,v) \quad \text{in } (0,\infty)\times\Omega, \\
\partial_{\mathfrak{n}} u = 0 = \partial_{\mathfrak{n}} v \quad \text{on } (0,\infty)\times\partial\Omega. \\
u(0,x) = u_0(x) \quad \text{in } \Omega, \\
v(0,x) = v_0(x) \quad \text{in } \Omega.
\end{gathered} \tag{1.1}
$$

under the following hypotheses:

(H1) $N\in\mathbb{N}$, $\Omega\subseteq\mathbb{R}^N$ bounded smooth domain, $p,q>\max\{2,N\}$, $d,r>0$;

(H2) $g\in C^2(\mathbb{R}^2_+,\mathbb{R})$, $\partial_j g<0$ for $j=1,2$, $g(0,0)>0$; $g$ is negative outside a bounded region.

Slow diffusion ($p,q>2$) arises in filtration, and (1.1) could, e.g., model the spread of microorganisms in lymph nodes. The case considered here can be thought of as competition between a species and one of its mutants. The crucial difference is the dispersal, but no spatial adaptation has taken place, and the fitness function differ at most by a constant factor $r>0$, in applications $r=1$.

Our main result states that every positive solution of (1.1) converges to one of the nontrivial equilibria. The same result has been obtained in [6] for linear

2010 *Mathematics Subject Classification.* 35K57, 35K65.
*Key words and phrases.* Two-species competition-diffusion system; slow dispersal; identical species; convergence to equilibria.

Published November 20, 2015.

diffusion $p = q = 2$. The case where spatial adaption has occurred is different (isolated equilibria) and has found quite some interest over the years. The reader is referred to the classical papers [8] and [5] for linear dispersal, where the "slower diffuser" persists and can invade, to [12] for nonlocal diffusion, and to [14] for linear vs. nonlocal dispersal. Other related papers are [9], [10], [11], [13], and [15] but slow dispersal has not been considered to our knowledge.

We remark that system (1.1) with degenerate operators ($p > 2$, $q > 2$) may involve a time dependent free boundary problem for some initial data (see e.g [1]), but this issue is not considered here.

The paper is organized as follows. The next section recalls some well-known results for the solution semiflow of (1.1) and outlines the proof for global existence and nonnegativity. These results are used in Section 3 to establish the convergence result.

## 2. Preliminaries

Let $r \in \{p, q\}$ and $A_r : L^2(\Omega) \supset \mathrm{dom}(A_r) \to L^2(\Omega)$ be the subdifferential of

$$w \mapsto \begin{cases} \frac{1}{r}\int_\Omega |\nabla w|^r & w \in W^{1,r}(\Omega) \\ \infty & w \in L^2(\Omega) \setminus W^{1,r}(\Omega) \end{cases},$$

then $A = (A_p, dA_q)$ is the realization of the principal (elliptic) part of (1.1) in $L^2(\Omega)$. $A$ is densely defined, m-accretive, and generates a completely continuous solution semigroup in $L^2(\Omega) \times L^2(\Omega)$. Therefore the standard theory of local Lipschitz perturbations of $A$ (cf. [18], e.g.) guarantees that (1.1) generates a local solution semiflow in $L^2(\Omega) \times L^2(\Omega)$, if $g$ is smoothly extended to $\mathbb{R}^2$. The reader is also referred to [17] for a general setting involving set-valued solution semiflows. Since one is dealing with a competition problem, one is interested in nonnegative solutions only. Thus, solutions of (1.1) satisfying $u(0, \cdot) \geq 0$, $v(0, \cdot) \geq 0$ should be global and nonnegative. For our main result it suffices to consider smooth initial conditions (regularity properties), hence one can assume for solutions $(u, v)$ of (1.1) on $[0, T]$ that $u \in L^p([0, T], W^{1,p}(\Omega)) \cap W^{1,p'}([0, T], W^{-1,p'}(\Omega))$ and $v \in L^p([0, T], W^{1,p}(\Omega)) \cap W^{1,p'}([0, T], W^{-1,p'}(\Omega))$. Moreover, if $p,\ q > N$ as assumed in (H1), one has $W^{1,r}(\Omega) \hookrightarrow C(\overline{\Omega})$ for $r \in \{p, q\}$.

**Lemma 2.1.** *Let* (H1)–(H2) *be satisfied,* $T > 0$, $u_0, v_0 \in \mathrm{dom}(A)$ *be positive, and* $(u, v)$ *be the solution of* (1.1) *on* $[0, T)$ *with* $(u(0, \cdot), v(0, \cdot)) = (u_0, v_0)$*. Then* $0 \leq u(t, \cdot) \leq \|u_0\|_\infty + \beta + 1$ *and* $0 \leq v(t, \cdot) \leq \|v_0\|_\infty + \beta + 1$ *for* $0 \leq t < T$*.*

*Proof.* It suffices to deal with the statements for $u$. Let $\sigma > g(0, 0)$ and $w(t, \cdot) = e^{-\sigma t}u(t, \cdot) \geq 0$ for $t \in [0, T]$. Then $w$ satisfies

$$\partial_t w - e^{(p-2)\sigma t}\Delta_p w + \sigma w - wg(e^{\sigma t}w, v) = 0. \tag{2.1}$$

Note that the function $h(t, x, y) := \sigma y - yg(e^{\sigma t}y, v(t, \cdot))$ is strictly increasing in $y$ in view of (H2) and the choice of $\sigma$. Weak $p-$Laplacian comparison theorems have been established beginning with [2], and the proof of proposition 2.2 in [3] or that of Lemma 4.9 in [16] (Dirichlet case) apply immediately. We also refer to [4] for a more general quasilinear operator. In fact, let $\phi$ be the solution of $\dot{\phi} = \phi g(\phi, 0)$, $\phi(0) = 0$, and $\psi(t) = e^{-\sigma t}\varphi(t)$, then $\psi$ satisfies

$$\dot{\psi}(t) + \sigma\psi(t) - \psi g(e^{\sigma t}\psi, 0) = 0,$$

hence
$$\dot\psi(t) + \sigma\psi(t) - \psi g(e^{\sigma t}\psi, v(t)) \ge 0$$
in view of (H2). Thus,
$$\begin{aligned}&\int_\Omega \Big( (\partial_t w(t,x) - \dot\psi(t))[w(t,x)-\psi(t)]^+ - e^{(p-2)\sigma t}\Delta_p w(t,x)[w(t,x)-\psi(t)]^+\\ &+ \sigma(w(t,x)-\psi(t))[w(t,x)-\psi(t)]^+\\ &- (w(t,x)g(e^{\sigma t}w(t,x), v(t,x)) - \psi g(e^{\sigma t}\psi, v(t,x)))[w(t,x)-\psi(t)]^+\Big)\,dx \le 0,\end{aligned}$$
hence the m-accretiveness of the $p$-Laplacian and $h$ monotone increasing imply that
$$\int_\Omega [w(t,x)-\psi(t)]^+\,dx \le \int_\Omega [w(0,x)-\psi(0)]^+\,dx = 0,$$
hence $w(t,\cdot) \le \psi(t)$ for $t \in [0,T)$, therefore,
$$u(t,\cdot) \le \phi(t) \le \|u_0\|_\infty + \beta + 1$$
for $t \in [0,T)$. The nonnegativity of $u$ follows from the same weak comparison argument and the fact that the constant 0 solves (2.1). $\square$

## 3. Main result

Let $Z := \{(y,z) \in \mathbb{R}_+^2 : y^2+z^2 > 0, g(y,z) = 0\}$. It follows from (H2) that there exists a $\beta > 0$ with $g(\beta,0) = 0$ and a strictly decreasing function $\gamma \in C^1([0,\beta],\mathbb{R}_+)$ with $Z = \{(y,\gamma(y)) : y \in [0,\beta]\}$. In fact, $\gamma'(y) = -\frac{\partial_1 g(y,\gamma(y))}{\partial_2 g(y,\gamma(y))}$ and in particular $\gamma'(\beta) < 0$.

**Theorem 3.1.** *Let* (H1)–(H2) *be satisfied,* $u_0, v_0 \in \operatorname{dom}(A)$ *be positive, and* $(u,v)$ *be the solution of* (1.1) *with* $(u(0,\cdot), v(0,\cdot)) = (u_0,v_0)$. *Then* $(u(t,\cdot), v(t,\cdot))$ *converges uniformly to some* $(\zeta,\eta) \in Z$.

*Proof.* Select $\epsilon \in (0,\beta)$ with $\epsilon < \min(\{\gamma(\xi) - \xi\gamma'(\xi) : 0 \le \xi \le \beta\})$. Set
$$\kappa(\xi) := \begin{cases} \epsilon - \gamma(\xi) & 0 \le \xi \le \beta \\ \epsilon/\xi/\beta & \xi > \beta \end{cases}$$
and
$$V_0(y,z) := \int_1^y \big(\frac{\kappa(\xi)}{\xi}\big)\,d\xi + \frac{1}{r}\int_1^z \big(1 - \frac{\epsilon}{\xi}\big)\,d\xi$$
for $(y,z) \in (0,\infty)\times(0,\infty)$.

Clearly, $V_0$ is continuously differentiable on $\mathbb{R}_+^2$ and has, as outlined in [6], the following properties:

- $V_0 \ge 0$;
- $V_0(y,z) \to \infty$, if $y \to 0+$ or $z \to 0+$;
- $V_0(y,z) \to \infty$, if $y \to \infty$ or $z \to \infty$;
- $\nabla V_0(y,z)\cdot(yg(y,z), rzg(y,z)) \le 0$ and "=" implies $(y,z) \in Z$.

The last statement follows from
$$\nabla V_0(y,z)\cdot(yg(y,z), rzg(y,z)) = \begin{cases} (\epsilon - \gamma(y))g(y,z) + (1-\frac{\epsilon}{z})zg(y,z) & 0 < y \le \beta \\ \frac{\epsilon}{\beta}yg(y,z) + (1-\frac{\epsilon}{z})zg(y,z) & y > \beta. \end{cases}$$
Thus, we obtain $-\gamma(y)g(y,z) + zg(y,z) = (z-\gamma(y))g(y,z)$ if $0 < y < \beta$, which is $\le 0$, since $\operatorname{sgn}(z-\gamma(y)) = -\operatorname{sgn}(g(x,y))$. Note that the expression is equal to 0, if

and only if $(y,z) \in Z$. If $y \geq \beta$ and $z > 0$, then $\frac{\epsilon}{\beta}y - \epsilon > 0$ and $g(y,z) < 0$, hence $\frac{\epsilon}{\beta}yg(y,z) + (1 - \frac{\epsilon}{z})zg(y,z) < 0$.

Set

$$V(\varphi,\psi) := \int_\Omega V_0(\varphi(x),\psi(x))\,dx \quad \text{for } \varphi,\psi \in L^\infty(\Omega).$$

Then

$$\begin{aligned}
&\frac{d}{dt}V(u,v)(t)\\
&= \int_\Omega \partial_1 V_0(u(t,x),v(t,x))u_t(t,x)\,dx + \int_\Omega \partial_2 V_0(u(t,x),v(t,x))v_t(t,x)\,dx\\
&= \int_\Omega \partial_1 V_0(u(t,x),v(t,x))\Delta_p u(t,x)\,dx + d\int_\Omega \partial_1 V_0(u(t,x),v(t,x))\Delta_q u(t,x)\,dx\\
&\quad + \int_\Omega \nabla V_0(u(t,x),v(t,x)) \cdot (u(t,x)g(u(t,x),v(t,x)), rv(t,x)g(u(t,x),v(t,x))\,dx.
\end{aligned}$$

Integration by parts shows that

$$\int_\Omega h(w(x))\Delta_p w(x) = -\int_\Omega h'(w(x))|\nabla w(x)|^p \leq 0,$$

if $h \in C^1(\mathbb{R})$ is nondecreasing and $w \in \operatorname{dom}(A_p)$. This and the corresponding $A_q$ statement imply

$$\begin{gathered}
\int_\Omega \partial_1 V_0(u(t,x),v(t,x))\Delta_p u(t,x)\,dx \leq 0,\\
d\int_\Omega \partial_1 V_0(u(t,x),v(t,x))\Delta_q u(t,x)\,dx \leq 0,
\end{gathered}$$

which yields that $\frac{d}{dt}V(u,v)(t) \leq 0$ and equal to zero, if and only if $(u,v)(t) \in Z$. Thus, the $\omega$-limit set of $(u,v)$ contains only pairs $(\zeta,\eta)$ with $(\zeta(x),\eta(x)) \in Z$ for $x \in \Omega$. Since the $\omega$-limit set is backward invariant (cf. [7]), each $(\zeta,\eta)$ is constant on $\Omega$.

Assume that $(a_j,\gamma(a_j)) \in \omega(u,v)$ for $j = 1,2$ and that $a_1 < a_2$, then $(\rho,\gamma(\rho)) \in \omega(u,v)$, and we can assume without loss of generality that $0 < a_1 < a_2 < \beta$. Moreover, $\rho \mapsto V_0(\rho,\gamma(\rho))$ is constant, hence $\frac{\epsilon-\gamma(\rho)}{\rho} + \frac{1}{r}\big(1 - \frac{\epsilon}{\gamma(\rho)}\big)\gamma'(\rho) = 0$ for $a_1 \leq \rho \leq a_2$. This yields $\gamma'(\rho) = \frac{r}{\rho} > 0$ for $a_1 \leq \rho \leq a_2$ which contradicts $\gamma' < 0$. $\square$

**Remark 3.2.** Our results imply that mutations affecting dispersal alone do not drive the original species into extinction.

**Acknowledgments.** The work by L. Tello is partially supported by the research projects MTM2013-42907-P and MTM2014-57113-P of Ministerio de Economía y Competitividad, Spain.

## References

[1] S. N. Antontsev, J. I. Díaz, S. Shmarev; *Energy methods for free boundary problems: Applications to Nonlinear PDEs and Fluid Mechanics.* Springer Science & Business Media, Boston, 2012.

[2] A. N. Carvalho, C. B. Gentile; *Comparison results for nonlinear parabolic equations with monotone principal part*, J. Math. Anal. Appl. **259** (2001), 319–337.

[3] A. Derlet, P. Takáč; *A quasilinear parabolic model for population evolution*, DEA **4** (2012), 121–136.

[4] J. I. Díaz, F. De Thelin; *On a nonlinear parabolic problem arising in some models related to turbulent flows*. Siam Journal on Mathematical Analysis, **25** (4) (1994), 1085–1111.

[5] J. Dockery, V. Hutson, K. Mischaikow, M. Pernarowski; *The evolution of slow dispersal rates: a reaction diffusion model*, J. Math. Biol., **37** (1998), 61–83.

[6] H. Engler, G. Hetzer; *Convergence to equilibria for a class of reaction-diffusion systems*, Osaka J. Math., **29** (1992), 471–481.

[7] J. Hale; *Asymptotic behavior of dissipative systems*, Mathematical Surveys and Monographs 25, AMS, Providence, RI, 1988.

[8] A. Hastings; *Can spatial variation alone lead to selection for dispersal?* Theor. Pop. Biol., **24** (1983), 244–251.

[9] G. Hetzer, T. Nguyen, W. Shen; *Effects of small spatial variation of the reproduction rate in a two species competition model*, Electronic Journal of Differential Equations, **2010** (2010), No. 160, pp. 1–16.

[10] V. Hutson, Y. Lou, K. Mischaikow; *Spatial heterogeneity of resources versus Lotka-Volterra dynamics*, J. Differential Equations, **185** (2002), 97–136.

[11] V. Hutson, Y. Lou, K. Mischaikow, P. Poláčik; *Competing species near a degenerate limit*, SIAM J. Math. Anal., **35** (2003), 453–491.

[12] V. Hutson, S. Martinez, K. Mischaikow, G. T. Vickers; *The evolution of dispersal*, J. Math. Biol., **47** (2003), 483–517.

[13] V. Hutson, K. Mischaikow, P. Poláčik; *The evolution of dispersal rates in a heterogeneous time-periodic environment*, J. Math. Biol., **43** (2001), 501–533.

[14] C.-Y. Kao, Y. Lou, W. Shen; *Random dispersal vs. nonlocal dispersal*, Discrete and Continuous Dynamical Systems, **26** (2010), 551–596.

[15] Y. Lou, S. Martínez, P. Poláčik; *Loops and branches of coexistence states in a Lotka-Volterra competition model*, J. Differential Equations, **230** (2006) 720–742.

[16] J. F. Padial, P. Takáč, L. Tello; *An antimaximum principle for a degenerate parabolic problem*, Advances in Differential Equations **15** (2010), 601–648.

[17] J. Simsen, C. B. Gentile; *On p-Laplacian differential inclusions - Global existence, compactness properties and asymptotic behavior*, Nonlinear Anal. **71** (2009), 3488–3500.

[18] I. I. Vrabie; *Compactness Methods for Nonlinear Evolutions*, Second ed., Pitman Monographs and Surveys in Pure and Applied Mathematics, New York, 1995.



Georg Hetzer

Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849, USA

*E-mail address*: hetzege@auburn.edu

Lourdes Tello

Department of Applied Mathematics, ETS Arquitectura. Universidad Politécnica de Madrid, 28040 Madrid, Spain

*E-mail address*: l.tello@upm.es