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STABILIZED ADAMS TYPE METHOD WITH A BLOCK EXTENSION FOR THE VALUATION OF OPTIONS

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Abstract. We construct a continuous stabilized Adams type method (CSAM) that is defined for all values of the independent variable on the range of interest. This continuous scheme has the ability to provide a continuous solution between all the grid points with a uniform accuracy comparable to that obtained at the grid points. Hence, discrete schemes which are recovered from the CSAM as by-products are combined to form a stabilized block Adams type method (SBAM). The SBAM is then extended on the entire interval and applied as a single block matrix equation for the valuation of options on a non-dividend-paying stock by solving a system resulting from the semi-discretization of the Black-Scholes model. The stability of the SBAM is discussed and the convergence of the block extension of the SBAM is given. A numerical example is given to show the accuracy of the method.

1. Introduction

The Black-Scholes option pricing model is one of the most celebrated achievements in financial economics in the previous four decades. The model gives the theoretical value of European style options on a non-dividend-paying stock given the stock price, the strike price, the volatility of the stock, the time to maturity, and the risk-free rate of interest. However, since it is optimal to exercise early an American put option on a non-dividend paying stock, the Black-Scholes formula cannot be used. Hull [3] argues that it is never optimal for an American call option on a non-dividend-paying stock to be exercised early. Therefore the Black-Scholes formula can be used to value American Style call options on non-dividend-paying stocks. In fact, no exact analytic formula for valuing American put options on non-dividend paying stocks exists. As a result, numerous numerical procedures are utilized. A discussion of some of these numerical techniques is found in Hull [3]. In addition to that, several other numerical procedures for solving the Black-Scholes model abound in the literature (see Chawla et al. [2] and Khaliq et al. [8]). Since there is the possibility of an early exercise, Khaliq et al. [8]) consider the pricing of an American put option as a free boundary problem. In effect, the early exercise feature of the American put option transforms the Black-Scholes linear differential

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equation into a non-linear type. In order to do away with the free and moving boundary, Khaliq et al. [8] add a small continuous penalty term to the Black-Scholes equation and treat the nonlinear penalty term explicitly. They conclude that their method maintains superior accuracy and stability properties when compared to standard methods that are based on the Newton-type iteration procedure in valuing American options.

Furthermore, Chawla et al. [2] employ a technique based on the Generalized Trapezoidal Formulas (GTF) and compare the computational performance of the scheme obtained with the Crank-Nicolson scheme for the case of European option pricing. They note that their GTF (1/3) scheme is superior to the Crank-Nicholson scheme. While all these techniques try to accomplish the same goal by solving the Black-Scholes differential equation for a particular derivative security, they are applied only after transforming the model to be forward in time. In this paper, we propose SBAM that is A-stable and applied to solve the model in its original form without transforming it into a forward parabolic equation. Thus, consider the Black-Scholes model

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{1.1}$$

subject to the initial/boundary conditions

$$\begin{gathered} V(0,t) = X, \\ V(S,t) \to 0 \quad \text{as } S \to \infty, \\ V(S,T) = \max(X - S, 0), \end{gathered}$$

where $V(S,t)$ denotes the value of the option, σ the volatility of the underlying asset, X, the exercise price, T the expiry, and r the interest rate.

The method considered in this article involves the method of line approach to solve (1.1) in which we discretize the space derivatives in such as way that the resulting system of ordinary differential equations is stable (see Lambert [9], Ramos and Vigo-Aguiar [12], and Cash [1]). We then discretize time by using the SBAM. In particular, we seek a solution in the strip (rectangle) $[a,b]\times[c,d]$ by first discretizing the variable S with mesh spacings $\Delta S = 1/M$,

$$S_m = m\Delta S, \quad m = 0, 1, \dots, M.$$

We then define $v_m(t) \approx V(S_m,t)$, $\mathbf{v}(t) = [v_0(t), v_1(t), \dots, v_{M-1}(t)]^T$, and replace the partial derivatives $\frac{\partial^2 V(S,t)}{\partial S^2}$ and $\frac{\partial V(S,t)}{\partial S}$ occurring in (1.1) by central difference approximations to obtain

$$\begin{gathered} \frac{\partial^2 V(S_m,t)}{\partial S^2} = [v(S_{m+1},t) - 2v(S_m,t) + v(S_{m-1},t)]/(\Delta S)^2; \\ \frac{\partial V(S_m,\ t)}{\partial S} = [v(S_{m+1},t) - v(S_{m-1},t)]/(\Delta S), \quad m = 0, 1, \dots, M-1. \end{gathered}$$

Problem (1.1) then leads to the resulting semi-discrete problem

$$\begin{aligned} \frac{dv_i(t)}{dt} &= -\frac{1}{2}\sigma^2 S_i^2 [v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)]/(\Delta S)^2 \\ &\quad - rS_i[v_{i+1}(t) - v_{i-1}(t)]/(\Delta S) + rv_i(t) = 0, \end{aligned}$$

which can be written in the form

$$\frac{d\mathbf{v}(t)}{dt} = \mathbf{f}(t, \mathbf{v}), \tag{1.2}$$

where $\mathbf{f}(t, \mathbf{v}) = \mathbf{A}\ \mathbf{v} + \mathbf{Q}$ and $\mathbf{A}$ is an $M-1 \times M-1$ matrix arising from the central difference approximations to the derivatives of S and $\mathbf{Q}$ is a vector of given constants. Problem (1.2) is now a system of ordinary differential equations which can be solved by the SBAM.

We will assume the scalar form of (1.2) for notational simplification and will return to the system at the implementation stage in Section 5. We note that the Adams Moulton is one of the most popular methods available for solving (1.2). The k-Adams Method is given by

$$v_{n+k} - v_{n+k-1} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{1.3}$$

where β_j are constants. We note that v_{n+j} is the numerical approximation to the analytical solution $v(t_{n+j})$, $f_{n+j} = f(t_{n+j}, v(t_{n+j}))$, $j = 0, \dots, k$.

For non-stiff problems, (1.3) performs excellently, while for stiff for problems, (1.3) is restricted by the step-size. For instance, for $k = 4$, (1.3) gives the standard $4-$step Adams-Moulton Method which is of order 5 and has a stability interval of $[-1.84, 0]$. The method (1.3) is implemented in a step-by-step fashion in which on the partition π_N, an approximation is obtained at t_n only after an approximation at t_{n-1} has been computed, where

$$\pi_N : a = t_0 < t_1 < \cdots < t_N = b, \quad t_n = t_{n-1} + h, \quad n = 1, \dots, N,$$

$h = \frac{b-a}{N}$ is the constant step-size of the partition of π_N, N is a positive integer, and n is the grid index.

Our objective is to propose a 4-step stabilized Adams type method given by

$$v_{n+4} - v_{n+3} = \frac{h}{48}(f_n - 2f_{n+1} - 4f_{n+2} + 34f_{n+3} + 19f_{n+4}) \tag{1.4}$$

with an extended stability interval of $[-6, 0]$. The method has order 4, since we are more interested in the stability than the order as the system resulting from the semi-discretization of (1.1) could be stiff. The discretization of (1.2) using only (1.4) leads to an indeterminate, hence we are required to look for additional methods to complete the system as discussed in Section 2. The method (1.4) is a by-product of the CSAM constructed in Section 2. The construction of the CSAM is enhanced by perturbing the differential by adding a perturbation term involving the shifted Chebyshev's polynomial $\top^*(t)$ of degree 4.

In what follows, we explain how $\top^*(t)$ is obtained. We recall that the trigonometric definition of the Chebychev's polynomials of degree m are

$$\top(\xi) = \cos\{m \arccos \xi\}, \quad \xi \in [-1, 1], \quad m = 1, 2, 3 \dots$$

which can also be defined by the recurrence relation $T_0(\xi) = 1$, $T_1(\xi) = \xi$, $T_{n+1}(\xi) = 2\xi T_n(\xi) - T_{n-1}(\xi)$. We then obtain the shifted Chebyshev's polynomials $\top^*(t)$ by transforming the Chebychev's polynomials $\top(\xi)$ defined on $[-1, 1]$ to the interval $[t_n, t_{n+4}]$. This is done by using the linear transformation $t = d_1\xi + d_2$, $\xi \in [-1, 1]$, $t \in [t_n, t_{n+4}]$, where d_1 and d_2 are chosen such that $t = t_n$ when $\xi = -1$ and $t = t_{n+4}$ when $\xi = 1$ (see Johnson and Riess [7]). It is easily shown that after a simple algebraic computation $\xi = \frac{2(t-t_n)}{t_{n+4}-t_n} - 1$. Thus, we define the shifted Chebyshev's polynomials of degree 4 on the interval $[t_n, t_{n+4}]$ as

$$\top^*(t) = \top(\xi) = \cos[m \cos^{-1}(\frac{2(t - t_n)}{t_{n+4} - t_n} - 1)]$$

which is used to generate $\top^*(t_{n+j})$, $j=0,\dots,4$ involved with the perturbation term η in (2.3).

The rest of the paper proceeds as follows. In Section 2, we construct the CSAM and use it to produce the SBAM in Section 3. The block extension of the SBAM is given in Section 4. In Section 5, the implementation of the method to solve the Black-Scholes model is given. Section 6 is devoted to a numerical example and the conclusion of the paper is given in Section 7.

2. Construction of CSAM, and formulation of SBAM

In this section, we develop a 4-step SAM for (1.1) on the interval from t_n to $t_{n+4} = t_n + 4h$, where h is the chosen step-length. We assume that the solution on the interval $[t_n, t_{n+4}]$ is initially locally approximated by the polynomial

$$U_k(t) = \alpha_{k-1}(t)v_{n+k-1} + h\sum_{j=0}^{k} \beta_j(t) f_{n+j} \tag{2.1}$$

where $k = 4$, $\alpha_3(t)$ and $\beta_j(t)$ are continuous coefficients.

We assume that $v_{n+j} = U_k(t_n + jh)$ is the numerical approximation to the analytical solution $v(t_{n+j})$, $v'_{n+j} = U'_k(t_n + jh)$ is an approximation to $v'(t_{n+j})$. We also note that $f_{n+j} = f(t_{n+j}, v_{n+j})$, $j = 0,\dots,k$. The continuous method (2.1) is piecewise continuous on $[a, b]$ and defined for all $t \in [a, b]$. That is, $U_k(t)$ is defined such that $U_k(t) = v(t) + O(h^{k+1}))$, $t \in (t_n, t_{n+k})$. Since $k = 4$, the polynomials $\{U_0(t), U_4(t),\dots,U_{N-4}(t)\}$, then define piecewise polynomials $U(t)$ which is also continuous on $[a, b]$. Hence, (2.1) has the ability to provide a continuous solution on $[a, b]$ with a uniform accuracy comparable to that obtained at the grid points (see [11]) and can also be used to produce additional discrete methods (see Onumanyi et al. [10]). In what follows, we state the theorem that facilitates the construction of the CSAM (2.1).

Theorem 2.1. *Let the following conditions be satisfied*

$$U_k(t_{n+j}) = v_{n+k-1}, \tag{2.2}$$

$$U'_k(t_{n+k}) = f_{n+j} + \eta\top^*(t_{n+j}), \quad j = 0,\dots,k\,. \tag{2.3}$$

Then, the continuous representations (1.2) *and* (1.3) *are equivalent to*

$$U_k(t) = \Phi_k^T \left(W_k^{-1}\right)^T P_k(t), \tag{2.4}$$

where

$$W_k = \begin{pmatrix} P_0(t_{n+3}) & \cdots & P_4(t_{n+3}) & 0 \\ P'_0(t_n) & \cdots & P'_4(t_n) & \top_0 \\ P'_0(t_{n+1}) & \cdots & P'_4(t_{n+1}) & \top_1 \\ P'_0(t_{n+2}) & \cdots & P'_4(t_{n+2}) & \top_2 \\ P'_0(t_{n+3}) & \cdots & P'_4(t_{n+3}) & \top_3 \\ P'_0(t_{n+4}) & \cdots & P'_4(t_{n+4}) & \top_4 \end{pmatrix},$$

$$\Phi_k = (y_{n+3}, f_n, f_{n+1}, f_{n+2},\dots, f_{n+k})^T,$$

$$P_k(t) = (P_0(t), P_1(t),\dots, P_k(t), 0)^T.$$

The proof of the above theorem is the same as in Jator et al. [5] with only minor notational modifications. Note that T denotes the transpose and $P_j(t) = t^j, j = 0,\dots,k$ are basis functions, and the perturbation term involves η as a parameter and $\top^*(t_{n+j})$ are obtained from $\top^*(t)$ which is the shifted Chebychev's polynomial of degree 4.

Remark 2.2. The continuous method (2.1) is obtained by solving a system of 6 equations resulting from conditions (2.2) and (2.3) given in Theorem 2.1. It is also used to obtain method (1.3) and three additional methods.

In particular, method (2.1) is evaluated at $\{t = t_n, t_{n+1}, t_{n+2}, t_{n+4}\}$ to produce the following methods.

$$\begin{aligned}
v_n - v_{n+3} &= \frac{h}{16}(-5f_n - 22f_{n+1} - 12f_{n+2} - 10f_{n+3} + f_{n+4}) \\
v_{n+1} - v_{n+3} &= \frac{h}{12}(f_n - 8f_{n+1}1 - 10f_{n+2} - 8f_{n+3} + f_{n+4}) \\
v_{n+2} - v_{n+3} &= \frac{h}{48}(f_n - 2f_{n+1} - 20f_{n+2} - 30f_{n+3} + 3f_{n+4}) \\
v_{n+4} - v_{n+3} &= \frac{h}{48}(f_n - 2f_{n+1} - 4f_{n+2} + 34f_{n+3} + 19f_{n+4})
\end{aligned} \tag{2.5}$$

2.1. **SBAM.** Method (2.5) is used to form the block SBAM as follows:

$$A_1 Y_{\mu+1} = A_0 Y_\mu + h[B_1 F_{\mu+1} + B_0 F_\mu] \tag{2.6}$$

where

$$\begin{gathered}
Y_{\mu+1} = (v_{n+1}, v_{n+2}, v_{n+3}, v_{n+4})^T, \quad Y_\mu = (v_{n-3}, v_{n-2}, v_{n-1}, v_n)^T, \\
F_\mu = (f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4})^T, \quad F_{\mu-1} = (f_{n-3}, f_{n-2}, f_{n-1}, f_n)^T
\end{gathered}$$

for $\mu = 0, \dots \Gamma$, where $\Gamma = N/4$ is the number of blocks and $n = 0, 4, \dots, N-4$, and A_i, B_i, $i = 0, 1$ are 4 by 4 matrices whose entries are given by the coefficients of (2.5). In particular,

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{12} \\ 0 & 0 & 0 & \frac{1}{48} \\ 0 & 0 & 0 & -\frac{5}{16} \\ 0 & 0 & 0 & \frac{1}{48} \end{pmatrix}, \quad B_1 = \begin{pmatrix} -\frac{2}{3} & -\frac{5}{6} & -\frac{2}{3} & \frac{1}{12} \\ -\frac{1}{24} & -\frac{5}{12} & -\frac{5}{8} & \frac{1}{16} \\ -\frac{11}{8} & -\frac{3}{4} & -\frac{5}{8} & \frac{1}{16} \\ -\frac{1}{24} & -\frac{1}{12} & \frac{17}{24} & \frac{19}{48} \end{pmatrix}.$$

Consistency of the block SBAM. The consistency of (2.6) is accomplished by rewriting it as

$$Y_{\mu+1} = A_1^{-1} A_0 Y_\mu + h A_1^{-1}[B_1 F_{\mu+1} + B_0 F_\mu] \tag{2.7}$$

and define the local truncation error of (2.7) as

$$L[z(t); h] = Z_{\mu+1} - (A_1^{-1} A_0 Z_\mu + h A_1^{-1}[B_1 \overline{F}_{\mu+1} + B_0 \overline{F}_\mu]) \tag{2.8}$$

where

$$\begin{gathered}
Z_{\mu+1} = ((v(t_{n+1}), v(t_{n+2}), v(t_{n+3}), v(t_{n+4}))^T, \\
\overline{F}_{\mu+1} = ((f(t_{n+1}, v(t_{n+2})), f(t_{n+3}, v(t_{n+4})), f(t_{n+1}, v(t_{n+2})), f(t_{n+3}, v(t_{n+4})))^T, \\
Z_\mu = (v(t_{n-3}), v(t_{n-2}), v(t_{n-1}), v(t_n))^T, \\
\overline{F}_\mu = ((f(t_{n-3}, v(t_{n-3})), f(t_{n-2}, v(t_{n-2})), f(t_{n-1}, v(t_{n-1})), f(t_n, v(t_n)))^T,
\end{gathered}$$

and $L[z(x);h] = (L_1[z(x);h], L_2[z(x);h], L_3[z(x);h], L_4[z(x);h])^T$ is a linear difference operator.

Thus, assuming that the arbitrary function $z(t)$ is the exact solution and is sufficiently differentiable; we can expand the terms in (12) as a Taylor series about the point t_n, to obtain the expression for the local truncation error (LTE) as

$$L[z(t);h] = O(h^5). \tag{2.9}$$

Thus, the method has order four.

Stability. The Linear-stability regions are obtained by applying (2.6) to the test equation $y' = \lambda y$ to give

$$Y_{\mu+1} = M(q)Y_\mu, \quad q = \lambda h, \tag{2.10}$$

where the stability matrix is

$$M(q) = (A_1 - qB_1)^{-1}(A_0 + qB_0) = \frac{12 + 24q + 20q^2 + 8q^3 + q^4}{12 - 24q + 20q^2 - 8q^3 + q^4}.$$

Definition 2.3. The block method (2.6) is said to be A-stability if for all $q \in \mathbb{C}^-$, $M(q)$ has a dominant eigenvalue $q_{\max}$ such that $|q_{\max}| \leq 1$; moreover, since $q_{\max}$ is a rational function, the real part of the zeros of $q_{\max}$ must be negative, while the real part of the poles of $q_{\max}$ must be positive.

A simple calculation gives the zeros as $\{-1-i, -1+i, -3-\sqrt{3}, -3+\sqrt{3}\}$ and the poles as $\{1-i, 1+i, 3-\sqrt{3}, 3+\sqrt{3}\}$ (see Figure 1).

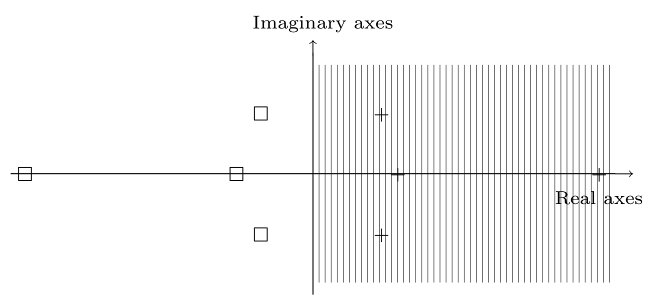

FIGURE 1. Stability region for the SBAM confirming A-stability

Remark 2.4. Method (2.6) can be used to solve initial value systems in a block by block fashion as in Jator [5]. In this paper, the approach in [5] is modified by first generating the blocks on the entire interval and then combining these blocks to form a single block matrix equation which is solved to provide the global solution of (1.2).

3. Block extension and implementation

Order Convergence. Since the block extension of (2.6) gives a global method, we discuss the convergence of the block extension of (2.6).

Theorem 3.1. *Let Y, Z be solution vectors formed by extending* (2.6) *from the interval $[t_0, t_4]$, to the intervals $[t_4, t_8], \dots, [t_{N-4}, t_N]$, and $E = Z - Y$, where Y is interpreted as an approximation of the solution vector for the system formed from the block extension of* (2.6) *whose exact solution is Z. If $e_i = |v(t_i) - v_i|$, where the exact solution $v(t)$ is several times differentiable on $[a, b]$ and if $\|E\| = \|Z - Y\|$, then, the SBAM is a fourth-order convergent method. That is $\|E\| = O(h^4)$.*

Proof. Let the matrices obtained from the block extension of (2.6) be defined as follows:

$$A = \begin{pmatrix}
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\cdots & 0\\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\cdots & 0\\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\cdots & 0\\
0 & 0 & -1 & 1 & & 0 & 0 & 0 & 0\cdots & 0\\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0\cdots & 0\\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0\cdots & 0\\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0\cdots & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0\cdots & 0\\
\vdots & & & & & & \ddots & & & \vdots\\
0 & 0 & 0 & 0\cdots & 0 & 0 & 1 & 0 & -1 & 0\\
0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 1 & -1 & 0\\
0 & 0 & 0 & 0\cdots & 0 & 1 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0 & -1 & 1
\end{pmatrix},$$

$$B = h\begin{pmatrix}
-\frac{2}{3} & -\frac{5}{6} & -\frac{2}{3} & \frac{1}{12} & 0 & 0 & 0 & 0 & 0\cdots & 0\\
-\frac{1}{24} & -\frac{5}{12} & -\frac{5}{8} & \frac{1}{16} & 0 & 0 & 0 & 0 & 0\cdots & 0\\
-\frac{11}{8} & -\frac{3}{4} & -\frac{5}{8} & \frac{1}{16} & 0 & 0 & 0 & 0 & 0\cdots & 0\\
-\frac{1}{24} & -\frac{1}{12} & \frac{17}{24} & \frac{19}{48} & 0 & 0 & 0 & 0 & 0\cdots & 0\\
0 & 0 & 0 & \frac{1}{12} & -\frac{2}{3} & -\frac{5}{6} & -\frac{2}{3} & \frac{1}{12} & 0\cdots & 0\\
0 & 0 & 0 & \frac{1}{48} & -\frac{1}{24} & -\frac{5}{12} & -\frac{5}{8} & \frac{1}{16} & 0\cdots & 0\\
0 & 0 & 0 & -\frac{5}{16} & -\frac{11}{8} & -\frac{3}{4} & -\frac{5}{8} & \frac{1}{16} & 0\cdots & 0\\
0 & 0 & 0 & \frac{1}{48} & -\frac{1}{24} & -\frac{1}{12} & \frac{17}{24} & \frac{19}{48} & 0\cdots & 0\\
\vdots & & & & & & \ddots & & & \vdots\\
0 & 0 & 0 & 0\cdots & 0 & \frac{1}{12} & -\frac{2}{3} & -\frac{5}{6} & -\frac{2}{3} & \frac{1}{12}\\
0 & 0 & 0 & 0\cdots & 0 & \frac{1}{48} & -\frac{1}{24} & -\frac{5}{12} & -\frac{5}{8} & \frac{1}{16}\\
0 & 0 & 0 & 0\cdots & 0 & -\frac{5}{16} & -\frac{11}{8} & -\frac{3}{4} & -\frac{5}{8} & \frac{1}{16}\\
0 & 0 & 0 & 0\cdots & 0 & \frac{1}{48} & -\frac{1}{24} & -\frac{1}{12} & \frac{17}{24} & \frac{19}{48}
\end{pmatrix},$$

$$C = (\frac{h}{12}f_0, \frac{h}{48}f_0, -y_0 - \frac{5h}{16}f_0, \frac{h}{48}f_0, 0, \dots, 0)^T\,.$$

The local truncation errors are

$$\begin{aligned}
\tau_{i+1} &= -\frac{13}{180}h^5 y^{(5)}(t_i + \theta_i) + O(h^6),\\
\tau_{i+2} &= -\frac{13}{360}h^5 y^{(5)}(t_i + \theta_i) + O(h^6),\\
\tau_{i+3} &= -\frac{1}{40}h^5 y^{(5)}(t_i + \theta_i) + O(h^6),\\
\tau_{i+4} &= -\frac{17}{360}h^5 y^{(5)}(t_i + \theta_i) + O(h^6),
\end{aligned} \tag{3.1}$$

for $i = 0, 4, \dots, N-4$, $|\theta_i| \le 1$. We further define the vectors

$$Z = (v(t_1), \dots, v(t_N))^T, \quad Y = (v_1, \dots, v_N)^T,$$
$$F = (f_1, \dots, f_N)^T, \quad L(h) = (\tau_1, \dots, \tau_N)^T,$$

where $L(h)$ is the local truncation error.

Let $E = Y - Z = (e_1, \dots, e_N)^T$. The exact form of the system given by the block extension of (2.6) is

$$AZ - BF(Z) + C + L(h) = 0, \tag{3.2}$$

and the approximate form of the system is

$$AY - BF(Y) + C = 0, \tag{3.3}$$

where Y is the approximation of the solution vector Z. Subtracting (3.2) from (3.3), we obtain

$$AE - BF(Y) + BF(Z) = L(h)\,. \tag{3.4}$$

Using the mean-value theorem, we write (3.4) as

$$(A - BJ)E = L(h),$$

where the Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial v_1} & \dots & \frac{\partial f_1}{\partial v_N} \\ \vdots & & \vdots \\ \frac{\partial f_N}{\partial v_1} & \dots & \frac{\partial f_N}{\partial v_N} \end{pmatrix}.$$

Let $M = -BJ$ be a matrix of dimension N. We have

$$(A + M)E = L(h), \tag{3.5}$$

and for sufficiently small h, the matrix $A + M$ is a monotone and thus invertible (see Jain and Aziz [4] and Jator and Li [6]). Therefore,

$$(A + M)^{-1} = D = (d_{i,j}) \ge 0, \quad \sum_{j=1}^{N} d_{i,j} = O(h^{-1}). \tag{3.6}$$

If $\|E\|$ is the norm of maximum global error and from (3.5), $E = (A + M)^{-1} L(h)$, using (3.6) and the truncation error vector $L(h)$, it follows that $\|E\| = O(h^4)$. Therefore, the TDM is an fourth-order convergent method. $\square$

3.1. **Implementation.** Recall that the semi-discretization of (1.1) is initially performed on the partition

$$\pi_M : \{c = S_0 < S_1 < \dots < S_M = d, \quad S_m = S_{m-1} + \Delta S\},$$

where $\Delta S = \frac{d-c}{M}$ is a constant step-size of the partition of π_M, $m = 1, 2, \dots, M$, M is a positive integer and m the grid index. The resulting system of ODEs (1.2) is then solved on the partition π_N. We emphasize the block extension of (2.6) lead to a single matrix of finite difference equations, which is solved to provide all the solutions of (1.2) on the entire grid given by the rectangle $[a, b] \times [c, d]$.

Step 1: Use the block extension of (2.6) to generated from the rectangles $[t_0, t_4] \times [c, d]$, to the rectangle $[t_4, t_8] \times [c, d], \dots, [t_{N-4}, t_N] \times [c, d]$.

Step 2: Solve the system obtained in step 1, noting that $v_m(t) \approx V(S_m,\ t)$ and $\mathbf{v}(t_n) = \mathbf{v}_n$, to obtain $\mathbf{v}_n = [V_{0,n}, V_{1,n}, \dots, V_{M-1,n}]^T$, $n = 1, 2, \dots, N$.

Step 3: The solution of (1.1) is approximated by the solutions in step 2 as $\mathbf{v}(t_n) = \mathbf{v}_n$, where $\mathbf{v}(t_n) = [V(S_0, t_n), V(S_1, t_n), \dots, V(S_{M-1}, t_n)]^T$, $n = 1, 2, \dots, N$.

4. Numeriacl examples

Our method was implemented Mathematica 8.0 enhanced by the feature NSolve[ ].

Example 4.1. Consider a five-month European call and put options on a non-dividend-paying stock when the stock price is \$50, the strike price is \$50, the risk-free interest rate is 10% per annum, and the volatility is 40% per annum. This example is taken from Hull [3].

To compute the call and put options we use the standard notation to denote $X = 50$, $S = 50$, $r = 0.10$, $\sigma = 0.40$, and $T = 0.4167$. The theoretical solutions for the prices of the European call and put options are given in Hull [3] as follows.

$$c = SN(d_1) - Xe^{-r(T-t)}N(d_2),$$
$$p = Xe^{-r(T-t)}N(-d_2 - SN(-d_1),$$

where

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = \frac{\ln(S/X) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t},$$

and $N(x)$ is the cumulative probability distribution function for the standard normal variable.

We note that Example 4.1 was chosen to illustrate that the SBAM does not exhibit oscillatory behavior near the exercise price which occurs in well known methods such as the Crank-Nicolson method (see [2] and [8]). In Figures 2 and 3, we plot the values of the call and put options with their corresponding exact solutions at expiration versus S respectively, while in Figures 4 and 5 we plot the the call and the put options with their corresponding exact solutions versus S and t. In all cases it is obvious that the SBAM gives an oscillation-free solution as the graphs are smooth through out.

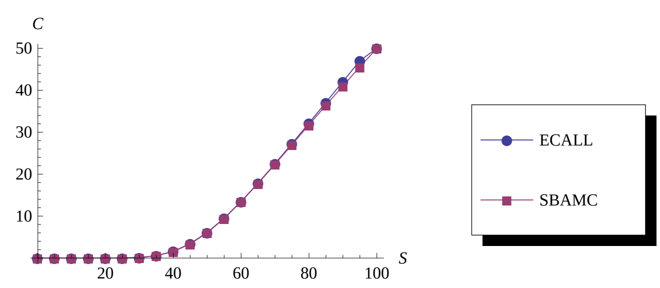

Figure 2. Call curves (C) at $t = 0$ for Example 4.1, where ECALL is the exact solution and SBAMC is the call option for the current method

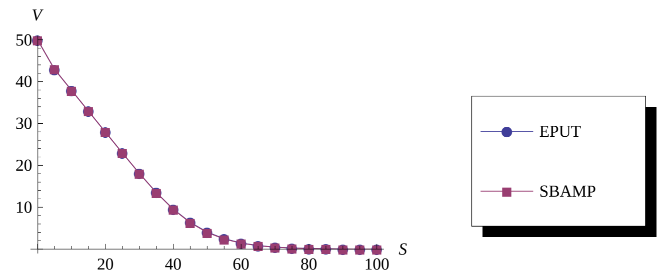

FIGURE 3. Put curves (V) at $t = 0$ for Example 4.1, where EPUT is the exact solution and SBAMP is the put option for the current method

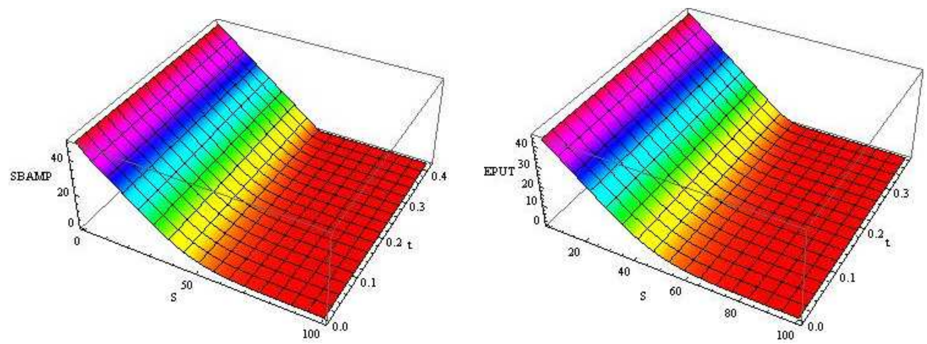

FIGURE 4. Approximate and exact solutions for the put option for Example 4.1 with $h = 1/12$, $\Delta S = 20$

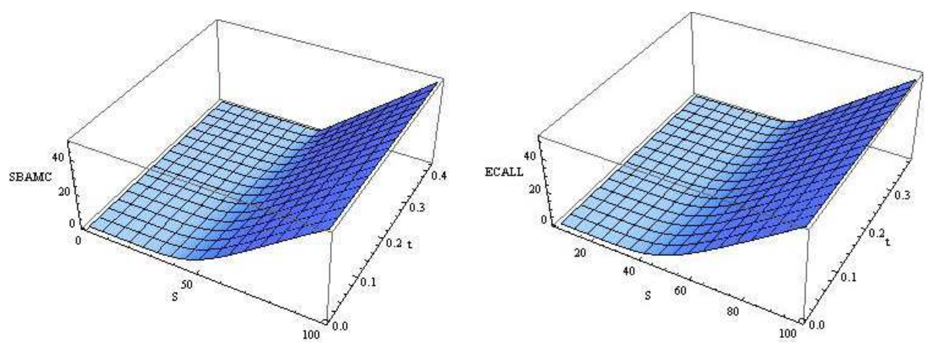

FIGURE 5. Approximate and exact solutions for the call option for Example 4.1 with $h = 1/12$, $\Delta S = 20$

Example 4.2. As our second test example, we solve the stiff parabolic equation (see Cash[1])

$$\frac{\partial V}{\partial t} = \kappa \frac{\partial^2 V}{\partial S^2}$$

$$u(0,t) = u(1,t), \quad u(S,0) = \sin \pi S + \sin \omega \pi S, \quad \omega \gg 1.$$

The exact solution $V(S,t) = e^{-\pi^2 \kappa t} \sin \pi S + e^{-\omega^2 \pi^2 \kappa t} \sin \omega \pi S$.

Cash [1] notes that as ω increases, equations of the type given in Example 4.2 exhibit characteristics similar to model stiff equations. Hence, the methods such as the Crank-Nicolson method which are not L_0-stable are expected to perform poorly. However, we found that the SBAM is not L_0-stable and performs well when applied to this problem. In Table 1, we display the results for $\kappa = 1$ and a range of values for ω.

ω	SBAM
1	3.23×10^{-6}
2	1.85×10^{-5}
10	1.62×10^{-6}
20	1.62×10^{-6}
30	1.62×10^{-6}
40	1.62×10^{-6}

Table 1. Errors of the method for Example 4.2 at $t = 1$ and $\Delta S = 1/10$, $h = 1/12$

To test for convergence, Example 4.2 was solved for various values of $h = \Delta S$ and the results for the global maximum absolute errors ($\text{Err} = \max |v_m(t_n) - V(S_m, t_n)|$) are reproduced in Table 2. We also give the rate of convergence (ROC) which is calculated using the formula $\text{ROC} = \log_2(\text{Err}^{2h}/\text{Err}^h)$, Err^h is the error obtained using the step size h. In general, the ROC shows that the order of the method is slightly greater than 2. This is expected since the central difference method used for the spatial discretization is of order 2 and hence affects the convergence of the SBAM which is of order 4 with respect to the time variable. In Figure 6, the solutions obtained using the SBAM are plotted versus S and t and are in good agreement with the plots given by the exact solution. Furthermore, In Figure 7, the errors (residuals) at all the grid points are generally small when the residuals are plotted versus S and t.

$N = M$	Err	ROC
4	1.15×10^{-1}	
8	1.92×10^{-2}	2.6
16	1.62×10^{-3}	2.5
32	6.82×10^{-4}	2.4
64	1.48×10^{-4}	2.2
128	3.67×10^{-5}	2.0
256	9.23×10^{-6}	2.0

Table 2. Err and ROC for Example 4.2

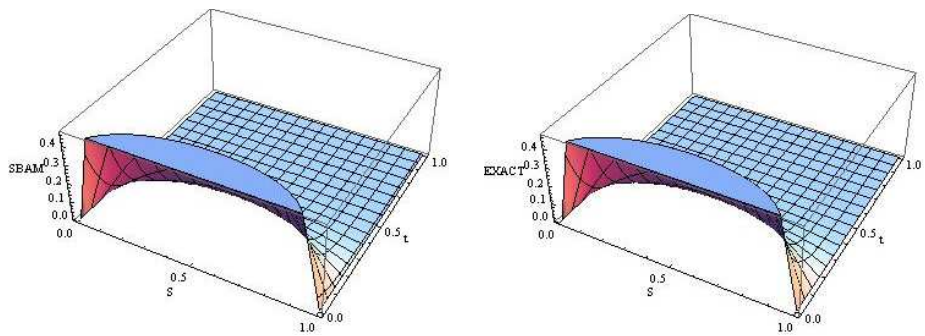

Figure 6. Approximate and exact solutions for Example 4.2 with $h = 1/128$, $\Delta S = 1/128$

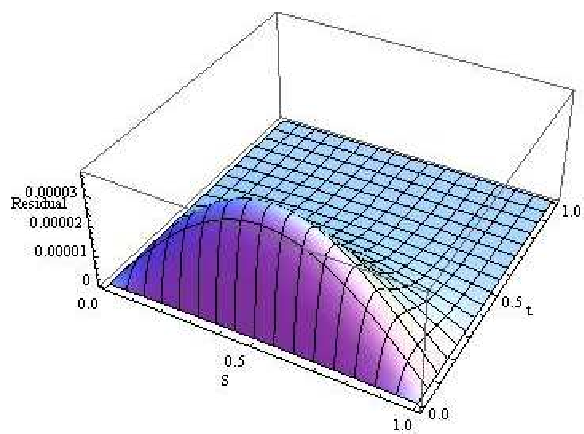

Figure 7. Residuals for $S = t = 1/128$ for Example 4.2

Conclusion. An SBAM with a block extension for solving the the Black-Scholes differential equation has been proposed. The method is shown to give oscillation-free solutions as illustrated in Figures 2–5. The method is also shown to be convergent and capable of solving stiff problems (see Table 2.

References

[1] J. R. Cash; *Two new finite difference schemes for parabolic equations*, SIAM J. Numer. Anal., 21 (1984) 433-446.

[2] M. M. Chawla, M. A. Al-Zanaidia, D. J. Evans; *Generalized Trapezoidal Formulas for the Black-Scholes Equation of Option Pricing*, International Journal of Computer Mathematics, 80 (2003) 1521-1526.

[3] J. C. Hull; *Options, Futures, and Other Derivatives, Fourth Edition*, Prentice Hall, Upper Saddle River, NJ, 2000.

[4] M. K. Jain, T. Aziz; *Cubic spline solution of two-point boundary Value with signifigant first derivatives*, Comp. Meth. Appl. Mech. Eng., 39 (1983) 83-91.

[5] S.N. Jator, S. Swindle, R. French; *Trigonometrically fitted block Numerov type method for* $y'' = f(x, y, y')$, Numer Algor., (2012): DOI 10.1007/s11075-012-9562-1.

[6] S. N. Jator, J. Li; *An algorithm for second order initial and boundary value problems with an automatic error estimate based on a third derivative method*, Numerical Algorithms, 59 (2012) 333-346.

[7] L. W. Johnson, R. D. Riess; *Numerical Analysis* (second edition), Addison-Wesley, Massachusetts, USA, 1982.

[8] A. Q. M. Khaliq, D. A. Voss, S. H. K. Kazmi; *A linearly implicit predictorcorrector scheme for pricing American options using a penalty method approach*, Journal of Banking & Finance, 30 (2006) 489-502.

[9] J. D. Lambert; *Numerical methods for ordinary differential systems,* John Wiley, New York, 1991.

[10] P. Onumanyi, U. W. Sirisena, S. N. Jator; *Continuous finite difference approximations for solving differential equations*, Inter. J. Compt. Maths. 72 (1999) 15-27.

[11] P. Onumanyi, D. O. Awoyemi, S. N. Jator, U. w. Sirisena; *New linear mutlistep methods with continuous coefficients for first order initial value problems*, J. Nig. Math. Soc. 13 (1994) 37-51.

[12] H. Ramos, J. Vigo-Aguiar; *A fourth-order Runge-Kutta method based on BDF-type Chebyschev approximations*, J. Comp. Appl. Math. 204 (2007) 124-136.

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