Ninth MSU-UAB Conference on Differential Equations and Computational Simulations.
Electronic Journal of Differential Equations, Conference 20 (2013), pp. 15–25.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

S-SHAPED BIFURCATION CURVES FOR LOGISTIC GROWTH AND WEAK ALLEE EFFECT GROWTH MODELS WITH GRAZING ON AN INTERIOR PATCH

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ABSTRACT. We study the positive solutions to the steady state reaction diffusion equations with Dirichlet boundary conditions of the form

$$-u'' = \begin{cases} \lambda[u - \frac{1}{K}u^2 - c\frac{u^2}{1+u^2}], & x \in (L, 1-L), \\ \lambda[u - \frac{1}{K}u^2], & x \in (0, L) \cup (1-L, 1), \end{cases}$$
$$u(0) = 0, \quad u(1) = 0$$

and

$$-u'' = \begin{cases} \lambda[u(u+1)(b-u) - c\frac{u^2}{1+u^2}], & x \in (L, 1-L), \\ \lambda[u(u+1)(b-u)], & x \in (0, L) \cup (1-L, 1), \end{cases}$$
$$u(0) = 0, \quad u(1) = 0.$$

Here, λ, b, c, K, and L are positive constants with $0 < L < \frac{1}{2}$. These types of steady state equations occur in population dynamics; the first model describes logistic growth with grazing, and the second model describes weak Allee effect with grazing. In both cases, u is the population density, $\frac{1}{\lambda}$ is the diffusion coefficient, and c is the maximum grazing rate. These models correspond to the case of symmetric grazing on an interior region. Our goal is to study the existence of positive solutions. Previous studies when the grazing was throughout the domain resulted in S-shaped bifurcation curves for certain parameter ranges. Here, we show that such S-shaped bifurcations occur even if the grazing is confined to the interior. We discuss the results via a modified quadrature method and *Mathematica* computations.

1. INTRODUCTION

In [8], the authors studied the nonlinear boundary-value problem

$$\begin{gathered} -\Delta u = \lambda[u - \frac{1}{K}u^2 - c\frac{u^2}{1+u^2}], \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{gathered} \tag{1.1}$$

Here, $\Delta u = \text{div}\,(\nabla u)$ is the Laplacian of u, and Ω is a smooth bounded region with $\partial\Omega \in C^2$. Also, λ, K, and c are positive constants where u is the population density

2000 *Mathematics Subject Classification.* 34B18.
Key words and phrases. Grazing on an interior patch; positive solutions;
S-shaped bifurcation curves.

Published October 31, 2013.

and $\frac{1}{\lambda}$ is the diffusion coefficient. The term $u - \frac{1}{K}u^2$ represents logistic growth; this means that as a given population grows, its per capita growth rate declines linearly (see [9] and [10]).

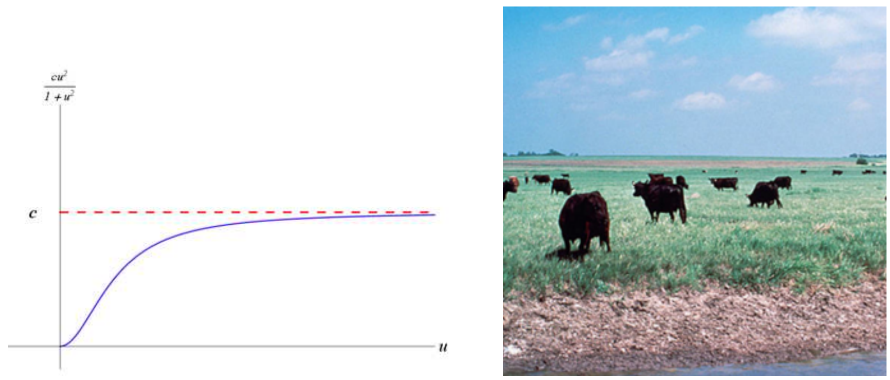


FIGURE 1. Grazing

It is interesting to study natural phenomena affecting the population such as grazing. Here, the term $c\frac{u^2}{1+u^2}$ corresponds to the grazing rate by a fixed number of grazers on the population, where the coefficient c is the maximum grazing rate (see Figure 1). This type of model can be used to describe several ecological systems such as the dynamics of fish (see [9] and [13]) and spruce budworm populations (see [14] and Figure 2).

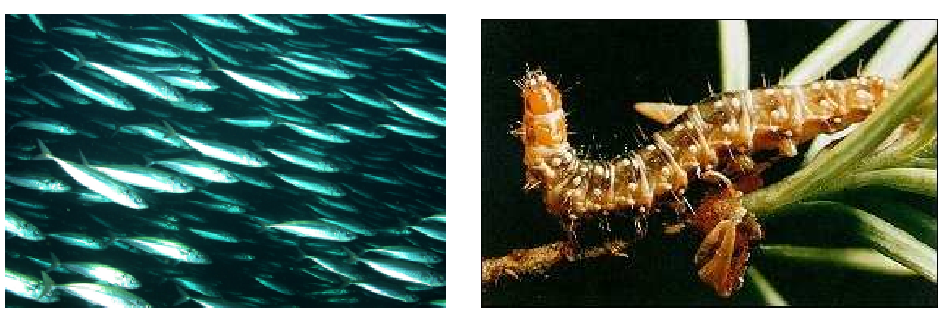


FIGURE 2. Examples of Fish and Spruce Budworms

The authors in [8] proved the existence of at least one positive solution for all $\lambda > \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the principal eigenvalue of the operator $-\Delta$ with Dirichlet boundary conditions. Also, the authors discuss the existence of at least three positive solutions for certain ranges of λ.

In [11], the authors studied the one-dimensional reaction diffusion model:

$$\begin{gathered} -u'' = \lambda[u(u+1)(b-u) - c\frac{u^2}{1+u^2}], \quad x \in (0,1), \\ u(0) = 0, \quad u(1) = 0. \end{gathered} \tag{1.2}$$

Here, λ, b, and c are all positive parameters where u is the population density and $\frac{1}{\lambda}$ is the diffusion coefficient. The term $u(u+1)(b-u)$ represents weak Allee effect (see [1] and [12]). Under a weak Allee effect, for small populations the per capita growth begins positive and initially increases. This differs from logistic growth whose per capita growth rate is decreasing. The initial increase in population growth can be caused by a number of factors such as shortage of mates or predator saturation. Two examples of populations that experience weak Allee effect are the apple snail and the smooth cordgrass plant (see [3] and [6] and Figure 3). Furthermore, the term $c\frac{u^2}{1+u^2}$ is the same grazing rate described previously.

Figure 3. Examples of an Apple Snail Shell with Eggs and Smooth Cordgrass

The authors in [11] were able to show the evolution of the bifurcation curve over a range of c-values, for a fixed value of b. In particular, they discuss S-shaped bifurcation curves for certain ranges of b and c.

We are interested in extending some of the results in [8] and [11] regarding the S-shaped bifurcation curve in the one dimensional case when the grazing is confined to an interior patch. Previous studies have been done examining population dynamics on split domains; by a split domain, we mean that phenomena such as grazing or harvesting are only allowed on part of the domain. In [2], the authors studied logistic growth with constant yield harvesting in both the symmetric and asymmetric cases. We are interested in pursuing a similar study by analyzing models which describe grazing of a fixed number of grazers within the interior of the domain on a logistically growing species and a species subject to weak Allee effect, respectively.

We study the positive solutions to the steady state reaction diffusion equations with Dirichlet boundary conditions of the form

$$
\begin{gathered}
-u'' = \begin{cases} \lambda[u - \frac{1}{K}u^2 - c\frac{u^2}{1+u^2}], & x \in (L, 1-L), \\ \lambda[u - \frac{1}{K}u^2], & x \in (0, L) \cup (1-L, 1), \end{cases} \\
u(0) = 0, \quad u(1) = 0
\end{gathered}
\tag{1.3}
$$

and

$$
\begin{gathered}
-u'' = \begin{cases} \lambda[u(u+1)(b-u) - c\frac{u^2}{1+u^2}], & x \in (L, 1-L), \\ \lambda[u(u+1)(b-u)], & x \in (0, L) \cup (1-L, 1), \end{cases} \\
u(0) = 0, \quad u(1) = 0.
\end{gathered}
\tag{1.4}
$$

Here, λ, b, c, K, and L are positive constants with $0 < L < \frac{1}{2}$. These models correspond to the case of symmetric grazing on an interior region. We follow the ideas used in [2], that is, modify the quadrature method discussed in [7] to analyze these problems.

In Section 2, we will discuss the quadrature method in the case of a split domain. Section 3 is concerned with presenting our computational results via *Mathematica* for the logistic growth model. In Section 4, we will discuss similar results for the weak Allee effect model.

2. Preliminaries

We consider a general model of the form

$$-u'' = \begin{cases} \lambda \tilde{f}(u), & x \in (L, 1-L), \\ \lambda f(u), & x \in (0, L) \cup (1-L, 1), \end{cases} \tag{2.1}$$
$$u(0) = 0, \quad u(1) = 0.$$

We will study solutions u that are symmetric about $x = 1/2$ such that $u(L^-) = u(L^+)$ and $u'(L^-) = u'(L^+)$ where $\|u\|_\infty = \rho$ and $\sigma(\rho) = u(L)$. A typical solution to (2.1) can be seen in Figure 4.

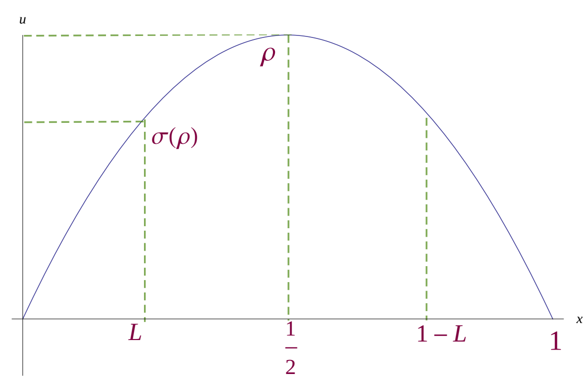


Figure 4. Typical Solution u

First, we will focus on the region $(L, \frac{1}{2})$. Suppose u is a positive solution to (2.1). On the interval $(L, \frac{1}{2})$, we have $-u'' = \lambda \tilde{f}(u)$.

Multiplying by u', integrating, and using the fact that $u'(1/2) = 0$ and $u(1/2) = \rho$, we obtain

$$u'(x) = \sqrt{2\lambda[\tilde{F}(\rho) - \tilde{F}(u)]} \quad L < x \le \frac{1}{2}$$

where $\tilde{F}(s) := \int_0^s \tilde{f}(t)dt$. Integrating again, we see that on the interval $(L, \frac{1}{2})$ solutions will satisfy

$$\int_{u(x)}^{\rho} \frac{1}{\sqrt{\tilde{F}(\rho) - \tilde{F}(v)}} dv = \sqrt{2\lambda}[\frac{1}{2} - x].$$

Let $\sigma := u(L)$. Then, if we evaluate the equation above at $x = L$, we have

$$\int_\sigma^\rho \frac{1}{\sqrt{\tilde{F}(\rho) - \tilde{F}(v)}} dv = \sqrt{2\lambda}[\frac{1}{2} - L].$$

Simplifying we obtain

$$\lambda = \Big[\frac{1}{\sqrt{2}(\frac{1}{2} - L)} \int_\sigma^\rho \frac{1}{\sqrt{\tilde{F}(\rho) - \tilde{F}(v)}} dv\Big]^2 =: G_1(\sigma, \rho). \tag{2.2}$$

Next, we consider the region $(0, L)$. On the interval $(0, L)$, we have $-u'' = \lambda f(u)$. If we use similar calculations as before and if $u'(0) = m$, then we obtain

$$u'(x) = \sqrt{2}\sqrt{\frac{m^2}{2} - \lambda F(u(x))}.$$

Integrating again we obtain

$$\int_0^{u(x)} \frac{1}{\sqrt{\frac{m^2}{2} - \lambda F(v)}} dv = \sqrt{2}x.$$

Letting $\sigma = u(L)$, we find that

$$\int_0^\sigma \frac{1}{\sqrt{\frac{m^2}{2} - \lambda F(v)}} dv = \sqrt{2}L. \tag{2.3}$$

Recall $u'(L^-) = u'(L^+)$ which implies

$$\frac{m^2}{2} = \lambda[\tilde{F}(\rho) - \tilde{F}(\sigma) + F(\sigma)].$$

Substituting the above into (2.3), we obtain

$$\lambda = \Big[\frac{1}{\sqrt{2}L} \int_0^\sigma \frac{1}{\sqrt{\tilde{F}(\rho) - \tilde{F}(\sigma) + F(\sigma) - F(v)}} dv\Big]^2 =: G_2(\sigma, \rho). \tag{2.4}$$

Therefore,

$$G_1(\sigma, \rho) = G_2(\sigma, \rho).$$

Thus, if u is a solution of (2.1) with $u(L) = \sigma(\rho)$ and $\|u\|_\infty = \rho$, then ρ and σ must satisfy $G_1(\sigma, \rho) = \lambda$ and $G_2(\sigma, \rho) = \lambda$.

Suppose that given a ρ, there exists $\sigma(\rho)$ such that $G_1(\sigma(\rho), \rho) = G_2(\sigma(\rho), \rho)$. Then

$$\begin{aligned} \sqrt{\lambda} &= \frac{1}{\sqrt{2}(\frac{1}{2} - L)} \int_{\sigma(\rho)}^\rho \frac{1}{\sqrt{\tilde{F}(\rho) - \tilde{F}(v)}} dv \\ &= \frac{1}{\sqrt{2}L} \int_0^{\sigma(\rho)} \frac{1}{\sqrt{\tilde{F}(\rho) - \tilde{F}(\sigma(\rho)) + F(\sigma(\rho)) - F(v)}} dv. \end{aligned} \tag{2.5}$$

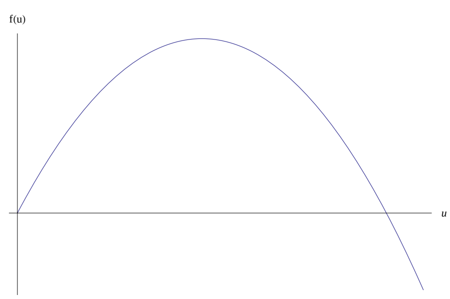
In fact, given λ and ρ such that (2.5) is satisfied where $\sigma(\rho)$ satisfies $G_1(\sigma, \rho) = G_2(\sigma, \rho)$, we can back track and use the Implicit Function Theorem to obtain a solution of the form seen in Figure 4. Further, $\lambda = G_1(\sigma(\rho), \rho)$ (or $\lambda = G_2(\sigma(\rho), \rho)$) provides the bifurcation diagram for these positive solutions. The following theorem summarizes the above discussion:

Theorem 2.1. *Let $\rho > 0$ and $\sigma(\rho) \in (0, \rho)$ be such that $G_1(\rho, \sigma) = G_2(\rho, \sigma)$. Then, (2.1) has a positive solution symmetric about $x = \frac{1}{2}$ with $\|u\|_\infty = \rho$ and $u(L) = \sigma(\rho)$ if and only if $\sqrt{\lambda} = G_1(\rho, \sigma(\rho))$ $(= G_2(\rho, \sigma(\rho)))$.*

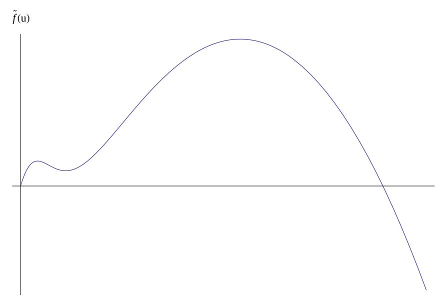
3. Results for logistic growth with grazing

In this section, we consider a model with logistic growth with grazing only in an interior region (see Figure 5),

$$-u'' = \begin{cases} \lambda \tilde{f}(u) = \lambda[u - \frac{1}{K}u^2 - c\frac{u^2}{1+u^2}], & x \in (L, 1-L), \\ \lambda f(u) = \lambda[u - \frac{1}{K}u^2], & x \in (0, L) \cup (1-L, 1), \end{cases} \tag{3.1}$$
$$u(0) = 0, \quad u(1) = 0.$$



(a) $f(u) = u - \frac{1}{K}u^2$



(b) $\tilde{f}(u) = u - \frac{1}{K}u^2 - c\frac{u^2}{1+u^2}$

Figure 5. Logistic growth with grazing only in an interior region

We know from [8] that for certain parameter ranges the authors were able to establish the occurrence of an S-shaped bifurcation curve for (1.1) when grazing was allowed over the entire domain (case when $L = 0$). In fact, they found that given $c < 2$ then for $K >> 1$ the bifurcation curve for model (1.1) will be S-shaped (see Figure 6). That is, there exist $m_1, m_2, m_3 > 0$ such that (1.1) has:

- no positive solution for $\lambda \in (0, m_1]$
- exactly one positive solution for $\lambda \in (m_1, m_2)$
- exactly two positive solutions for $\lambda = m_2$
- exactly three positive solutions for $\lambda \in (m_2, m_3)$
- exactly two positive solutions for $\lambda = m_3$
- exactly one positive solution for $\lambda \in (m_3, \infty)$

We hope to obtain similar results even when grazing is restricted to an interior patch. To plot the bifurcation curve for our model, we use *Mathematica*. For a fixed K, c, and L, we input a ρ value and use *Mathematica* to solve for the value of σ where $G_1(\sigma, \rho) = G_2(\sigma, \rho)$.

Given a ρ, does there exist a unique $\sigma = \sigma(\rho)$ such that $G_1(\sigma, \rho) = G_2(\sigma, \rho)$? From our computations, we observe that for any given ρ-value there will be a unique σ (see Figure 7). Our calculations imply that G_1 and G_2 will intersect only once for any given ρ-value; the value where the intersection occurs is σ.

Once we find σ, we substitute the values of σ and ρ back into either G_1 or G_2. Thus, for a given ρ, we can find the corresponding value of λ which we then use to plot the bifurcation curve. From our *Mathematica* computations, we observe that

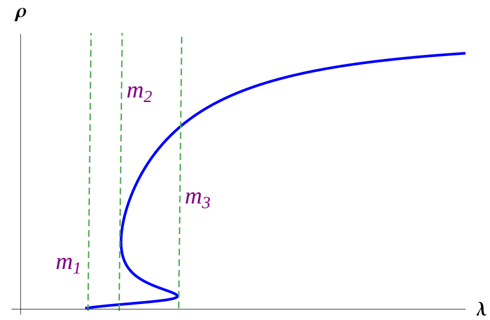


FIGURE 6. Symmetric grazing: λ versus ρ

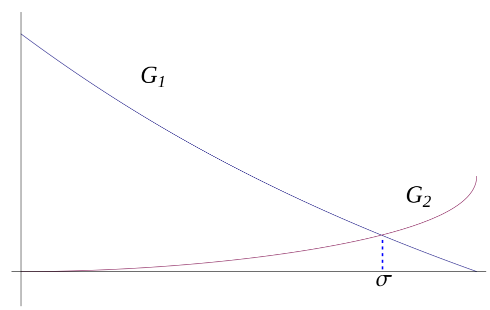
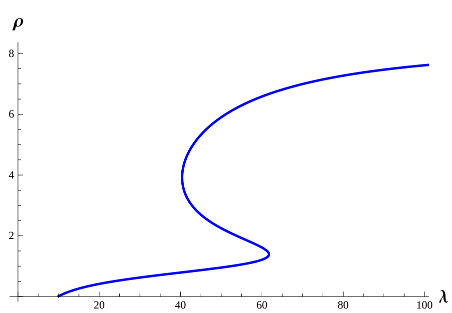


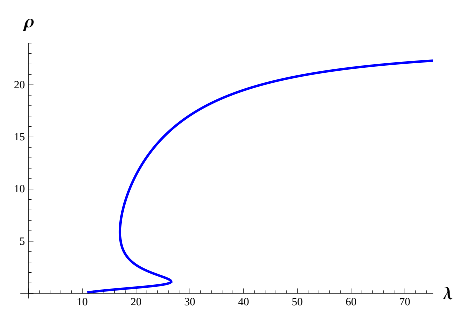
FIGURE 7. Graph of G_1 and G_2

even when the grazing is confined to a small interior patch an S-shaped bifurcation curve similar to the results from [8] still occurs.

For example, we fix $c = 1.5$, and we let $L = .05, .25, .4$, and $.4999$ to show the evolution of the bifurcation curve as grazing is restricted to a smaller interior region. For each case, we find $K >> 1$ such that an S-shape bifurcation curve occurs (see Figures 8 and 9).

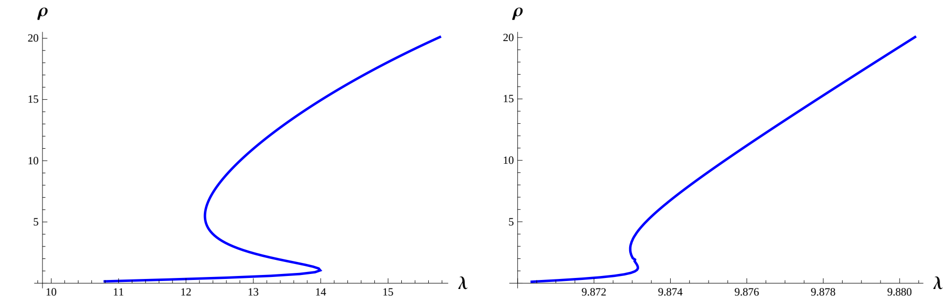


(a) $K = 10$, $c = 1.5$, and $L = .05$



(b) $K = 25$, $c = 1.5$, and $L = .25$

FIGURE 8. Asymmetric grazing: λ versus ρ



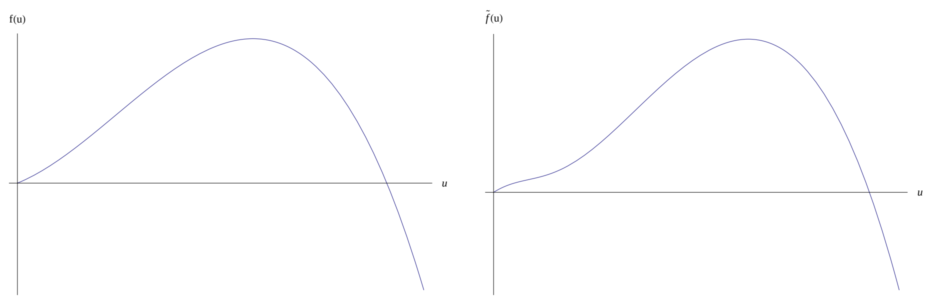
(a) $K = 50$, $c = 1.5$, and $L = .40$ (b) $K = 16000$, $c = 1.5$, and $L = .4999$

Figure 9. Asymmetric grazing: λ versus ρ

4. Results for weak Allee effect with grazing

We will now consider a model with weak Allee effect with grazing only in an interior region (see Figure 10):

$$-u'' = \begin{cases} \lambda\tilde{f}(u) = \lambda[u(u+1)(b-u) - c\frac{u^2}{1+u^2}], & x \in (L, 1-L), \\ \lambda f(u) = \lambda[u(u+1)(b-u)], & x \in (0, L) \cup (1-L, 1), \end{cases} \tag{4.1}$$
$$u(0) = 0, \quad u(1) = 0.$$



(a) $f(u) = u(u+1)(b-u)$ (b) $\tilde{f}(u) = u(u+1)(b-u) - c\frac{u^2}{1+u^2}$

Figure 10. Weak Allee effect with grazing only in an interior region

In [11], the authors found the occurrence of an S-shaped bifurcation curve for certain parameter ranges when grazing was through out the entire domain (case when $L = 0$). Specifically, they noted that if $b > b_0$ (some) and $c \in (b-1, c_0$ (some)) then the bifurcation curve for model (1.1) will be S-shaped (see Figure 11). That is, there exist $m_1, m_2, m_3 > 0$ such that (1.1) has

- no positive solution for $\lambda \in (0, m_1)$
- exactly one positive solution for $\lambda = m_1$

- exactly two positive solutions for $\lambda = (m_1, m_2]$
- exactly three positive solutions for $\lambda \in (m_2, m_3)$
- exactly two positive solutions for $\lambda = m_3$
- exactly one positive solution for $\lambda \in (m_3, \infty)$

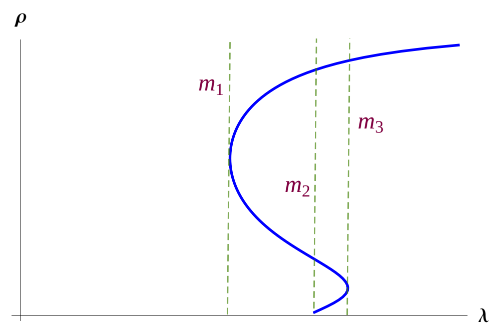


FIGURE 11. Symmetric grazing: λ versus ρ

The bifurcation curve for model (1.1) can be calculated using a similar method as in Section 4. For a given ρ, we must find a σ such that $G_1(\sigma, \rho) = G_2(\sigma, \rho)$; then, substituting ρ and σ back into G_1 or G_2 will give us λ. We use this information to then plot the bifurcation curve of λ versus ρ with *Mathematica*. As we observed with the logistic model, we have computational evidence that for a given ρ there will be a unique σ value (see Figure 12).

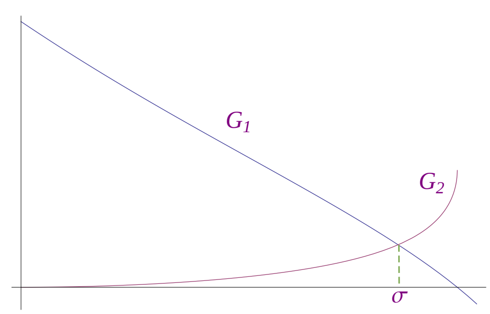


FIGURE 12. Graph of G_1 and G_2

Based on our observations using *Mathematica*, we find that even when the domain is split and grazing is only allowed within the interior an S-shaped bifurcation curve similar to the results from [11] will still occur.

For instance, we fix $b = 5$, and we let $L = .01, .15, .25$, and $.40$ to show the evolution of the bifurcation curve as grazing is restricted to a smaller interior patch. In each case, we find $c \in (b - 1, c_0(\text{some}))$ such that an S-shape bifurcation curve occurs (see Figures 13 and 14).

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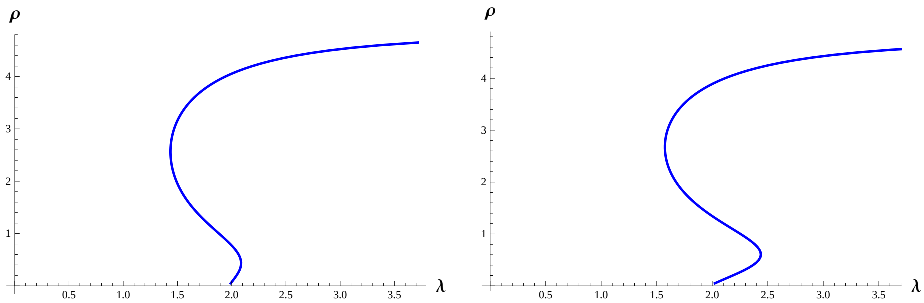
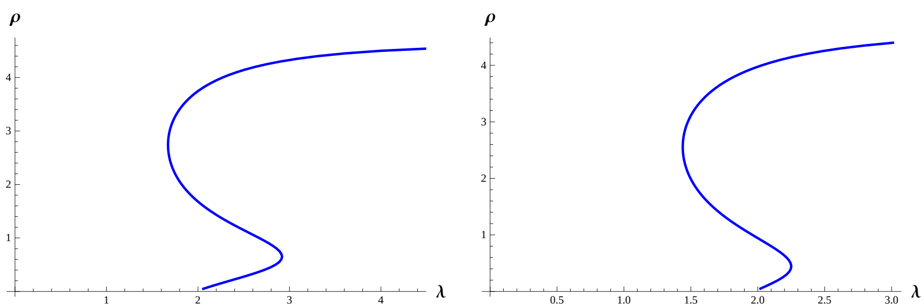


(a) $b = 5$, $c = 5$, and $L = .01$ (b) $b = 5$, $c = 7$, and $L = .15$

FIGURE 13. Asymmetric grazing: λ versus ρ



(a) $b = 5$, $c = 10$, and $L = .25$ (b) $b = 5$, $c = 15$, and $L = .40$

FIGURE 14. Asymmetric grazing: λ versus ρ

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