Eighth Mississippi State - UAB Conference on Differential Equations and Computational Simulations. *Electronic Journal of Differential Equations*, Conf. 19 (2010), pp. 257–266.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

LIPSCHITZ CONSTANTS FOR POSITIVE SOLUTIONS OF SECOND-ORDER ELLIPTIC EQUATIONS

ANTONIO VITOLO

Abstract. We are concerned with positive solutions of second order fully nonlinear elliptic equations. Here we present Lipschitz estimates, in the viscosity setting, and bounds for optimal constants.

1. Introduction and statement of the results

In a series of papers [6, 7, 8], jointly with Italo Capuzzo Dolcetta, the author proved the inequalities

$$C^{-1}|Du(x)| \le \begin{cases} \sqrt{u(0)M} & \text{if } 2|x| \le \sqrt{\frac{2u(0)}{M}} \le R \\ \frac{u(0)}{R} + MR & \text{if } 2|x| \le R \le \sqrt{\frac{2u(0)}{M}}\,, \end{cases} \tag{1.1}$$

for a positive constant C, when $u \ge 0$ and

$$|F(x, Du(x), D^2u(x))| \le M$$

in the ball B_R of radius $R > 0$ centered at the origin.

Here F is a second-order uniformly elliptic operator satisfying suitable assumptions, that will be made precise in the sequel.

The estimate (1.1) extends the Glaeser's inequality [10],

$$|u'(0)| \le \begin{cases} \sqrt{2u(0)M} & \text{if } R \ge \sqrt{\frac{2u(0)}{M}} \\ \frac{u(0)}{R} + MR & \text{if } R < \sqrt{\frac{2u(0)}{M}} \end{cases} \tag{1.2}$$

for non-negative C^2-functions u with $|u''| \le M$ in $[-R, R]$, which is a local version of the Landau inequality [13]

$$|u'(0)| \le \sqrt{2 \sup_{\mathbb{R}} |u|\, M} \tag{1.3}$$

for C^2-functions u such that $|u''| \le M$ on the whole real axis, see [15] and [16] for other variants. Applications of this kind of inequalities can be found for instance in [17] and [4].

2000 *Mathematics Subject Classification.* 35J60, 45M20, 49L25.
Key words and phrases. Fully nonlinear second order elliptic equations; positive solutions; viscosity solutions.
©2010 Texas State University - San Marcos.
Published September 25, 2010.

In the case of a linear second-order uniformly elliptic operator

$$Lu := a_{ij}(x)D_{ij}u + b_i(x)D_iu$$

the inequalities (1.1) for non-negative functions $u \in C^2(\overline{B}_R)$ such that

$$|Lu| \le M$$

are due to Yan Yan Li and Louis Nirenberg [14]. They also observe that $C = \sqrt{2}$ is optimal for $n = 1$ in the first of (1.1) and the best constant in higher dimensions is not known even for the Laplace operator.

Concerning this, denote by B_R the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^n$ and consider the set $\mathfrak{F}_n$ of all pairs (u, B_R) such that $u \in C^2(\overline{B}_R)$ and

$$u \ge 0 \quad \text{in } B_R, \quad \sup_{B_R} |\Delta u| := M_R \ge \frac{2u(0)}{R^2}\,.$$

For $(u, B_R) \in \mathfrak{F}_n$ let $C_u(R)$ be the greatest lower bound of the positive real numbers C such that

$$|Du(0)| \le C\sqrt{u(0)M_R}\,. \tag{1.4}$$

If $u(0) = 0$, then u has a local mininum at $x = 0$ so that $Du(0) = 0$ and $C_u(R) = 0$. The same happens if $M_R = 0$. Otherwise

$$C_u(R) = \frac{|Du(0)|}{\sqrt{u(0)\,M_R}}\,.$$

Denote by $\overline{C}_n$ the optimal constant (in the class $\mathfrak{F}_n$)

$$\overline{C}_n := \sup_{(u,B_R)\in\mathfrak{F}_n} C_u(R)\,.$$

Note that, if $(u, B_R) \in \mathfrak{F}_n$ is such that $M_R > \frac{2u(0)}{R^2}$, by continuity we find $r \in (0, R)$ such that $M_r = \frac{2u(0)}{r^2}$. Since obviously $(u, B_r) \in \mathfrak{F}_n$ and $C_u(r) \ge C_u(R)$, then $\overline{C}_n$ can be computed as the least upper bound of $C_u(R)$ over all the pairs $(u, B_R) \in \mathfrak{F}_n$ such that

$$\sup_{B_R} |\Delta u| := M_R = \frac{2u(0)}{R^2}\,.$$

Note also that we can also include in $\mathfrak{F}_2$ the limit case $u \ge 0$ in $B_\infty = \mathbb{R}$ and $M_\infty = 0$. By the Liouville theorem we shall have $C_u(\infty) = 0$.

According to the above notations, the quoted result of Li and Nirenberg [14] can be reformulated saying that

$$\overline{C}_1 = \sqrt{2}.$$

The following Example shows that for $n \ge 2$ we have instead

$$\overline{C}_n \ge 3/2.$$

Example 1.1. For the polynomial

$$u(x, y) = x^2 - \frac{1}{5}\,y^2 - 2x + \frac{10}{9},$$

we have $\Delta u = 8/5$ and $u \ge 0$ in B_R with $R = \frac{5}{6}\sqrt{2}$. Therefore, $u \in \mathfrak{F}_2$ and

$$C_u(R) = \frac{|Du(0, 0)|}{\sqrt{u(0, 0)\,\Delta u}} = \frac{3}{2}\,.$$

In Section 3 we see that this is the optimal constant in the class $\mathfrak{F}_{2,2}$ of the pairs $(u, B_R) \in \mathfrak{F}_2$ such that u is a polynomial of degree ≤ 2 in B_R, i.e.

$$\sup_{(u,B_R)\in\mathfrak{F}_{2,2}} C_u(R) = 3/2.$$

From the qualitative viewpoint we will consider fully nonlinear second-order uniformly elliptic operators

$$F(x, Du, D^2u)$$

with at most linear growth in the gradient, see the definition in the next Section. Here Du and D^2u denote the gradient vector and the Hessian matrix of u.

This is the case of linear second-order uniformly elliptic operators

$$Lu = \mathrm{Tr}(A(x)D^2u) + b_i(x)D_iu\,,$$

where the $A(x) := [a_{ij}(x)]$ has eigenvalues in $[\lambda, \Lambda]$ and $|b_i(x)| \leq b_0$ for positive constants $\lambda \leq \Lambda$ (ellipticity constants) and b_0 (first order constant).

Different examples of fully nonlinear elliptic operators are the upper and lower envelopes of linear uniformly elliptic operators L_k or L_{hk} with (positive) ellipticity constants λ and Λ $(\geq \lambda)$, for instance Bellman operators

$$F(x, Du, D^2u) := \inf_k L_ku$$

and Isaacs operators

$$F(x, Du, D^2u) := \sup_h \inf_k L_{hk}u\,,$$

arising in optimal control problems and differential games. Taking the upper and lower envelope of the totality of linear uniformly elliptic operators with (positive) ellipticity constants Λ and Λ $(\geq \lambda)$, we obtain the maximal and minimal Pucci operators $\mathcal{P}^+_{\lambda,\Lambda}(D^2u)$ and $\mathcal{P}^+_{\lambda,\Lambda}(D^2u)$, such that

$$\mathcal{P}^+_{\lambda,\Lambda}(Z) := \Lambda\,\mathrm{Tr}(Z^+) - \lambda\,\mathrm{Tr}(Z^-)\,,$$
$$\mathcal{P}^-_{\lambda,\Lambda}(Z) := \lambda\,\mathrm{Tr}(Z^+) - \Lambda\,\mathrm{Tr}(Z^-)$$

for all $n\times n$ real symmetric matrices Z. Note that $\mathcal{P}^{\pm}_{1,1}(D^2u)$ is the Laplace operator Δu. For more examples and results about elliptic differential operators we refer to [9] and [3].

In [7] and [8], based on the perturbation method of Caffarelli [2], qualitative Glaeser's type results in B_R are obtained by restraining the oscillations of the main term with respect to the x-variable. There, letting

$$\beta(x, y) := \sup_{X\neq 0} \frac{|F(x, 0, X) - F(y, 0, X)|}{|X|}\,,$$

it is required, for some $\tau \in (0, \frac{1}{2})$,

$$\sup_{y\in B_{R/2}} \Big(⨍_{B_{\tau R}(y)} |\beta(x, y)|^n\,dx\Big)^{1/n} \leq \theta \tag{1.5}$$

with a suitably small positive constant θ.

Remark 1.2. The integral condition (1.5) allows a generalization the Cordes-Nirenberg type estimates here below.

Let L be a linear second-order uniformly elliptic operator with ellipticity constants λ, Λ and first order constant b_0, and $f \in L^\infty(B_1)$. For any $\alpha \in (0,1)$ there exists θ such that, if

$$|a_{ij}(x) - a_{ij}(y)| \le \theta, \quad x, y \in \overline{B}_{1/2}, \quad |x-y| < r_0 \quad (< \tfrac{1}{2}),$$

then a bounded solution u of the equation

$$Lu = f(x) \quad \text{in } B_1$$

is $C^{1,\alpha}(\overline{B}_{1/2})$ and

$$\|u\|_{C^{1,\alpha}(\overline{B}_{1/2})} \le C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}),$$

see [2]. This is the case of continuous coefficients $a_{ij}(x)$ with a sufficiently (uniformly) small modulus of continuity.

The integral condition (1.5) is used by Caffarelli [2] to have $C^{1,\alpha}$ estimates for viscosity solutions, see also [19], [18] and [8]. Suppose now that

$$\beta(x,y) = \mu(|x-y|),$$

where $\mu : [0, 2R] \to [0, +\infty)$. Combined with the uniform ellipticity, from which (recall that $F(x,0,0) = 0$)

$$\begin{aligned} \beta(x,y) &\le \frac{|F(x,0,X) - F(x,0,0)|}{|X|} + \frac{|F(y,0,0) - F(y,0,X)|}{|X|} \\ &\le 2\,\frac{\mathcal{P}^+_{\lambda,\Lambda}(X)}{|X|} \le 2\Lambda, \end{aligned}$$

this yields

$$⨍_{B_{r_0}(y)} \beta(x,y)^n \, dx \le n\, 2^{n-1} \Lambda^{n-1} \int_0^{r_0} \frac{\mu(\sigma)}{\sigma} \, d\sigma.$$

for $r_0 \in (0, R)$.

Hence, according to condition (1.5), in order to have $C^{1,\alpha}$-estimates we should require the latter integral to be small.

However, if we only ask for Lipschitz estimates, then by Ishii-Lions [11] we may just assume that integral to be finite.

Theorem 1.3. *Let F be uniformly elliptic with ellipticity constants $\lambda > 0$, $\Lambda \ge \lambda$ and first order constant $b_0 > 0$. Suppose that F is continuous, $F(x,0,0) = 0$ and*

$$|F(x,\xi,X) - F(y,\xi,X)| \le \mu(|x-y|)|X| + \omega(|x-y|)|\xi| \tag{1.6}$$

for $x, y \in B_R$, $\xi \in \mathbb{R}^n$, $X \in \mathcal{S}^n$, where μ, ω are non-negative real functions such that

$$\int_0^{2R} \frac{\mu(\sigma)}{\sigma} \, d\sigma < +\infty$$

and $\lim_{\sigma \to 0^+} \omega(\sigma) = 0$. Let $u \in C(\overline{B}_R)$ be a viscosity solution of the equation

$$F(x, Du, D^2u) = f(x)$$

in B_R with $f \in C(B_R)$ such that $\|f\|_{L^\infty(B_R)} = M < +\infty$.

Then u is locally Lipschitz continuous in B_R. If in addition we assume that $u \geq 0$, then there exists a positive constant $C = C(n, \lambda, \Lambda, b_0 R, \mu, \omega)$ such that

$$C^{-1} \limsup_{y \to x} \frac{|u(x) - u(y)|}{|x - y|} \leq \begin{cases} \sqrt{u(0)M} & \text{if } 2|x| \leq \sqrt{\frac{2u(0)}{M}} \leq R \\ \frac{u(0)}{R} + MR & \text{if } 2|x| \leq R \leq \sqrt{\frac{2u(0)}{M}}\,. \end{cases} \tag{1.7}$$

Remark 1.4. By Rademacher's Theorem it follows from Theorem 1.3 that the inequalities (1.1) hold true almost everywhere for $x \in \overline{B}_{R/2}$ with a positive constant C depending on $n, \lambda, \Lambda, b_0 R, \mu, \omega$, and everywhere if u is differentiable in $\overline{B}_{R/2}$.

This Theorem is based on a result of Ishii-Lions [11, Theorem VII.2] which yields the Lipschitz continuity of viscosity solutions under assumptions which are guaranteed by (1.6).

The proof will be given in Section 2 after a few preliminaries about viscosity solutions. In Section 3 we will be concerned with a lower bound for the optimal constant, as mentioned in advance.

2. Lipschitz inequalities for viscosity solutions

We start with the basic notations and definitions. Throughout the paper $B_r(y)$ will be a ball of radius $r > 0$ centered at $y \in \mathbb{R}^n$ and $B_r := B_r(0)$. By $\mathcal{S}^n$ we denote the set of $n \times n$ real symmetric matrices endowed with the partial ordering induced by semidefinite positiveness.

Let Ω be a domain (open connected set) of $\mathbb{R}^n$, then $F : \Omega \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ is said uniformly elliptic in Ω with ellipticity constants $\lambda > 0$ and $\Lambda \geq \lambda$ and first order constant $b_0 > 0$ if

$$\mathcal{P}^-_{\lambda,\Lambda}(Z) - b_0|\zeta| \leq F(x, \xi + \zeta, X + Z) - F(x, \xi, X) \leq \mathcal{P}^+_{\lambda,\Lambda}(Z) + b_0|\zeta|$$

for all $x \in \Omega$, $\xi, \zeta \in \mathbb{R}^n$ and $X, Z \in \mathcal{S}^n$.

A viscosity subsolution u of the equation $F(x, Du, D^2u) = f(x)$ is a function $u \in C(\Omega)$ such that for all $(y, \varphi) \in \Omega \times C^2(B_r(y))$ a local maximum for $u - \varphi$ at y implies

$$F(y, D\varphi(y), D^2\varphi(y)) \geq f(y).$$

Similarly for a viscosity supersolution $u \in C(\Omega)$, a local minimum for $u - \varphi$ at y will imply

$$F(y, D\varphi(y), D^2\varphi(y)) \leq f(y).$$

The viscosity solutions are both viscosity subsolutions and supersolutions.
We recall that in the above definition we may equivalently require that the local maximum or minimum is equal zero, so that the graph of the test function φ touches above or below, respectively, that one of the solution u. For a widespread treatment of viscosity solutions we refer to [5] and [3].

Proof of Theorem 1.3. It is sufficient to consider the case $R = 1$, we can use a rescaling argument for arbitrary $R > 0$.
From [11, Theorem VII.2], which provides the Lipschitz continuity of u, we deduce for $\|u\|_{L^\infty(B_{3/4})}, \|f\|_{L^\infty(B_{3/4})} \leq 1$ the inequality

$$|u(x) - u(y)| \leq C|x - y|, \quad x, y \in \overline{B}_{1/2}, \tag{2.1}$$

with a positive constant C depending on n, λ, Λ, b_0, μ and ω. For general u and f, we set

$$K := \|u\|_{L^\infty(B_{3/4})} + \|f\|_{L^\infty(B_{3/4})}$$

and $v = K^{-1}u$. Since $F(x, Du, D^2u) = f(x)$ then

$$G(x, Dv, D^2v) = g(x),$$

where

$$G(x, \xi, X) = K^{-1}F(x, K\xi, KX), \quad g(x) = K^{-1}f(x).$$

Since G satisfies, as F does, the structure conditions of [11, Theorem VII.2] with $\|v\|_{L^\infty(B_{3/4})}, \|g\|_{L^\infty(B_{3/4})} \le 1$, then from (2.1) we have

$$\sup_{x,y\in B_{1/2},\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|} \le C(\|u\|_{L^\infty(B_{3/4})} + \|f\|_{L^\infty(B_{3/4})})\,. \tag{2.2}$$

For $u \ge 0$ in B_1, using the Harnack inequality (see for instance [3, 12, 1]) we get

$$\sup_{x,y\in B_{1/2},\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|} \le C(u(0) + \|f\|_{L^\infty(B_1)}) \tag{2.3}$$

with a possibly different constant $C > 0$. Next, for each point $x_0 \in \overline{B}_{1/2}$ we localize the equation in $B_r(x_0)$ with $r \in (0, \frac{1}{2})$ and rescale setting

$$u(x) = v(r^{-1}(x - x_0)), \quad x \in B_r(x_0).$$

Then $v \in C(\overline{B}_1)$ satisfies the equation

$$G(y, Dv, D^2v) = g(y) \tag{2.4}$$

where

$$G(y, \eta, Y) := r^2F(x_0 + ry, r^{-1}\eta, r^{-2}Y), \quad g(y) = r^2f(x_0 + ry).$$

Note that G is uniformly elliptic in B_1 with elliptic constants λ, Λ and first order constant coefficient b_0r. Moreover G satisfies (1.6) with $R = 1$ and slightly modified μ and ω, namely

$$|G(x, \xi, X) - G(y, \xi, X)| \le \mu(r|x-y|)|X| + r\omega(r|x-y|)|\xi|.$$

So by (2.3) we can infer that

$$\sup_{x,y\in B_{r/2}(x_0),\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|} \le C\Big(\frac{u(x_0)}{r} + r\|f\|_{L^\infty(B_1)}\Big). \tag{2.5}$$

From this, again by the Harnack inequality, it follows that

$$\limsup_{x\to x_0} \frac{|u(x)-u(x_0)|}{|x-x_0|} \le C\Big(\frac{u(0)}{r} + \frac{Mr}{2}\Big), \quad r \le 1\,, \tag{2.6}$$

for a possibly larger constant C. Therefore, minimizing the right-hand side by the choice $r = \sqrt{\frac{2u(0)}{M}}$ if $u(0) \le \frac{M}{2}$, $r = 1$ otherwise, we obtain the result in the case $R = 1$, as it suffices. □

3. Computation for optimal constants

Let u be a C^2-function in B_R and $M_u(x)$ be the maximum modulus of the eigenvalues of $D^2u(x)$. The natural extension of the Glaeser's one-dimensional inequality (1.2) to higher dimensions would be the following.

Suppose that $u(x) \geq 0$, $\sup_{x \in B_R} M_u(x) = M \geq \frac{2u(0)}{R^2}$. Using the Taylor's formula for $|x| < R$ we have

$$0 \leq u(x) \leq u(0) + Du(0) \cdot x + \frac{1}{2} M|x|^2.$$

If $Du(0) \neq 0$, letting $x = r\omega$ with $\omega = -\frac{Du(0)}{|Du(0)|}$ and $r \in (0, R)$, we get

$$|Du(0)| \leq \frac{u(0)}{r} + \frac{Mr}{2}.$$

Optimizing the right-hand side to get the inequalities (1.1) we see that the interpolation inequality (1.4) continues to hold with $C = \sqrt{2}$ and $M_R = M$ as in the one-dimensional case.

If we assume instead

$$M = \sup_{x \in B_R} |\Delta u(x)|,$$

we will find in general larger constants $C = C_u(R)$ in dimension $n > 1$. This is not the case of convex or concave functions, because $|\Delta u| \leq M$ implies $M_u(x) \leq M$ and hence again $C_u(R) = \sqrt{2}$, but it may happen as soon as the eigenvalues of D^2u have different sign, as Example 1.1 shows.

The remaining part of this Section is essentially devoted to prove that $C_u := C_u(R) \leq \frac{3}{2}$ for any $(u, B_R) \in \mathfrak{F}_{2,2}$. Indeed, let

$$u(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33}$$

be a quadratic polynomial in $\mathfrak{F}_2$, namely

$$u \geq 0 \quad \text{in } B_R, \quad |\Delta u| = 2|a_{11} + a_{22}| = M, \quad a_{33} = \tfrac{1}{2}MR^2.$$

We may suppose $a_{11} + a_{22} \geq 0$, otherwise we consider the polynomial

$$v(x, y) = u(x, y) - 2(a_{11} + a_{22})x^2.$$

Let λ_1 and λ_2 be the eigenvalues of the quadratic form

$$q(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2.$$

Since $\lambda_1\lambda_2 \geq 0$ implies either the convexity or the concavity of u, from the above discussion we still have in this case $C_u \leq \sqrt{2}$.

Thus we are left with $\lambda_1\lambda_2 < 0$, and we may assume, eventually interchanging the axes with each other:

$$-\lambda_1 < \lambda_2 < 0 < \lambda_1\,.$$

Introducing the parameter $\alpha = -\lambda_2/\lambda_1$, the above reads $0 < \alpha < 1$.

Remark 3.1. As a matter of fact, since $u \in \mathfrak{F}_2$, we cannot have

$$\frac{1}{2} < \alpha < 1.$$

To see this, it is convenient to put u in the form

$$u(x, y) = \lambda_1(x - x_0)^2 + \lambda_2(y + y_0)^2 + c$$

with $x_0 \geq 0$, $y_0 \geq 0$. By positivity in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ we have

$$\begin{aligned} &\lambda_1 r^2\cos^2\theta + \lambda_2 r^2\sin^2\theta - 2\lambda_1 x_0 r\cos\theta \\ &+ \lambda_2 y_0 r\sin\theta + \lambda_1 x_0^2 + \lambda_2 y_0^2 + c \geq 0 \end{aligned} \tag{3.1}$$

for all $r \in (0,R)$ and $\theta \in [0,2\pi]$. Since $u \in \mathfrak{F}_2$,

$$\lambda_1 x_0^2 + \lambda_2 y_0^2 + c =: u(0,0) = \frac{1}{2}MR^2 := (\lambda_1+\lambda_2)R^2, \tag{3.2}$$

and substituting in (3.1) we obtain

$$\begin{aligned} &\lambda_1 R^2\cos^2\theta + \lambda_2 R^2\sin^2\theta - 2\lambda_1 x_0 R\cos\theta \\ &+ 2\lambda_2 y_0 R\sin\theta + (\lambda_1+\lambda_2)R^2 \geq 0. \end{aligned} \tag{3.3}$$

Dividing both the sides of (3.3) by λ_1, we obtain

$$(1-\alpha+\cos^2\theta - \alpha\sin^2\theta)R \geq 2\left(x_0\cos\theta + \alpha y_0\sin\theta\right). \tag{3.4}$$

Since $y_0 \geq 0$, computing the above for $\theta = \pi/2$ we deduce that $\alpha \leq \frac{1}{2}$, as claimed.

For $\alpha = 1/2$, again taking $\theta = \pi/2$ in (3.4), we have $y_0 = 0$ and (3.4) implies

$$\frac{3}{2}\cos^2\theta - \frac{2x_0}{R}\cos\theta \geq 0, \quad \theta \in [0,2\pi],$$

which implies $x_0 = 0$. Therefore, the standard equation is

$$u(x,y) = \lambda_1 x^2 + \lambda_2 y^2 + c,$$

so $Du(0,0) = 0$ and consequently $C_u = 0$. Hence, by Remark 3.1, we are left with $\alpha \in (0,\frac{1}{2})$. In this case we derive from (3.4) the lower bound

$$R \geq 2 \max_{\theta\in[0,\pi/2]} \frac{x_0\cos\theta + \alpha y_0\sin\theta}{1-\alpha+\cos^2\theta-\alpha\sin^2\theta}. \tag{3.5}$$

to estimate

$$C_u = \frac{|Du(0,0)|}{\sqrt{u(0,0)\Delta u}} = \frac{\sqrt{2}}{R}\,\frac{\sqrt{\lambda_1^2x_0^2+\lambda_2^2y_0^2}}{\lambda_1+\lambda_2} = \frac{\sqrt{2}}{R}\,\frac{\sqrt{x_0^2+\alpha^2y_0^2}}{1-\alpha}\,. \tag{3.6}$$

If $x_0 = 0$ (with $y_0 \neq 0$, otherwise $C_u = 0$), using the above inequalities with $\theta = \pi/2$ and recalling that $\alpha < 1/2$, we have

$$C_u \leq \frac{\sqrt{2}}{2}\,\frac{1-2\alpha}{1-\alpha} \leq \frac{1}{\sqrt{2}}\,.$$

Next, we set $\xi = y_0/x_0$ and $t = \tan\theta$. If $x_0 > 0$ then, combining inequalities (3.5) and (3.6) once more, we obtain

$$\begin{aligned} C_u &\leq \sqrt{2}\inf_{\theta\in(0,\pi/2)} \frac{1-\alpha+\cos^2\theta-\alpha\sin^2\theta}{2(x_0\cos\theta+\alpha y_0\sin\theta)}\,\frac{\sqrt{x_0^2+\alpha^2y_0^2}}{1-\alpha} \\ &\leq \sqrt{2}\sup_{0<\alpha<1/2,\,\xi>0}\inf_{t>0}\frac{\sqrt{1+\alpha^2\xi^2}}{1-\alpha}\,\frac{2-\alpha+(1-2\alpha)t^2}{2(1+\alpha\xi t)\sqrt{1+t^2}} \\ &\leq \sqrt{2}\quad\inf_{t>0}\sup_{0<\alpha<1/2,\,\xi>0}\frac{\sqrt{1+\alpha^2\xi^2}}{1-\alpha}\,\frac{2-\alpha+(1-2\alpha)t^2}{2(1+\alpha\xi t)\sqrt{1+t^2}} \\ &\leq \sqrt{2}\quad\inf_{t>0}\frac{\sigma(t)}{2\sqrt{1+t^2}}\,, \end{aligned}$$

where

$$\sigma(t) := \sup_{0<\alpha<1/2,\, \xi>0} \frac{2-\alpha+(1-2\alpha)t^2}{1-\alpha}\,\frac{\sqrt{1+\xi^2}}{1+\xi t}.$$

Observing that

$$\sigma(t) \le \begin{cases} 3/t & \text{if } t \le 1 \\ 2+t^2 & \text{if } t > 1 \end{cases}$$

and choosing $t = 1$ we finally get

$$C_u \le \sqrt{2}\,\inf_t \frac{\sigma(t)}{2\sqrt{1+t^2}} \le \frac{3}{2}\,.$$

The above discussion shows that $C = 3/2$ is an upper bound for the optimal constant in $\mathfrak{F}_{2,2}$. By Example 1.1 we conclude that $\sup_{(u,B_R)\in\mathfrak{F}_{2,2}} C_u(R) = 3/2$.

References

[1] M. E. Amendola, L. Rossi and A. Vitolo; Harnack Inequalities and ABP Estimates for Nonlinear Second Order Elliptic Equations in Unbounded Domains, *Abstract and Applied Analysis* **2008** (2008), Article ID 178534, 1–19.

[2] L. A. Caffarelli; Interior a priori estimates for solutions of fully nonlinear equations, *Annals of Mathematics* **130** (1989), 189–213.

[3] L. A. Caffarelli and X. Cabrè; Fully Nonlinear Elliptic Equations, American Mathematical Society Colloquium Publications **43**, Providence, Rhode Island, 1995.

[4] M. W. Certain and T. G. Kurtz; Landau-Kolmogorov inequalities for semigroups and groups, *Proc. Amer. Math Soc.*, **63** (1977), 226–230.

[5] M. G. Crandall, H. Ishii and P. L. Lions; User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc.* **27** (1992), 1–67.

[6] I. Capuzzo Dolcetta and A. Vitolo; Gradient and Hölder estimates for positive solutions of Pucci type equations, *C. R., Math., Acad. Sci. Paris* **346** (2008), 527–532.

[7] I. Capuzzo Dolcetta and A. Vitolo; $C^{1,\alpha}$ and Glaeser type estimates, *Rend. Mat., Ser VII* **29** (2009), 17–27.

[8] I. Capuzzo Dolcetta and A. Vitolo; Glaeser's type gradient estimates for non-negative solutions of fully nonlinear elliptic equations, *Dyn. Contin. Discrete Impuls. Syst., Ser. A.* **28** n.2, Dedicated to Louis Nirenberg on the Occasion of his 85th Birthday, Part I (2010), 539–557.

[9] D. Gilbarg and N. S. Trudinger; Elliptic Partial Differential Equations of Second Order, 2-nd ed., Grundlehren der Mathematischen Wissenschaften No. 224, Springer-Verlag, Berlin-New York, 1983.

[10] G. Glaeser; Racine carrée d'une fonction différentiable, *Ann. Ist. Fourier* **13** (1963), 203–207.

[11] H. Ishii and P. L. Lions; Viscosity solutions of fully nonlinear second order elliptic partial differential equations, *J. Differential Equations* **83** (1990), 26–78.

[12] S. Koike and Takahashi; Remarks on regularity of viscosity solutions for fully nonlinear uniformly elliptic PDEs with measurable ingredients, *Adv. Differential Equations* **7** (2002), 493–512.

[13] E. Landau; Einige Ungleichungen für zweimal differenzierbare Funktionen, *Proc. London Math. Soc.*, **13** (1913), 43–49.

[14] Y. Y. Li and L. Nirenberg; Generalization of a well-known inequality, *Progress in Nonlinear Differential Equations and Their Applications* **66** (2005), 365–370.

[15] V. G. Maz'ya and A. Kufner; Variations on the theme of the inequality $(f')^2 \le 2f \sup|f''|$, *Manuscripta Math.*, **56** (1986), 89–104.

[16] V. G. Maz'ya and T. O. Shaposhnikova; Sharp pointwise interpolation inequalities for derivatives, *Funct. Anal. Appl.*, **36** (2002), 30–48.

[17] L. Nirenberg and F. Treves; Solvability of a first order linear partial differential equation, *Comm. Pure Appl. Math.*, **16** (1963), 331–351.

[18] A. Swiech, $W^{1,p}$-Interior estimates for solutions of fully nonlinear, uniformly elliptic equations, *Adv. Differential Equations* **2** (1997), 1005–1027.

[19] N. S. Trudinger; Hölder gradient estimates for fully nonlinear elliptic equations, *Proc. Royal Soc. Edinburgh* **108A** (1988), 57–65.

Antonio Vitolo
Department of Mathematics and Computer Science, University of Salerno, p. Grahamstown, I-84084 Fisciano (SA), Italy
E-mail address: vitolo@unisa.it