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BOUNDED SOLUTIONS OF NONLINEAR PARABOLIC EQUATIONS

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Abstract. We are concerned with bounded solutions existing for all times for nonlinear parabolic equations with possibly nonlinear boundary conditions. A counterexample shows that, without an additional condition, a (weak) maximum principle does not hold for linear problems defined on the entire real line in time. We consider solutions bounded for all times and derive a (weak) maximum principle which is valid on the entire real line. Using comparison techniques, *a priori* estimates and approximation methods, we prove the existence and, in some cases, positivity and uniqueness of bounded solutions existing for all times.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded, open and connected set with boundary $\partial\Omega$. We consider nonlinear second order parabolic boundary value problems of the form

$$\begin{gathered}\frac{\partial u}{\partial t}(x,t) - Lu(x,t) = f(x,t,u) \quad \text{in } \Omega \times \mathbb{R},\\ \mathcal{B}u = \varphi(x,t,u) \quad \text{on } \partial\Omega \times \mathbb{R},\\ \sup_{\Omega\times\mathbb{R}} |u(x,t)| < \infty,\end{gathered} \tag{1.1}$$

where L is a second order, uniformly elliptic differential operator with time dependent coefficients and $\mathcal{B}$ is a linear first-order boundary operator. The coefficients of the operators L and $\mathcal{B}$ are (locally) Hölder continuous and bounded. We are interested in bounded solutions existing for all times. Steady-state solutions, time-periodic as well as almost-periodic solutions are special cases of bounded solutions existing for all times.

Many papers have been devoted to the study of solutions of parabolic equations existing for all times. To the best of our knowledge, the time-dependent bounded coefficients case was initiated by Fife in [5] for linear equations with Dirichlet boundary conditions. (We also refer to Cannon [3, pp. 101–110] for the one-dimensional heat equation with constant coefficients.) Some recent results and a bibliography may be found in Castro and Lazer [4], Fife [5], Hess [9], Krylov [10] and Shen and

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Yi [17], among others. We also refer to [7, 15] and references therein for the ordinary differential equations case. In [5, 10] a linear problem is considered where it is assumed that the coefficients of the differential equations and all data are bounded and uniformly Hölder continuous. In [9], the existence of periodic solutions for nonlinear problems was proved by assuming that the coefficients of the operator and all the data are time-periodic. In the periodic case considered in [9], the boundary conditions were still linear and time-independent. Since we are dealing with solutions existing for all times for equations with possibly nonlinear boundary conditions, many tools used for compact or semi-infinite time interval are not directly applicable, mainly due to the lack of compactness. Also, unlike ordinary differential equations, forward bounded solutions to nonlinear parabolic equations cannot be extended in the past unless very stringent conditions are imposed.

This paper is organized as follows. In Section 2, we first show with a counterexample that the (weak) maximum principle fails when solutions exist for all times, even when the coefficients in the equation are very smooth. We then establish L^∞-*a priori* estimates for bounded solutions to linear boundary value problems and derive a weak maximum principle which is valid on the entire real line in time. The counterexample shows that an additional condition is needed for the maximum or comparison principle to hold. We then formulate the general assumptions and state our main result concerning the existence and, in some cases, positivity and uniqueness of bounded solutions existing for all times for nonlinear problems with (possibly) nonlinear boundary conditions. In Section 3, we prove some auxiliary results which are needed for the proof of our main result. Using these results with comparison techniques, Gagliardo-Nirenberg type interpolation inequalities, *a priori* estimates obtained herein and approximation methods, we prove our main result. In the proof of our main result, we use an approximation argument. However, a delicate point in the proof lies in the obtainment of the *a priori* estimates for the derivatives of the approximating solutions since we are dealing with solutions existing for all times, and hence there is some lack of compactness. A couple of examples are given at the end of the paper.

2. Main results

One of the principal ingredients used in the study of second order parabolic partial differential operators is the (weak) maximum or comparison principle. We show with a counterexample that this principle fails when solutions exist for all times, even when the coefficients are very smooth. This lack of maximum principle is in sharp contrast with the initial-boundary value problem, the time-periodic boundary value problem, or the steady-state (elliptic) problem for which solutions exist for all times as well (see e.g. [1, 9, 14, 16]). Therefore, consider the linear boundary-value problem

$$\begin{gathered}\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \lambda u = 0 \quad \text{in } (0,\pi)\times\mathbb{R},\\ u(0,t) = 0 = u(\pi,t) \quad \text{for all } t\in\mathbb{R},\end{gathered}$$

where $\lambda \in \mathbb{R}$. Letting $u = -e^{-\gamma t}\sin x$, with $\gamma > 1+\lambda$, one has that $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \lambda u > 0$ in $(0,\pi)\times\mathbb{R}$, whereas $u < 0$ in $(0,\pi)\times\mathbb{R}$. Thus, the (weak) maximum (or comparison) principle does not hold; i.e., there is no positivity or order-preservation of the operator solution, even when $\lambda > 0$. An analysis of this counterexample suggests that one has to consider a condition on the behavior of the function $u(x,t)$ at $-\infty$ in time. In our case, we will consider functions that are bounded for all times. As illustrated by this counterexample, this condition is needed for the comparison result, if any, to hold on the entire real line in time. Since our results hold true for more general operators with time-dependent coefficients, we first introduce some notation and general assumptions.

Throughout this paper all functions are real-valued. We denote by Ω a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and closure $\overline{\Omega}$. We assume that $\partial\Omega$ belongs to $C^{2+\mu}$ with $\mu \in (0,1)$. We consider the second order parabolic operator in $\Omega \times \mathbb{R}$ given by

$$\frac{\partial u}{\partial t} - Lu, \tag{2.1}$$

where

$$Lu := \sum_{i,j=1}^{N} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u$$

with symmetric positive definite coefficient-matrix (a_{ij}). We assume that

- (i) $a_{ij}, b_i, c \in C^{\mu,\mu/2}_{\rm loc}(\overline{\Omega}\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R})$;
- (ii) there are constants $c_0 \geq 0$ and $\gamma_0 > 0$ such that for all $(x,t) \in \overline{\Omega}\times\mathbb{R}$, $c(x,t) \leq -c_0$ and $\sum_{i,j=1}^N a_{ij}(x,t)\xi_i\xi_j \geq \gamma_0|\xi|^2$ for all $\xi \in \mathbb{R}^N$.

For every $x \in \partial\Omega$, we denote by $\eta(x) := (\eta_1(x),\dots,\eta_N(x))$ the (unit) outer normal to $\partial\Omega$ at x. Let $\nu = (\nu_1,\dots,\nu_N)$ be such that for every i, $\nu_i \in C^{1+\mu,(1+\mu)/2}_{\rm loc}(\partial\Omega \times \mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R})$ and for all $(x,t) \in \partial\Omega\times\mathbb{R}$, $\sum_{i=1}^N \nu_i(x,t)\eta_i(x) > 0$; i.e., ν is an outward pointing nowhere tangent vector on $\partial\Omega$. Let $\frac{\partial u}{\partial\nu} := \sum_{i=1}^N \frac{\partial u}{\partial x_i}(x,t)\nu_i(x,t)$ denote the outward directional derivative of u with respect to ν on the boundary $\partial\Omega$.

Let ϵ denote a variable which takes on the values 0 and 1 only. We define the boundary operator $\mathcal{B}_\epsilon$ by

$$\mathcal{B}_\epsilon u := \epsilon\frac{\partial u}{\partial\nu} + \alpha(x,t)u, \tag{2.2}$$

where $\alpha \in C^{1+\mu,(1+\mu)/2}_{\rm loc}(\partial\Omega\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R})$ such that for all $(x,t) \in \partial\Omega\times\mathbb{R}$, $\alpha(x,t) \geq \alpha_0 \geq 0$. The constant α_0 is such that $\alpha_0 > 0$ if $\epsilon = 0$, and $\alpha_0 \geq 0$ if $\epsilon = 1$. Moreover, we assume that

$$c_0 + \alpha_0 > 0; \tag{2.3}$$

which implies that the coefficients $c(x,t)$ and $\alpha(x,t)$ do not vanish simultaneously. Thus, for $\epsilon = 0$, $\mathcal{B}_0 u$ is a Dirichlet boundary condition, whereas for $\epsilon = 1$, $\mathcal{B}_1 u$ corresponds to a Neumann or a regular oblique derivative boundary condition.

We denote by

$$a_+ := \max\{a,0\} \quad \text{and} \quad a_- := \max\{-a,0\}.$$

We first obtain L^∞-*a priori* estimates for solutions to linear boundary value problems by assuming that the solutions are bounded for all times. As a consequence we derive a maximum (or comparison) principle which is valid on the entire real line. These results will play an important role in the proofs of our main results. Some of the techniques used in the proof of the following proposition were somewhat inspired by [10].

Proposition 2.1. *Let $u \in C^{2,1}(\Omega \times \mathbb{R}) \cap C^{\epsilon,0}_{\rm loc}(\overline{\Omega} \times \mathbb{R}) \cap L^{\infty}(\Omega \times \mathbb{R})$, where ϵ is either 0 or 1. Then there exists a constant K such that*

$$\sup_{\Omega\times\mathbb{R}} |u_{\pm}| \le K\Big(\sup_{\Omega\times\mathbb{R}} \big|\big(\frac{\partial u}{\partial t} - Lu\big)_{\pm}\big| + \sup_{\partial\Omega\times\mathbb{R}} \big|(\mathcal{B}_\epsilon u)_{\pm}\big| \Big). \tag{2.4}$$

Thus,

$$\sup_{\Omega\times\mathbb{R}} |u| \le K\Big(\sup_{\Omega\times\mathbb{R}} \big|\frac{\partial u}{\partial t} - Lu\big| + \sup_{\partial\Omega\times\mathbb{R}} \big|\mathcal{B}_\epsilon u\big| \Big).$$

The constant K depends only on the dimension N, the parabolicity constant γ_0, diam (Ω), *and the L^∞-bounds of the coefficients of the operators L and $\mathcal{B}_\epsilon$.*

It follows immediately from Proposition 2.1 that the (weak) maximum principle holds.

Corollary 2.2 (Weak Comparison Principle)**.** *Suppose the conditions of Proposition 2.1 are met. Assume that $\frac{\partial u}{\partial t} - Lu \ge 0$ in $\Omega \times \mathbb{R}$ and that $\mathcal{B}_\epsilon u \ge 0$ on $\partial\Omega \times \mathbb{R}$. Then $u \ge 0$ in $\overline{\Omega} \times \mathbb{R}$.*

Proof of Proposition 2.1. Since the inequalities in (2.4) are proved in a similar manner, it suffices to prove only the first inequality and for the sign + only. We set $\frac{\partial u}{\partial t} - Lu := f$, $\mathcal{B}_\epsilon u := \varphi$, and denote by $F_+ := \sup_{\Omega\times\mathbb{R}} |f_+(x,t)|$ and $\Phi_+ := \sup_{\partial\Omega\times\mathbb{R}} |\varphi_+(x,t)|$. We will consider only the case when F_+ and Φ_+ are finite. Otherwise, the above inequalities hold true automatically.

Assume that $\epsilon = 1$ and that $\alpha(x,t) \ge \alpha_0 > 0$ (i.e., a regular oblique derivative condition). We first consider the special case when $c(x,t) \le -c_0 < 0$. For $n \in \mathbb{N}$, consider the set $\Omega \times (-n, T)$. Owing to a possibility of letting $T > 0$ go to ∞, we assume that $T < \infty$. Now, define

$$w(t) := e^{-c_0 t}, \quad r_n := \frac{1}{w(-n)} \sup_{\Omega} u_+(x,-n).$$

For the function

$$v(x,t) = \frac{1}{c_0} F_+ + \frac{1}{\alpha_0}\Phi_+ + r_n w(t) - u(x,t),$$

we have that

$$\begin{aligned} \frac{\partial v}{\partial t} - Lv &= -c(x,t)\Big(\frac{1}{c_0}F_+ + \frac{1}{\alpha_0}\Phi_+\Big) + r_n w(t)\left(-c_0 - c(x,t)\right) - \Big(\frac{\partial u}{\partial t} - Lu\Big) \\ &\ge F_+ - f \ge 0 \quad \text{in } \Omega \times (-n,T), \end{aligned}$$

$$\begin{aligned} v(x,-n) &= \frac{1}{c_0}F_+ + \frac{1}{\alpha_0}\Phi_+ + r_n w(-n) - u(x,-n) \\ &\ge \sup_{\Omega} u_+(x,-n) - u(x,-n) \ge 0 \quad \text{in } \Omega, \end{aligned}$$

and

$$\mathcal{B}_1 v = \alpha(x,t)\left(\frac{1}{c_0}F_+ + \frac{1}{\alpha_0}\Phi_+ + r_n w(t)\right) - \mathcal{B}_1 u \ge \Phi_+ - \varphi \ge 0 \quad \text{on } \partial\Omega \times (-n,T].$$

By the standard maximum principle (see e.g. [16, 6, 12]) we have that $v \geq 0$ on $\overline{\Omega} \times [-n, T]$. Therefore,

$$\begin{aligned} u(x,t) &\leq \frac{1}{c_0} F_+ + \frac{1}{\alpha_0} \Phi_+ + r_n w(t) \\ &= \frac{1}{c_0} F_+ + \frac{1}{\alpha_0} \Phi_+ + e^{-c_0 t} e^{-c_0 n} \sup_{\Omega} u_+(x, -n), \quad \forall (x,t) \in \overline{\Omega} \times [-n, T]. \end{aligned}$$

Since $u \in L^\infty(\Omega \times \mathbb{R})$, it follows that $e^{-c_0 n} \sup_\Omega u_+(x, -n) \to 0$ as $n \to \infty$. We deduce that $u(x,t) \leq \frac{1}{c_0} F_+ + \frac{1}{\alpha_0} \Phi_+$ for every $(x,t) \in \Omega \times \mathbb{R}$. Thus,

$$\sup_{\Omega \times \mathbb{R}} |u_+| \leq K \Big(\sup_{\Omega \times \mathbb{R}} \big| \big(\frac{\partial u}{\partial t} - Lu\big)_+ \big| + \sup_{\partial\Omega \times \mathbb{R}} |(\mathcal{B}_\epsilon u)_+| \Big).$$

Note that the bound is independent of the time T.

For the more general case when $c(x,t) \leq 0$ for all $(x,t) \in \Omega \times \mathbb{R}$, we consider the auxiliary function $u(x,t) = z(x) w(x,t)$ where z is a positive bounded function on $\overline{\Omega}$ to be determined. A direct calculation shows that w satisfies

$$\begin{gathered} \frac{\partial w}{\partial t} - \Big(\sum_{i,j=1}^n a_{ij}(x,t) \frac{\partial^2 w}{\partial x_i x_j} + \sum_{i=1}^n \tilde{b}_i(x,t) \frac{\partial w}{\partial x_i} + \big(\frac{1}{z} Lz\big) w \Big) = \frac{f}{z} \quad \text{in } \Omega \times \mathbb{R}, \\ \frac{\partial w}{\partial \nu} + \Big(\alpha(x,t) + \frac{1}{z} \frac{\partial z}{\partial \nu} \Big) w = \frac{\varphi}{z} \quad \text{on } \partial\Omega \times \mathbb{R}, \end{gathered}$$

where $\tilde{b}_i = \frac{1}{z}(a_{ij} + a_{j\,i}) \frac{\partial z}{\partial x_j} + b_i$. We pick $z(x) = A + y(x)$, where y is a bounded function in $\overline{\Omega}$ which satisfies $Ly < 0$ in $\Omega \times \mathbb{R}$ (and which, without loss of generality, may be chosen such that $y \geq \kappa$ in $\overline{\Omega} \times \mathbb{R}$ for some constant $\kappa > 0$ depending only on $N, \gamma_0, \Omega,$ the L^∞-bounds of the coefficients of L , $\mathcal{B}_1$ and ν), and let A be a positive constant chosen sufficiently large such that $\alpha(x,t) + \frac{1}{z} \frac{\partial z}{\partial \nu} \geq \frac{1}{2} \alpha_0$. (There are several examples of such a function z, see [8, p. 32], [10, p. 77 and p. 108]). This reduces to the case discussed above. Applying the result above to w we get that

$$\sup_{\Omega \times \mathbb{R}} |u_+| \leq K \Big(\sup_{\Omega \times \mathbb{R}} |f_+(x,t)| + \sup_{\partial\Omega \times \mathbb{R}} |\varphi_+(x,t)| \Big).$$

Notice that for $\epsilon = 0$ (i.e., the Dirichlet boundary condition) we proceed in the same way as for the regular oblique derivative case. (Also see [10, p. 107-108].)

Now, for $\epsilon = 1$ and $\alpha \equiv 0$ (i.e., the Neumann boundary condition), we assume that $c(x,t) \leq -c_0 < 0$ as stipulated in (2.3). We reduce the problem to the case with regular oblique derivative boundary condition by choosing an auxiliary function $u(x,t) = z(x) w(x,t)$ where now $z(x) = A + y(x)$ and y satisfies the Laplace-Dirichlet equation $\Delta y = 1$ in Ω with $y = 0$ on $\partial\Omega$. Choosing the constant A sufficiently large such that $Lz = c(x,t)A + Ly < 0$ in $\Omega \times \mathbb{R}$ and $z = A + y > 0$, it follows from the standard maximum principle that $0 < \frac{\partial y}{\partial \nu} = \frac{\partial z}{\partial \nu}$. The estimates (2.4) are therefore obtained in a way similar to the regular oblique derivative case. The proof is complete. $\square$

In what follows, we will need the following notation for ordered real-valued functions. Let S be a nonempty set, if $u, v : S \to \mathbb{R}$ are two functions such that $u(s) \leq v(s)$ for every $s \in S$, then we write $u \leq v$. Finally, by an order-interval $[u, v]$ between u and v we mean the set of all functions $w : S \to \mathbb{R}$ such that $u \leq w \leq v$.

We now consider the nonlinear boundary-value problem

$$\begin{gathered}\frac{\partial u}{\partial t}(x,t) - Lu(x,t) = f(x,t,u) \quad \text{in } \Omega\times\mathbb{R},\\ \mathcal{B}_\epsilon u = \Phi_\epsilon(x,t,u) \quad \text{on } \partial\Omega\times\mathbb{R},\\ \sup_{\Omega\times\mathbb{R}} |u(x,t)| < \infty.\end{gathered} \tag{2.5}$$

We will use the following definitions for bounded sub- and super-solutions.

Definition 2.3. A function $\underline{u} \in C^{2,1}(\Omega\times\mathbb{R}) \cap C^{\epsilon+\mu,(\epsilon+\mu)/2}_{\rm loc}(\overline{\Omega}\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R})$, where ϵ is either 0 or 1, is a subsolution of (2.5) if

(1) $\frac{\partial \underline{u}}{\partial t} - L\underline{u} \le f(x,t,\underline{u})$ in $\Omega\times\mathbb{R}$, and
(2) $\mathcal{B}_\epsilon \underline{u} \le \Phi_\epsilon(x,t,\underline{u})$ on $\partial\Omega\times\mathbb{R}$.

A supersolution of (2.5) is defined by reversing the inequality signs in (1) and (2). In order to state the main result for the nonlinear equation (2.5), we assume the following conditions on the reaction function f and the boundary term Φ_ϵ.

The reaction function f satisfies the following conditions.

(A1) $f \in C^\mu_{\rm loc}(\overline{\Omega}\times\mathbb{R}\times\mathbb{R})$; that is, for $[c,d]\subset\mathbb{R}$ with $X = \overline{\Omega}\times\mathbb{R}\times[c,d]$, there is a constant $K(X)$ such that $|f(x,t,u) - f(y,s,v)| \le K(|x-y|^2 + |t-s| + |u-v|^2)^{\mu/2}$ for all $(x,t,u),(y,s,v)\in X$.

(A2) There is a constant $M_0\in\mathbb{R}$ such that $|f(x,t,0)| \le M_0$ for all $(x,t)\in\overline{\Omega}\times\mathbb{R}$.

The function $\Phi_\epsilon(x,t,u) = (1-\epsilon)\varphi_0(x,t) + \epsilon\varphi_1(x,t,u)$ satisfies the following conditions.

(A3)
- If $\epsilon = 0$, then $\Phi_0 = \varphi_0 \in C^{2+\mu,(2+\mu)/2}_{\rm loc}(\partial\Omega\times\mathbb{R}) \cap L^\infty(\partial\Omega\times\mathbb{R})$.
- If $\epsilon = 1$, then $\Phi_1 = \varphi_1 \in C^{1+\mu}_{\rm loc}(\partial\Omega\times\mathbb{R}\times\mathbb{R})$ is such that (A2) is satisfied; that is, the functions $\varphi_1, \frac{\partial\varphi_1}{\partial x}, \frac{\partial\varphi_1}{\partial u}$ satisfy (A1) and (A2).

Note that conditions (A1)–(A2) are fulfilled by any function of the form $f(x,t,u) = p(x,t)g(u)$ where $p\in C^{\mu,\mu/2}(\overline{\Omega}\times\mathbb{R})$ and $g\in C^\mu_{\rm loc}(\mathbb{R})$. (A similar observation holds for the boundary term Φ_1.)

It should also be pointed out that (A1) and (A2) imply that f sends sets bounded in u into bounded sets; that is,

(A2') for every $r>0$, there is $M_r>0$ such that $|f(x,t,u)|\le M_r$ for all $(x,t,u)\in\overline{\Omega}\times\mathbb{R}\times[-r,r]$.

In addition, (A3) implies that the function φ_1 is locally Lipschitz in u, uniformly in (x,t); that is,

(A3') for $[c,d]\subset\mathbb{R}$, there is a constant $\varrho_1 = \varrho_1([c,d])>0$ such that $|\varphi_1(x,t,u) - \varphi_1(x,t,v)| \le \varrho_1|u-v|$ for all $u,v\in[c,d]$ and all $(x,t)\in\partial\Omega\times\mathbb{R}$.

Our main result for (2.5) is given by the following theorem in which we assume the following one-sided (local) Lipschitz condition.

(LL) Given $c,d\in\mathbb{R}$ with $c\le d$, there is a constant $k_0\ge 0$ such that $f(x,t,u) - f(x,t,v) \ge -k_0(u-v)$ for all (x,t,u), $(x,t,v)\in\overline{\Omega}\times\mathbb{R}\times[c,d]$ with $v\le u$.

Theorem 2.4. *Let* (A1)–(A3) *and* (LL) *hold. Suppose that there exist a supersolution $\overline{u}$ and a subsolution $\underline{u}$ of* (2.5) *such that $\underline{u}\le\overline{u}$ in $\overline{\Omega}\times\mathbb{R}$. Then* (2.5) *has a least one solution $u\in C^{2+\mu,(2+\mu)/2}_{\rm loc}(\overline{\Omega}\times\mathbb{R})$ such that $\underline{u}\le u\le\overline{u}$ in $\overline{\Omega}\times\mathbb{R}$. Moreover, there exist a minimal solution v^* and a maximal solution u^* in $[\underline{u},\overline{u}]$; that is, if w is any solution of* (2.5) *such that $\underline{u}\le w\le\overline{u}$, then $v^*\le w\le u^*$.*

In the proof of Theorem 2.4, we will use an approximation argument. However, the main difficulty lies in the obtainment of the required *a priori* estimates for the derivatives of the solutions since we are dealing with solutions existing for all times, and hence there is some lack of compactness.

As an immediate consequence of Theorem 2.4, we have the following corollary on the existence of positive solutions that are bounded for all times.

Corollary 2.5 (Positive Solutions)**.** *Assume that the assumptions in Theorem 2.4 are satisfied. Suppose that f, Φ_ϵ are nonnegative and there exists a supersolution $\overline{u}$ of* (2.5) *such that $0 \le \overline{u}$ in $\overline{\Omega} \times \mathbb{R}$. Then* (2.5) *has a nonnegative solution $u \in C^{2+\mu,(2+\mu)/2}_{\rm loc}(\overline{\Omega}\times\mathbb{R})$ such that $u \le \overline{u}$ in $\overline{\Omega}\times\mathbb{R}$. Moreover there exist nonnegative minimal solution v^* and maximal solution u^* in $[0, \overline{u}]$; that is, if w is any solution of* (2.5) *such that $0 \le w \le \overline{u}$, then $v^* \le w \le u^*$.*

Notice that if $f(t,\cdot,0) \not\equiv 0$ in the above corollary, then it immediately follows from Nirenberg's Strong Maximum Principle for parabolic equations that $v^*(x,t) > 0$ in $\Omega \times \mathbb{R}$. We cannot assert that the solution obtained in Theorem 2.4 is unique. However, in order to guarantee uniqueness of solutions to (2.5), one way is to require that f and Φ_ϵ be monotone nonincreasing in u.

Proposition 2.6 (Uniqueness)**.** *Let $\underline{u}, \overline{u}$ be ordered subsolution and supersolution of* (2.5) *and suppose that f and Φ_ϵ are nonincreasing in u, for $u \in [\underline{u}, \overline{u}]$. Then* (2.5) *has at most one solution u such that $\underline{u} \le u \le \overline{u}$.*

Proof. Let u, v be two solutions of (2.5) with $\underline{u} \le u, v \le \overline{u}$. We need to show that $u = v$. Indeed, first set $U := \{(x,t) \in \Omega \times \mathbb{R} : u(x,t) < v(x,t)\}$. By the monotonicity of f and Φ_ϵ, we get that

$$
\begin{gathered}
\frac{\partial(u-v)}{\partial t} - L(u-v) \ge 0 \quad \text{in } U,\\
\mathcal{B}_\epsilon(u-v) \ge 0 \quad \text{on } \partial U \cap (\partial\Omega \times \mathbb{R}),\\
u - v = 0 \quad \text{on } \partial U \cap (\Omega \times \mathbb{R}),\\
\sup_{\Omega\times\mathbb{R}} |u-v| < \infty.
\end{gathered}
$$

If U is bounded below in time, then by the classical maximum principle (see e.g. [16]) it follows that $u \ge v$ in U. This contradiction to the definition of U implies that U should be unbounded below in time. But, by using an argument similar to that in the proof of Proposition 2.1, we get that $u \ge v$ in U; which is again a contradiction. Hence, U is empty. Reversing the role of u and v, we deduce that $u = v$ on $\overline{\Omega} \times \mathbb{R}$. The proof is complete. □

Remark 2.7. An analysis of the proof of Theorem 2.4 will show that the condition (A1) on the function f can (slightly) be generalized by assuming the following conditions (A1') and (A1"). (Notice that (A1') is a local condition in $t \in \mathbb{R}$.)

- (A1') $f \in C^\mu_{\rm loc}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R})$; that is, for $a,b,c,d \in \mathbb{R}$ with $X = \overline{\Omega} \times [a,b] \times [c,d]$, there exists a constant $K(X)$ such that $|f(x,t,u) - f(y,s,v)| \le K(|x-y|^2 + |t-s| + |u-v|^2)^{\mu/2}$, for all $(x,t,u),(y,s,v) \in X$.
- (A1") f is locally Hölder in u uniformly in x and t; that is, there exists $\varrho_0 > 0$ such that
  $|f(x,t,u) - f(x,t,v)| \le \varrho_0 |u-v|^\mu$ for all $u,v \in [c,d]$ and all $(x,t) \in \overline{\Omega} \times \mathbb{R}$.

3. Preliminary Results and Proof of the Main Result

To prove the main result stated above, we need some auxiliary results on the linear problem. In the following result, we use Proposition 2.1 to obtain existence and uniqueness of solutions for the linear problem. This result plays an important role in the approximation argument used in the proof of the nonlinear problem. (It should be observed that, in contrast to [5, 10], we assume that the coefficients in the linear operator are only *locally* Hölder continuous in time.)

Proposition 3.1. *Consider the linear boundary-value problem*

$$\begin{gathered}\frac{\partial u}{\partial t} - Lu = f \quad \text{in } \Omega\times\mathbb{R},\\ \mathcal{B}_\epsilon u = \varphi \quad \text{on } \partial\Omega\times\mathbb{R},\\ \sup_{\Omega\times\mathbb{R}} |u(x,t)| < \infty,\end{gathered} \tag{3.1}$$

where $f \in C^{\mu,\mu/2}_{\rm loc}(\overline{\Omega}\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R})$ *and* $\varphi \in \mathrm{W}^{2-\epsilon-\frac{1}{p},(2-\epsilon-\frac{1}{p})/2}_{p,\rm loc}(\partial\Omega\times\mathbb{R}) \cap L^\infty(\partial\Omega\times\mathbb{R})$ *with* $p = \frac{N+2}{1-\mu}$. *Then there exists a unique function* $u \in C^{2,1}(\Omega\times\mathbb{R}) \cap C^{1+\mu,(1+\mu)/2}_{\rm loc}(\overline{\Omega}\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R})$ *satisfying* (3.1).

Proof. Uniqueness follows immediately from Corollary 2.2. We now proceed to prove the existence. For every $n \in \mathbb{N}$, pick a cut-off function $\xi_n \in C^\infty(\mathbb{R})$ such that $0 \le \xi_n \le 1$, and $\xi_n(s) = 1$ if $s \ge -n$, $\xi_n(s) = 0$ if $s \le -(n+1)$. Define $f_n(x,t) = \xi_n f(x,t)$, for all $(x,t) \in \Omega\times\mathbb{R}$, $\varphi_n(x,t) = \xi_n\varphi(x,t)$, for all $(x,t) \in \partial\Omega\times\mathbb{R}$. It follows that $f_n \in C^{\mu,\mu/2}_{\rm loc}(\overline{\Omega}\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R})$ and $\varphi_n \in \mathrm{W}^{2-\epsilon-\frac{1}{p},(2-\epsilon-\frac{1}{p})/2}_{p,\rm loc}(\partial\Omega\times\mathbb{R}) \cap L^\infty(\partial\Omega\times\mathbb{R})$.

Fix $n \in \mathbb{N}$. Let $T_n = -(n+1)$ and $T \ge -n$, and consider the initial-boundary value problem

$$\begin{gathered}\frac{\partial u}{\partial t} - Lu = f_n \quad \text{in } \Omega\times(T_n,T],\\ \mathcal{B}_\epsilon u = \varphi_n \quad \text{on } \partial\Omega\times(T_n,T],\\ u(x,T_n) = 0 \quad \forall\, x \in \overline{\Omega}.\end{gathered} \tag{3.2}$$

It follows from [11, pp. 341-343 and p. 351] that the problem (3.2) has a (unique) solution $w_n \in W^{2,1}_p(\Omega\times(T_n,T))$. We extend w_n by setting $w_n(x,t) = 0$ for all $(x,t) \in \overline{\Omega}\times(-\infty,T_n]$. Thus, $w_n \in W^{2,1}_{p,\rm loc}(\Omega\times\mathbb{R})$. It follows from the Imbedding Theorem that $w_n \in C^{1+\mu,(1+\mu)/2}_{\rm loc}(\overline{\Omega}\times\mathbb{R})$. Moreover, by the (interior) regularity of generalized solutions [11, pp. 223-224], we get that $w_n \in C^{2,1}(\Omega\times\mathbb{R})$. Using the *a priori* estimate in Proposition 2.1 we get that $\sup_{\Omega\times\mathbb{R}}|w_n| < M$, where M is independent of n.

Next, we will prove that a subsequence of $\{w_n\}$ converges (on compact sets) to a solution u of the linear problem (3.1). Indeed, consider $Q_1 = \Omega\times(-1,1)$ and $X = \Omega\times(-2,2)$ with $\overline{X} = \overline{\Omega}\times[-2,2]$. For each $n \in \mathbb{N}$, define $z_n(x,t) = \zeta(t)w_n(x,t)$ for all $(x,t) \in \overline{\Omega}\times[-2,2]$, where $\zeta \in C^\infty(\mathbb{R})$ is a cut-off function such that $0 \le \zeta \le 1$ and $\zeta(s) = 0$ if $s \le -2$, $\zeta(s) = 1$ if $s \ge -(2-\delta)$ with $0 < \delta < 1$. Observe that

$z_n = w_n$ in $\overline{\Omega} \times [-1,1]$ and z_n satisfies the initial-boundary value problem

$$\begin{gathered} \frac{\partial z}{\partial t} - Lz = h_n \quad \text{in } \Omega \times (-2,2], \\ \mathcal{B}_\epsilon z = \zeta \varphi_n \quad \text{on } \partial\Omega \times (-2,2], \\ z(x,-2) = 0 \quad \forall\, x \in \overline{\Omega}, \end{gathered} \tag{3.3}$$

where $h_n = \frac{d\zeta}{dt} w_n + \zeta f_n$. We have that $h_n \in L^p(X)$ and

$$\zeta \varphi_n \in \mathrm{W}_{p,\mathrm{loc}}^{2-\epsilon-\frac{1}{p},\left(2-\epsilon-\frac{1}{p}\right)/2}(\partial\Omega \times (-2,2)).$$

From the solvability results for linear problems [11, pp. 341–343 and p. 351], it follows that (3.3) has a unique solution $z_n \in \mathrm{W}_p^{2,1}(X)$ and that

$$|z_n|_{W_p^{2,1}(X)} \le K\Big(|h_n|_{L^p(X)} + |\zeta\varphi_n|_{W_p^{2-\epsilon-\frac{1}{p},\left(2-\epsilon-\frac{1}{p}\right)/2}(\partial\Omega\times(-2,2))}\Big) \tag{3.4}$$

for all $n \in \mathbb{N}$, where K depends only on X. Since w_n and f_n are uniformly bounded, it follows that there is a constant $C > 0$ such that $|h_n|_{L^p(X)} <$ C for all $n \in \mathbb{N}$. Since for n sufficiently large $\zeta\varphi_n = \varphi_n = \varphi \in W_p^{2-\epsilon-\frac{1}{p},\left(2-\epsilon-\frac{1}{p}\right)/2}(\partial\Omega \times (-2,2))$, it follows that $|z_n|_{W_p^{2,1}(Q_1)} < C$. We claim that $\{z_n\}$ has a subsequence which converges to a solution of the boundary-value problem in Q_1.

Indeed, define $T : \big(W_p^{2,1}(Q_1), |\cdot|_{W_p^{2,1}(Q_1)}\big) \to \big(L^p(Q_1), |\cdot|_{L^p(Q_1)}\big)$ by $T(v) := \frac{\partial v}{\partial t} - Lv$. Clearly, T is a continuous linear operator, and hence is weakly continuous (see e.g. [2, pp. 39]). Since $W_p^{2,1}(Q_1)$ is a reflexive Banach space which is compactly imbedded into $C^{1+\mu,(1+\mu)/2}(\overline{Q}_1)$ and $|z_n|_{W_p^{2,1}(Q_1)} \le C$, it follows that there is a subsequence $\{w_{1n}\}$ of $\{z_n\}$ such that $w_{1n} \to u_1$ in $C^{1+\mu,(1+\mu)/2}(\overline{Q}_1)$ and $w_{1n} \rightharpoonup u_1$ in $W_p^{2,1}(Q_1)$. This implies that $T(w_{1n}) \rightharpoonup T(u_1)$. But, for n sufficiently large, one has that $T(w_{1n}) = f$ in Q_1. Therefore, by the uniqueness of the limit, we deduce that $T(u_1) = f$ in Q_1. Moreover, $\mathcal{B}_\epsilon w_{1n} \to \mathcal{B}_\epsilon u_1$ in $C^{\mu,\mu/2}(\partial\Omega \times [-1,1])$ and, since $\mathcal{B}_\epsilon w_{1n} = \varphi$ on $\partial\Omega \times [-1,1]$, we get that $\mathcal{B}_\epsilon u_1 = \varphi$. Thus, u_1 is a solution of the boundary value problem $\frac{\partial z}{\partial t} - Lz = f$ in $\Omega \times (-1,1)$, $\mathcal{B}_\epsilon z = \varphi$ on $\partial\Omega \times [-1,1]$ with $\sup_{\Omega\times[-1,1]} |z(x,t)| < \infty$. By the regularity of generalized solutions [11, pp. 223-224], one has that $u_1 \in C^{2,1}(\Omega \times (-1,1))$. Thus $u_1 \in C^{1+\mu,(1+\mu)/2}(\bar{\Omega} \times [-1,1]) \cap C^{2,1}(\Omega \times (-1,1))$ and $\sup_{\overline{\Omega}\times[-1,1]} |u_1| < M$.

Next, for $k \ge 2$, set $Q_k = \Omega \times (-k,k)$ and consider instead the subsequence denoted by $\{w_{(k-1)n}\}$. Using a similar argument as above, we get a subsequence $\{w_{kn}\}$ of $\{w_{(k-1)n}\}$ such that $\{w_{kn}\}$ converges to u_k in $C^{1+\mu,(1+\mu)/2}(\overline{\Omega} \times [-k,k])$. Moreover, u_k satisfies the boundary value problem

$$\begin{gathered} \frac{\partial u}{\partial t} - Lu = f \quad \text{in } \Omega \times (-k,k), \\ \mathcal{B}_\epsilon u = \varphi \quad \text{on } \partial\Omega \times [-k,k], \\ \sup_{\Omega\times[-k,k]} |u(x,t)| < \infty. \end{gathered}$$

As above $u_k \in C^{1+\mu,(1+\mu)/2}(\bar{\Omega} \times [-k,k]) \cap C^{2,1}(\Omega \times (-k,k))$ and $\sup_{\overline{\Omega}\times[-k,k]} |u_k| < M$.

Now, by the diagonalization argument, choose the sequence $\{w_{jj}\}$ located on the 'diagonal.' Observe that, by construction, $\{w_{jj}\}$ is a subsequence of $\{w_{kn}\}_{n=1}^\infty$ for $k \le j$, and hence is a subsequence of $\{w_n\}$. We shall prove that $\{w_{jj}\}$ converges to a solution u of (3.1). Indeed, let $\overline{\Omega} \times [-k,k]$ and $\varepsilon > 0$. Since $\{w_{kn}\}$ converges to u_k in $C^{1+\mu,(1+\mu)/2}(\overline{\Omega} \times [-k,k])$, as $n \to \infty$, it follows that there exists $N = N(k) \in \mathbb{N}$ such that for all $n \geq N, |w_{kn} - u_k|_{C^{1+\mu,(1+\mu)/2}(\overline{\Omega}\times[-k,k])} < \varepsilon$. Using the fact that $w_{jj} \in \{w_{kn}\}_{n=1}^{\infty}$ for all $j \geq k$, we get that for all $j \geq \max\{k, N\}$, $|w_{jj} - u_k|_{C^{1+\mu,(1+\mu)/2}(\overline{\Omega}\times[-k,k])} < \varepsilon$. Thus $\{w_{jj}\}$ is subsequence of $\{w_n\}$ and it converges to a function u in $C^{1+\mu,(1+\mu)/2}(\overline{\Omega} \times [-k,k])$, where $u|_{\overline{\Omega}\times[-k,k]} = u_k$. Since $k \in \mathbb{N}$ is arbitrarily chosen, $u \in C^{1+\mu,(1+\mu)/2}_{\rm loc}(\overline{\Omega} \times \mathbb{R}) \cap C^{2,1}(\Omega \times \mathbb{R})$ with $\sup_{\Omega\times\mathbb{R}} |u| \leq M$, and u satisfies the linear problem (3.1). The proof is complete. $\square$

To obtain the *a priori* estimates needed in the proof of the nonlinear problem, we will need the following interpolation inequalities of Gagliardo-Nirenberg type (see e.g. [13] for the proof).

Lemma 3.2. *Let $\Omega \times I \subset \mathbb{R}^n \times \mathbb{R}$ and $1 \leq p < \infty$, where I is a bounded open interval. Then, there is a constant $C > 0$ such that for all $u \in W^{2,1}_p(\Omega \times I)$ one has*

$$|u|_{W^{1,1/2}_p(\Omega\times I)} \leq C|u|_{W^{2,1}_p(\Omega\times I)}|u|_{L^p(\Omega\times I)}. \tag{3.5}$$

Moreover, for every $\varepsilon > 0$,

$$|u|_{W^{1,1/2}_p(\Omega\times I)} \leq C\Big(\varepsilon|u|_{W^{2,1}_p(\Omega\times I)} + \frac{1}{4\varepsilon}|u|_{L^p(\Omega\times I)}\Big). \tag{3.6}$$

We are in a position to prove our main result contained in Theorem 2.4. Delicate *a priori* estimates on the derivatives of the approximating solutions are derived in the proof.

Proof of Theorem 2.4. Setting $k = \max(\varrho_1, k_0)$, it follows from (A3') and (LL) that, for $(x,t) \in \overline{\Omega}\times\mathbb{R}$, the functions $\Phi_\epsilon(x,t,w)+kw$ and $f(x,t,w)+kw$ are nondecreasing in w in the interval $[\underline{u}, \overline{u}]$. Moreover, (A2') implies that $f(\cdot,\cdot,w) \in L^\infty(\Omega \times \mathbb{R})$ whenever $w \in [\underline{u}, \overline{u}]$. To prove the existence of the solutions u^* and v^* of (2.5), we proceed with a (linear) approximation as follows. First, we construct monotone sequences $\{u_n\}$ and $\{v_n\}$ successively from the (linear) iteration process

$$\begin{gathered}
\frac{\partial u_n}{\partial t} - Lu_n + ku_n = f(x,t,u_{n-1}) + k\,u_{n-1} \quad \text{in } \Omega \times \mathbb{R},\\
\mathcal{B}_\epsilon u_n + \epsilon k\,u_n = \Phi_\epsilon(x,t,u_{n-1}) + \epsilon k\,u_{n-1} \quad \text{on } \partial\Omega \times \mathbb{R},\\
\sup_{\Omega\times\mathbb{R}} |u_n(x,t)| < \infty,
\end{gathered} \tag{3.7}$$

where for $n = 1$, we set $u_0 = \overline{u}$. Since $f(\cdot,\cdot,\overline{u}) + k\overline{u} \in C^{\mu',\mu'/2}_{\rm loc}(\overline{\Omega} \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R})$ and $\Phi_\epsilon(\cdot,\cdot,\overline{u}) + \epsilon k\,\overline{u} \in C^{2-\epsilon+\mu',(2-\epsilon+\mu')/2}_{\rm loc}(\partial\Omega \times \mathbb{R}) \cap L^\infty(\partial\Omega \times \mathbb{R})$ with $\mu' \leq \mu^2$, it follows from Proposition 3.1 that (3.7) has a unique solution $u_1 \in C^{2,1}(\Omega \times \mathbb{R}) \cap C^{1+\mu',(1+\mu')/2}_{\rm loc}(\overline{\Omega} \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R})$ which is such that $\underline{u} \leq u_1 \leq \overline{u}$ by Corollary 2.2. For $n \geq 2$, a similar argument shows that (3.7) has a unique solution $u_n \in C^{2,1}(\Omega \times \mathbb{R}) \cap C^{1+\mu,(1+\mu)/2}_{\rm loc}(\overline{\Omega} \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R})$ such that $\underline{u} \leq u_n \leq u_{n-1} \leq \overline{u}$ in $\overline{\Omega} \times \mathbb{R}$. In a similar manner, it is shown that if we set $u_0 = \underline{u}$, we have $v_1 \in C^{2,1}(\Omega\times\mathbb{R})\cap C^{1+\mu',(1+\mu')/2}_{\rm loc}(\overline{\Omega}\times\mathbb{R})\cap L^\infty(\Omega\times\mathbb{R})$, $v_n \in C^{2,1}(\Omega\times\mathbb{R})\cap C^{1+\mu,(1+\mu)/2}_{\rm loc}(\overline{\Omega}\times\mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R})$ for $n \geq 2$, with $\underline{u} = v_0 \leq v_1 \leq v_2 \leq \ldots \leq v_{n-1} \leq v_n \leq \ldots \leq u_n \leq u_{n-1} \leq \ldots \leq u_2 \leq u_1 \leq u_0 = \overline{u}$. Since the sequences $\{u_n\}$ and $\{v_n\}$ are monotone and (uniformly) bounded, the pointwise limits

$$u^*(x,t) = \lim_{n\to\infty} u_n(x,t), \quad v^*(x,t) = \lim_{n\to\infty} v_n(x,t)$$

exist with $\underline{u} \le v^* \le u^* \le \overline{u}$. We now proceed to show that u^* and v^* are solutions of (2.5).

For that purpose, consider $Q_1 = \Omega \times (-1,1)$ and $Q_2 = \Omega \times (-2,2)$. For each $n \in \mathbb{N}$, define $z_n(x,t) = \zeta(t)u_n(x,t)$ for all $(x,t) \in \overline{\Omega} \times [-2,2]$, where $\zeta \in C^\infty(\mathbb{R})$ is a cut-off function such that $0 \le \zeta \le 1$ and $\zeta(s) = 0$ if $s \le -2$, $\zeta(s) = 1$ if $s \ge -(2-\delta)$ with $0 < \delta < 1$. Observe that $z_n = u_n$ in $\overline{\Omega} \times [-1,1]$ and satisfies the linear initial-boundary value problem

$$\begin{gathered} \frac{\partial z_n}{\partial t} - Lz_n + kz_n = \frac{d\zeta}{dt} u_n + \zeta g_n \quad \text{in } \Omega \times (-2,2], \\ \mathcal{B}_\epsilon z_n + \epsilon k z_n = \zeta \Psi_n \quad \text{on } \partial\Omega \times (-2,2], \\ z_n(x,-2) = 0 \quad \text{on } \overline{\Omega}, \end{gathered} \tag{3.8}$$

where, for each $n \in \mathbb{N}$, $g_n = f(\cdot,\cdot,u_{n-1}) + k\, u_{n-1}$ and $\Psi_n = \Phi_\epsilon(\cdot,\cdot,u_{n-1}) + \epsilon k\, u_{n-1}$. By the solvability results for linear IBVP [11, pp. 341–343 and p. 351], it follows that the linear IBVP (3.8) has a unique solution $z_n \in \mathrm{W}^{2,1}_p(Q_2)$ where $p = \frac{N+2}{1-\mu}$. Moreover,

$$|z_n|_{W^{2,1}_p(Q_2)} \le K_0\Big(\big|\frac{d\zeta}{dt}u_n + \zeta g_n\big|_{L^p(Q_2)} + |\zeta\Psi_n|_{W_p^{2-\epsilon-\frac{1}{p},(2-\epsilon-\frac{1}{p})/2}(\partial\Omega\times(-2,2))}\Big), \tag{3.9}$$

for all $n \in \mathbb{N}$, where K_0 is a constant which depends on Q_2. Observe that for $\epsilon = 0$, we get immediately that $|z_n|_{W^{2,1}_p(Q_2)} \le$ const, for all n, since φ_0 does not depend on n. To show that $|z_n|_{W^{2,1}_p(Q_2)} \le$ const for $\epsilon = 1$, we proceed as follows. Using (A3) we compute $|\zeta\Psi_n|_{W_p^{1-\frac{1}{p},(1-\frac{1}{p})/2}(\partial\Omega\times(-2,2))}$ to get that

$$|\zeta\Psi_n|_{W_p^{1-\frac{1}{p},(1-\frac{1}{p})/2}(\partial\Omega\times(-2,2))} \le \hat{C}\Big(1 + |z_{n-1}|_{W_p^{1-\frac{1}{p},(1-\frac{1}{p})/2}(\partial\Omega\times(-2,2))}\Big), \tag{3.10}$$

where $\hat{C}$ is independent of n since $|\zeta\Psi_n|_{L^p(\partial\Omega\times(-2,2))} \le$ const, for all $n \in \mathbb{N}$. Combining (3.10) with (3.9) we obtain that

$$|z_n|_{W^{2,1}_p(Q_2)} \le C_0\Big(1 + |z_{n-1}|_{W_p^{1-\frac{1}{p},(1-\frac{1}{p})/2}(\partial\Omega\times(-2,2))}\Big),$$

where C_0 is independent of n but depends on $|\frac{d\zeta}{dt}u_n + \zeta g_n|_{L^p(Q_2)}$, $|\zeta\Psi_n|_{L^p}$, and $\overline{\Omega} \times [-2,2]$. Using the continuity of the trace operator, we deduce that

$$|z_n|_{W^{2,1}_p(Q_2)} \le K\left(1 + |z_{n-1}|_{W^{1,1/2}_p(\Omega\times(-2,2))}\right), \tag{3.11}$$

where K does not depend on n. By the interpolation inequality (3.6), we get that

$$|z_n|_{W^{2,1}_p(Q_2)} \le K\Big(1 + C\varepsilon|z_{n-1}|_{W^{2,1}_p(Q_2)} + \frac{C}{4\varepsilon}|z_{n-1}|_{L^p(Q_2)}\Big). \tag{3.12}$$

Now, we proceed inductively as follows. It follows from (3.11) that

$$|z_1|_{W^{2,1}_p(Q_2)} \le K\Big(1 + |\zeta\overline{u}|_{W^{1,1/2}_p(\Omega\times(-2,2))}\Big); \tag{3.13}$$

which when combined with the inequality (3.12) implies that

$$\begin{aligned}|z_2|_{W_p^{2,1}(Q_2)} &\le K\Big(1 + C\varepsilon|z_1|_{W_p^{2,1}(X)} + \frac{C}{4\varepsilon}|z_1|_{L^p(Q_2)}\Big)\\ &\le K\Big(1 + KC\varepsilon + KC\varepsilon|\zeta\overline{u}|_{W_p^{1,1/2}(Q_2)} + \frac{C}{4\varepsilon}|z_1|_{L^p(Q_2)}\Big).\end{aligned}$$

Proceeding by induction, we have that for every $n \in \mathbb{N}$ with $n \ge 2$,

$$\begin{aligned}&|z_n|_{W_p^{2,1}(Q_2)}\\ &\le K\Big(\sum_{i=0}^{n-1}(KC\varepsilon)^i + (KC\varepsilon)^{n-1}|\zeta\overline{u}|_{W_p^{1-\frac{1}{p},(1-\frac{1}{p})/2}(\partial\Omega\times(-2,2))} + \frac{MC}{4\varepsilon}\sum_{i=0}^{n-2}(KC\varepsilon)^i\Big),\end{aligned}$$

where K depends on C_0 and $\overline{\Omega}\times[-2,2]$, and the constant $M \ge M_n = |z_n|_{L^p(Q_2)}$ for all $n \in \mathbb{N}$. Therefore, we obtain the following estimate which involves a geometric series

$$|z_n|_{W_p^{2,1}(Q_2)} \le \Big(K + K|\zeta\overline{u}|_{W_p^{1-\frac{1}{p},(1-\frac{1}{p})/2}(\partial\Omega\times(-2,2))} + \frac{MCK}{4\varepsilon}\Big)\sum_{i=0}^{\infty}(KC\varepsilon)^i.$$

Thus,

$$|z_n|_{W_p^{2,1}(Q_2)} \le \tilde{C}, \quad \text{for all } n \in \mathbb{N},$$

provided $\varepsilon > 0$ is chosen sufficiently small such that $KC\varepsilon < 1$.

Now, we need to show that in Q_1 the sequence $\{z_n\} = \{u_n\}$ has a subsequence which converges to a solution of the problem (2.5). Indeed, define $T : \big(W_p^{2,1}(Q_1), |\cdot|_{W_p^{2,1}(Q_1)}\big) \to \big(L^p(Q_1), |\cdot|_{L^p(Q_1)}\big)$ by $T(v) := \frac{\partial v}{\partial t} - Lv + kv$. Hence, T is (weakly) closed. Since $W_p^{2,1}(Q_1)$ is a reflexive space which is compactly imbedded into $C^{1+\mu,(1+\mu)/2}(\overline{Q}_1)$ and $|z_n|_{W_p^{2,1}(Q_1)} \le \tilde{C}$ for all n, there is a subsequence $\{u_{1n}\}$ of $\{z_n\} = \{u_n\}$ such that $u_{1n} \rightharpoonup v_1$ in $W_p^{2,1}(Q_1)$ and $u_{1n} \to v_1$ in $C^{1+\mu,(1+\mu)/2}(\overline{Q}_1)$ as $n \to \infty$. Moreover, since T is (weakly) closed and $T(u_{1n}) = g_n \to f(\cdot,\cdot,v_1) + k\,v_1$ uniformly in Q_1, it follows that $T(v_1) = f(\cdot,\cdot,v_1) + k\,v_1$. In addition, $\mathcal{B}_\epsilon u_{1n} + \epsilon k u_{1n} \to \mathcal{B}_\epsilon v_1 + \epsilon k\, v_1$ in $C^{\mu,\mu/2}(\partial\Omega\times[-1,1])$ and $\mathcal{B}_\epsilon u_{1n} + \epsilon k\, u_{1n} = \Psi_n \to \Phi_\epsilon(\cdot,\cdot,v_1)) + \epsilon k\, v_1$ uniformly on $\partial\Omega\times[-1,1]$; which implies that $\mathcal{B}_\epsilon v_1 + \epsilon k v_1 = \Phi_\epsilon(\cdot,\cdot,v_1) + \epsilon k\, v_1$. Thus, v_1 satisfies the following nonlinear BVP

$$\begin{gathered}\frac{\partial v_1}{\partial t} - Lv_1 + kv_1 = f(x,t,v_1) + k\,v_1 \quad \text{in } \Omega\times(-1,1),\\ \mathcal{B}_\epsilon v_1 + \epsilon k\, v_1 = \Phi_\epsilon(x,t,v_1) + \epsilon k\, v_1 \quad \text{on } \partial\Omega\times[-1,1],\\ \sup_{\Omega\times[-1,1]}|v_1(x,t)| < \infty.\end{gathered}$$

By the interior regularity of generalized solutions [11, pp. 223-224], $v_1 \in C^{2,1}(\Omega\times(-1,1))$. Thus, $v_1 \in C^{1+\mu,(1+\mu)/2}(\overline{\Omega}\times[-1,1]) \cap C^{2,1}(\Omega\times(-1,1)) \cap L^\infty(\Omega\times\mathbb{R})$.

Next, for $k \ge 2$ let $Q_k = \Omega\times(-k,k)$. Consider the subsequence $\{u_{(k-1)n}\}$, and use an argument similar to the above to extract a subsequence $\{u_{kn}\}$ of $\{u_{(k-1)n}\}$ such that $u_{kn} \to v_k$ in $C^{1+\mu,(1+\mu)/2}(\overline{\Omega}\times[-k,k])$ and such that v_k satisfies the nonlinear equation

$$\begin{gathered}
\frac{\partial v_k}{\partial t} - Lv_k + k\,v_k = f(x,t,v_k) + kv_k \quad \text{in } \Omega\times(-k,k),\\
\mathcal{B}_\epsilon v_k + \epsilon k v_k = \Phi_\epsilon(x,t,v_k) + \epsilon k v_k \quad \text{on } \partial\Omega\times[-k,k],\\
\sup_{\Omega\times[-k,k]} |v_k| < \infty.
\end{gathered}$$

Note that by construction, $v_k|_{\overline{\Omega}\times[-(k-1),k-1]} = v_{k-1}$ for all $k\ge 2$; that is, v_k is an extension of v_{k-1}.

Using a 'diagonalization' process and proceeding as in the proof of Proposition 3.1, we choose a subsequence $\{u_{jj}\}$ (located on the 'diagonal' of the subsequences $\{u_{kn}\}_{n=1}^\infty$) which converges to the function v in $C^{1+\mu,(1+\mu)/2}(\overline{\Omega}\times[-k,k])$, where $v|_{\overline{\Omega}\times[-k,k]} = v_k$. Therefore, $v\in C^{1+\mu,(1+\mu)/2}_{\rm loc}(\overline{\Omega}\times\mathbb{R})\cap C^{2,1}(\Omega\times\mathbb{R})$, $\sup_{\Omega\times\mathbb{R}}|v|\le M$ and v satisfies (2.5). By the uniqueness of the (pointwise) limit we have that $v=u^*$. By the regularity properties of solutions to parabolic problems, we have that $u^*\in C^{2+\mu,(2+\mu)/2}_{\rm loc}(\overline{\Omega}\times\mathbb{R})\cap L^\infty(\Omega\times\mathbb{R})$. Similar arguments show that $v^*\in C^{2+\mu,(2+\mu)/2}_{\rm loc}(\overline{\Omega}\times\mathbb{R})\cap L^\infty(\Omega\times\mathbb{R})$ and that it is also a solution of (2.5). Thus, $\underline{u}\le v^*\le u^*\le\overline{u}$.

We finally establish that u^* and v^* are maximal and minimal solutions respectively in the interval $[\underline{u},\overline{u}]$. Let w be a solution of (2.5) with $\underline{u}\le w\le\overline{u}$, then the functions w, $\underline{u}$ are ordered supersolution and subsolution. The above conclusion implies that $\underline{u}\le v^*\le w$. A similar reasoning leads to $w\le u^*\le\overline{u}$. Thus $u^*\le w\le v^*$, and the proof is complete. □

We conclude this section with a couple of examples.

A Fisher-Dirichlet problem with time-dependent bounded coefficients. Consider the boundary value problem

$$\begin{gathered}
\frac{\partial u}{\partial t} - \Delta u = u(a(x,t)-b(x,t)u) \quad \text{in } \Omega\times\mathbb{R},\\
u=0 \quad \text{on } \partial\Omega\times\mathbb{R},\\
\sup_{\Omega\times\mathbb{R}}|u(x,t)|<\infty,
\end{gathered}\tag{3.14}$$

where $a,b\in C^{\mu,\mu/2}_{\rm loc}(\overline{\Omega}\times\mathbb{R})$ with $\lambda_1<\alpha\le a(x,t)\le A$, $0<\beta\le b(x,t)\le B$, $\forall\ (x,t)\in\Omega\times\mathbb{R}$, for some constants $\alpha,\beta,A,B\in\mathbb{R}$, where λ_1 is the principal eigenvalue of the Laplace operator with homogeneous Dirichlet boundary condition and associated eigenfunction φ. Choosing $\underline{u}(x,t)=\varepsilon\varphi(x)$ where $0<\varepsilon<(\alpha-\lambda_1)/B$, and $\overline{u}(x,t)=C$ where $C\in\mathbb{R}$ with $C\ge A/\beta$, it follows from Corollary 2.5 that (3.14) has a positive solution u such that $\underline{u}\le u\le\overline{u}$ in $\Omega\times\mathbb{R}$. Thus, u does not tend to zero as $t\to\pm\infty$.

A Neumann problem with nonlinear boundary conditions. Consider the boundary value problem

$$\begin{gathered}
\frac{\partial u}{\partial t} - \Delta u = u^n(a(x,t)-b(x,t)u^{2k+1}) \quad \text{in } \Omega\times\mathbb{R},\\
\frac{\partial u}{\partial\nu} = u^m\,(\delta-u) \quad \text{on } \partial\Omega\times\mathbb{R},\\
\sup_{\Omega\times\mathbb{R}}|u(x,t)|<\infty,
\end{gathered}\tag{3.15}$$

where $n, k, m \in \mathbb{N}$, $0 < \delta \in \mathbb{R}$ are fixed. We assume that $a, b \in C^{\mu,\mu/2}_{\rm loc}(\overline{\Omega} \times \mathbb{R})$ with $0 < \alpha \leq a(x,t) \leq A$, $0 < \beta \leq b(x,t) \leq B$, for all $(x,t) \in \Omega \times \mathbb{R}$, for some constants $\alpha, \beta, A, B \in \mathbb{R}$. Choosing $\underline{u}(x,t) = D$ where $0 < D < \delta$ such that $D^{2k+1} < \alpha/B$, and $\overline{u}(x,t) = C$ where $C \in \mathbb{R}$ with $C \geq \max(1 + A/\beta, \delta)$, it follows from Corollary 2.5 that (3.15) has a positive solution u such that $\underline{u} \leq u \leq \overline{u}$ in $\Omega \times \mathbb{R}$. Thus, u does not tend to zero as $t \to \pm\infty$.

References

[1] H. Amann; *On the existence of positve solutions of nonlinear elliptic boundary value problems,* Indiana Univ. Math. J. **21** (1971), 125-146.

[2] H. Brézis; “Analyse Fonctionnelle: Théorie et Applications,” Masson, Paris, 1983.

[3] J. R. Cannon; “The One-Dimensional Heat Equation,” Encyclopedia of Mathematics and its Applications, Vol. 23, Addison-Wesley Publishing Company, Inc, CA, 1984.

[4] A. Castro and A. Lazer; *Results on periodic solutions of parabolic equations suggested by elliptic theory,* Boll. Uni. Mat. Ital. B (6) **I** (1982), 1089 – 1104.

[5] P. Fife; *Solutions of parabolic boundary problems existing for all time,* Arch. Rational Mech. Anal. **16** (1964), 155 - 186.

[6] A. Freidman; “Partial Differenial Equations of Parabolic Type,” Prentice-hall, NJ, 1964.

[7] G. N. Galanis and P. K. Palamides; *Global positive solutions of a generalized logistic equations with bounded and unbounded coefficients*, Electronic Journal of Differential Equations **2003** (2003), 1-13.

[8] Q. Han and F. Lin; “Elliptic Partial Differential Equations,” Courant Institute of Mathematical Sciences, New York, 1997.

[9] P. Hess; “Periodic-parabolic Boundary Value Problems and Positivity,” Longman Group, UK, 1991.

[10] N. V. Krylov; “Lectures on Elliptic and Parabolic Equations in Hölder Spaces,” Graduate Studies in Mathematics 12, AMS, Providence, RI, 1996.

[11] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural’ceva; “Linear and Quasilinear Equations of Parabolic Type,” AMS Translations of Mathematical Monographs, Vol. 23, Providence, Rhodes Island, 1968.

[12] G. M. Lieberman; “Second Order Parabolic Differential Equations,” World Scientific, NJ, 1996.

[13] N. Mavinga; “Nonlinear Second Order Parabolic and Elliptic Equations with Nonlinear Boundary Conditions,” PhD Dissertation, University of Alabama at Birmingham, 2008.

[14] J. Mawhin and K. Schmitt; *Corrigendum: Upper and lower solutions and semilinear second order elliptic equations with non-linear boundary conditions*, Proc. Royal Soc. Edinburgh, **A 100** (1985), 361.

[15] M. N. Nkashama; *Dynamics of logistic equations with non-autonomous bounded coefficients*, Electronic Journal of Differential Equations **2000** (2000), No. 02, 1-8.

[16] M. H. Protter and H. F. Weinberger; “Maximum Principles in Differential Equations,” Prentice- Hall, Englewood Cliffs, NJ, 1967.

[17] W. Shen and Y. Yi; *Convergence in almost periodic Fisher and Kolmogorv models,* J. Math. Biol. **37** (1998), 84-102.

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