16th Conference on Applied Mathematics, Univ. of Central Oklahoma,
Electronic Journal of Differential Equations, Conf. 07, 2001, pp. 47–59.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

A new a priori estimate for multi-point boundary-value problems *

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Abstract

Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Caratheodory's conditions and $e(t) \in L^1[0,1]$. Let $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and $a_i \in \mathbb{R}$ for $i = 1, 2, \ldots, m-2$ be given. A priori estimates of the form

$$\|x\|_\infty \leq C\|x''\|_1, \quad \|x'\|_\infty \leq C\|x''\|_1,$$

are needed to obtain the existence of a solution for the multi-point boundary-value problem

$$x''(t) = f(t, x(t), x'(t)) + e(t), \quad 0 < t < 1,$$
$$x(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i),$$

using Leray Schauder continuation theorem. The purpose of this paper is to obtain a new a priori estimate of the form $\|x\|_\infty \leq C\|x''\|_1$. This new estimate then enables us to obtain a new existence theorem. Further, we obtain a new a priori estimate of the form $\|x\|_\infty \leq C\|x''\|_1$ for the three-point boundary-value problem

$$x''(t) = f(t, x(t), x'(t)) + e(t), \quad 0 < t < 1,$$
$$x'(0) = 0, \quad x(1) = \alpha x(\eta),$$

where $\eta \in (0,1)$ and $\alpha \in \mathbb{R}$ are given. The estimate obtained for the three-point boundary-value problem turns out to be sharper than the one obtained by particularizing the m-point boundary value estimate to the three-point case.

1 Introduction

Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Caratheodory's conditions and $e(t) \in L^1[0,1]$. Let $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and $a_i \in \mathbb{R}$ for

* *Mathematics Subject Classifications:* 34B10, 34B15, 34G20.
Key words: Three-point boundary-value problem, m-point boundary-value problem, a-priori estimates, Leray-Schauder Continuation theorem, Caratheodory's conditions.

Published July 20, 2001.

$i = 1, 2, \ldots, m-2$ be given. Let us consider the problem of existence of a solution for the multi-point boundary-value problem

$$\begin{gathered} x''(t) = f(t, x(t), x'(t)) + e(t), 0 < t < 1, \\ x(0) = 0, x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i). \end{gathered} \tag{1}$$

In [2] the author and Sergei Trofimchuk had studied this problem earlier and obtained existence results using the Leray-Schauder continuation theorem. Now, to apply the Leray-Schauder continuation theorem requires a priori estimates of the form

$$\|x\|_\infty \le C\|x''\|_1, \quad \|x'\|_\infty \le C\|x''\|_1.$$

For a function $x(t) \in W^{2,1}(0,1)$ with $x(0) = 0$, $x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$, and $\sum_{i=1}^{m-2} a_i \xi_i \neq 1$, Gupta and Trofimchuk obtained the a priori estimate

$$\|x'\|_\infty \le \frac{1}{1-\tau}\|x''\|_1,$$

where, $0 \le \tau < 1$ is suitable constant defined by a_i, and ξ_i, $i = 1, 2, \ldots, m-2$. Using, then the estimate $\|x\|_\infty \le \|x'\|_\infty$, for functions $x(t) \in W^{2,1}(0,1)$ with $x(0) = 0$, they obtained the estimate

$$\|x\|_\infty \le \frac{1}{1-\tau}\|x''\|_1.$$

The purpose of this paper is to obtain a new and sharper estimate $\|x\|_\infty \le C\|x''\|_1$ for $x(t) \in W^{2,1}(0,1)$ with $x(0) = 0$, $x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$, and $\sum_{i=1}^{m-2} a_i \xi_i \neq 1$. This new estimate then enables us to obtain a new existence theorem for the above boundary-value problem. Further, we obtain a new a priori estimate of the form $\|x\|_\infty \le C\|x''\|_1$ for the three-point boundary-value problem

$$\begin{gathered} x''(t) = f(t, x(t), x'(t)) + e(t), \quad 0 < t < 1, \\ x'(0) = 0, \quad x(1) = \alpha x(\eta), \end{gathered} \tag{2}$$

where $\eta \in (0,1)$ and $\alpha \in \mathbb{R}$ are given. The estimate obtained for the three-point boundary-value problem turns out to be sharper than the one obtained by particularizing the m-point boundary-value estimate to the three-point case. These a priori estimates have been motivated by the results of [1].

2 A priori estimates

We begin this section by first describing an estimate obtained by Gupta and Trofimchuk. Let $a_i \in \mathbb{R}$, $\xi_i \in (0,1)$, $i = 1, 2, \ldots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, with $\sum_{i=1}^{m-2} a_i \xi_i \neq 1$, be given. Let $x(t) \in W^{2,1}(0,1)$ be such that $x(0) = 0$, $x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$. Let us write the condition $x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$

in symmetric form $\sum_{i=1}^{m-1} a_i x(\xi_i) = 0$ by setting $a_{m-1} = -1$ and $\xi_{m-1} = 1$. Then the assumption $\sum_{i=1}^{m-2} a_i \xi_i \neq 1$ is equivalent to $\sum_{i=1}^{m-1} a_i \xi_i \neq 0$. Let us, define, for i, $j = 1, 2, \ldots, m-1$,

$$\begin{aligned} \sigma_{ij} &= a_i(\xi_i - \xi_j) \quad \text{for } i \neq j\,, \\ \sigma_{jj} &= (\sum_{i=1}^{m-1} a_i)\xi_j\,. \end{aligned}$$

We observe that

$$\sum_{i=1}^{m-1} \sigma_{ij} = \sum_{i=1}^{m-1} a_i \xi_i \neq 0, \text{ for } j = 1, 2, \ldots, m-1\,.$$

For $a \in \mathbb{R}$, setting $a_+ = \max(a, 0)$ and $a_- = \max(-a, 0)$ so that $a = a_+ - a_-$, $|a| = a_+ + a_-$, we see that

$$\sum_{i=1}^{m-1} (\sigma_{ij})_+ \neq \sum_{i=1}^{m-1} (\sigma_{ij})_-. \tag{3}$$

We, next, define

$$\sigma_+^j = \sum_{i=1}^{m-1} (\sigma_{ij})_+, \sigma_-^j = \sum_{i=1}^{m-1} (\sigma_{ij})_- \quad \text{for } j = 1, 2, \ldots, m-1\,,$$

and

$$\tau = \min\{\frac{\sigma_+^j}{\sigma_-^j}, \frac{\sigma_-^j}{\sigma_+^j} : j = 1, 2, \ldots, m-1\}. \tag{4}$$

We, note, that $0 \leq \tau < 1$ in view of (3).

Proposition 1 *Let $a_i \in \mathbb{R}$, $\xi_i \in (0,1)$, $i = 1, 2, \ldots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, with $\sum_{i=1}^{m-2} a_i \xi_i \neq 1$, be given. Then for $x(t) \in W^{2,1}(0,1)$ with $x(0) = 0$, $x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$ we have*

$$\|x\|_\infty \leq \frac{1}{1-\tau} \|x''\|_1, \tag{5}$$

where τ is as given in (4).

We refer the reader to [2] for a proof of this proposition.

Theorem 2 *Let $a_i \in \mathbb{R}$, $\xi_i \in (0,1)$, $i = 1, 2, \ldots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, with $\sum_{i=1}^{m-2} a_i \xi_i \neq 1$, $\sum_{i=1}^{m-2} a_i \neq 1$, be given. Then for $x(t) \in W^{2,1}(0,1)$ with $x(0) = 0$, $x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$ we have*

$$\|x\|_\infty \leq C \|x''\|_1, \tag{6}$$

where

$$C = \min\{\frac{1}{1-\tau},\, C_1\},$$

with τ *as defined in (4),*

$$C_1 = \max\{C_2, \frac{1}{1-\tau}\sum_{i=1}^{m-2}|\frac{a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2}a_i}|\},$$

and C_2 *as defined below in (12).*

Proof Let $\xi_{m-1}=1$, $a_{m-1}=-1$ so that the condition $x(1)=\sum_{i=1}^{m-2}a_i x(\xi_i)$ may be written in the symmetric form $\sum_{i=1}^{m-1}a_i x(\xi_i)=0$ and $\sum_{i=1}^{m-1}a_i\neq 0$. Since $x(t)\in W^{2,1}(0,1)$ there exists a $c\in[0,1]$ such that $\|x\|_\infty=|x(c)|$. We may assume that $x(c)>0$, by replacing $x(t)$ by $-x(t)$, if necessary. Next, since $x(0)=0$, we see that $c\in(0,1]$. In case, $c\in(0,1)$ we must have $x'(c)=0$. Applying, now, the Taylor's formula with integral remainder after the second term at each ξ_i, $i=1,2,\ldots,m-1$, to get

$$x(\xi_i)=x(c)+r_i, \tag{7}$$

where

$$r_i=\int_c^{\xi_i}(\xi_i-s)x''(s)ds\le 0, \tag{8}$$

$i=1,2,\ldots,m-1$. Multiplying the equation (7) by a_i, $i=1,2,\ldots,m-1$, and adding the resulting equations we obtain

$$0=\sum_{i=1}^{m-1}a_i x(\xi_i)=\sum_{i=1}^{m-1}a_i x(c)+\sum_{i=1}^{m-1}a_i r_i. \tag{9}$$

Now, equations (8), (9) imply that

$$\begin{aligned}0<x(c)&=-\frac{1}{\sum_{i=1}^{m-1}a_i}\sum_{i=1}^{m-1}a_i r_i=-\sum_{i=1}^{m-1}(\frac{a_i}{\sum_{i=1}^{m-1}a_i})\int_c^{\xi_i}(\xi_i-s)x''(s)ds\\ &\le\sum_{i=1}^{m-1}(\frac{a_i}{\sum_{i=1}^{m-1}a_i})_+|\int_c^{\xi_i}(\xi_i-s)x''(s)ds|. \end{aligned}\tag{10}$$

We, next, observe that

$$|\int_c^{\xi_i}(\xi_i-s)x''(s)ds|\le|\xi_i-c|\int_c^{\xi_i}|x''(s)|ds\le|\xi_i-c|\int_0^1|x''(s)|ds,$$

for $i = 1, 2, \ldots, m-1$. We thus see from (8) that

$$\begin{aligned}\|x\|_\infty = x(c) &\leq \sum_{i=1}^{m-1}(\frac{a_i}{\sum_{i=1}^{m-1} a_i})_+|\int_c^{\xi_i}(\xi_i - s)x''(s)ds| \\ &\leq \sum_{i=1}^{m-1}(\frac{a_i}{\sum_{i=1}^{m-1} a_i})_+|\xi_i - c|\int_0^1 |x''(s)|ds \\ &\leq \max_{u\in[0,1]}(\sum_{i=1}^{m-1}(\frac{a_i}{\sum_{i=1}^{m-1} a_i})_+|\xi_i - u|)\int_0^1 |x''(s)|ds. \qquad (11)\end{aligned}$$

Since, now, $\sum_{i=1}^{m-1}(\frac{a_i}{\sum_{i=1}^{m-1} a_i})_+|\xi_i - u|$ is a piecewise linear function, its maximum value is attained at one of the points, 0, ξ_j, $j = 1, 2, \ldots, m-1$. Accordingly, we get

$$\begin{aligned}&\max_{u\in[0,1]}(\sum_{i=1}^{m-1}(\frac{a_i}{\sum_{i=1}^{m-1} a_i})_+|\xi_i - u|) \\ &= \max\left\{\begin{array}{c}\sum_{i=1}^{m-1}\xi_i(\frac{a_i}{\sum_{i=1}^{m-1} a_i})_+, \\ \sum_{i=1, i\neq j}^{m-1}(\frac{a_i}{\sum_{i=1}^{m-1} a_i})_+|\xi_i - \xi_j|, j = 1, 2, \ldots, m-1,\end{array}\right\} \qquad (12) \\ &= \max\left\{\begin{array}{c}\sum_{i=1}^{m-2}\xi_i(\frac{a_i}{1-\sum_{i=1}^{m-2} a_i})_- + (\frac{1}{1-\sum_{i=1}^{m-2} a_i})_+, \\ \sum_{i=1, i\neq j}^{m-2}(\frac{a_i}{1-\sum_{i=1}^{m-2} a_i})_-|\xi_i - \xi_j| + (\frac{1}{1-\sum_{i=1}^{m-2} a_i})_+(1-\xi_j), \\ j = 1, 2, \ldots, m-2, \\ \sum_{i=1}^{m-2}(\frac{a_i}{1-\sum_{i=1}^{m-2} a_i})_-(1-\xi_i)\end{array}\right\} \equiv C_2.\end{aligned}$$

Accordingly, when $x(c) = \|x\|_\infty$ with $c \in (0, 1)$ we see that

$$\|x\|_\infty \leq C_2\|x''\|_1. \qquad (13)$$

Let, now, $c = 1$ so that $\|x\|_\infty = x(1)$. We, then, see that there exists a λ_i, for each $i = 1, 2, \ldots, m-2$, such that

$$x(1) - x(\xi_i) = (1 - \xi_i)x'(\lambda_i). \qquad (14)$$

It follows from equations (14) that

$$(\sum_{i=1}^{m-2} a_i - 1)x(1) = \sum_{i=1}^{m-2} a_i(x(1) - x(\xi_i)) = \sum_{i=1}^{m-2} a_i(1 - \xi_i)x'(\lambda_i).$$

Accordingly, we get

$$\begin{aligned}\|x\|_\infty &= x(1) = \sum_{i=1}^{m-2} \frac{a_i(1-\xi_i)}{\sum_{i=1}^{m-2} a_i - 1} x'(\lambda_i)\\ &\le \sum_{i=1}^{m-2} \Big|\frac{a_i(1-\xi_i)}{\sum_{i=1}^{m-2} a_i - 1}\Big| \|x'\|_\infty\\ &\le \Big(\frac{1}{1-\tau}\sum_{i=1}^{m-2} \Big|\frac{a_i(1-\xi_i)}{\sum_{i=1}^{m-2} a_i - 1}\Big|\Big)\|x''\|_1. \qquad (15)\end{aligned}$$

Thus from estimates (13), (15) we obtain

$$\|x\|_\infty \le \max\{C_2, \frac{1}{1-\tau}\sum_{i=1}^{m-2} \Big|\frac{a_i(1-\xi_i)}{\sum_{i=1}^{m-2} a_i - 1}\Big|\}\|x''\|_1 \equiv C_1\|x''\|_1. \qquad (16)$$

The estimate (6) is now immediate since $\|x\|_\infty \le \frac{1}{1-\tau}\|x''\|_1$, from Proposition 1. This completes the proof of Theorem 2. □

Remark 3 Let $\eta \in (0,1)$, $\alpha \in \mathbb{R}$ with $\alpha\eta \neq 1$ be given. It was proved earlier by Gupta and Trofimchuk for $x(t) \in W^{2,1}(0,1)$ with $x(0) = 0$, $x(1) = \alpha x(\eta)$ that

$$\begin{aligned}\|x\|_\infty &\le \|x''\|_1 \quad \text{if } \alpha \le 1,\\ \|x\|_\infty &\le \frac{1-\eta}{1-\alpha\eta}\|x''\|_1 \quad \text{if } \alpha\eta < 1 \text{ and } \alpha > 1,\\ \|x\|_\infty &\le \frac{\alpha-1}{\alpha\eta-1}\|x''\|_1 \quad \text{if } \alpha > 1 \text{ and } \alpha\eta > 1,\end{aligned}$$

so that

$$\begin{aligned}\tau &= 0 \quad \text{if } \alpha \le 1,\\ \frac{1}{1-\tau} &= \frac{1-\eta}{1-\alpha\eta} \quad \text{if } \alpha > 1 \text{ and } \alpha\eta < 1,\\ \frac{1}{1-\tau} &= \frac{\alpha-1}{\alpha\eta-1} \quad \text{if } \alpha > 1 \text{ and } \alpha\eta > 1\,.\end{aligned}$$

Remark 4 Let us note that for $x(t) \in W^{2,1}(0,1)$ with $x(0) = 0$, $x(1) = \alpha x(\eta)$ the constant C_2 defined in (12) is given by

$$C_2 = \max\{\eta(\frac{\alpha}{1-\alpha})_- + (\frac{1}{1-\alpha})_+, (\frac{1}{1-\alpha})_+(1-\eta), (\frac{\alpha}{1-\alpha})_-(1-\eta)\}.$$

It follows that

$$C_2 = \begin{cases} \max\{\frac{1+|\alpha|\eta}{1+|\alpha|}, \frac{|\alpha|(1-\eta)}{1+|\alpha|}\} & \text{for } \alpha \le 0,\\ \frac{1}{1-\alpha} & \text{for } 0 \le \alpha < 1,\\ \max\{\frac{\alpha\eta}{\alpha-1}, \frac{\alpha(1-\eta)}{\alpha-1}\} & \text{for } \alpha > 1\,. \end{cases}$$

Next, we see from the definition of C_1 in (16) and (3) that

$$C_1 = \begin{cases} \max\{\frac{1+|\alpha|\eta}{1+|\alpha|}, \frac{|\alpha|(1-\eta)}{1+|\alpha|}\} & \text{for } \alpha \leq 0, \\ \frac{1}{1-\alpha} & \text{for } 0 \leq \alpha < 1, \\ \max\{\frac{\alpha\eta}{\alpha-1}, \frac{\alpha(1-\eta)^2}{(\alpha-1)(1-\alpha\eta)}\} & \text{for } \alpha\eta < 1 \text{ and } \alpha > 1, \\ \max\{\frac{\alpha\eta}{\alpha-1}, \frac{\alpha(1-\eta)}{(\alpha\eta-1)}\} & \text{for } \alpha\eta > 1 \text{ and } \alpha > 1\,. \end{cases}$$

Finally, we see that for $x(t) \in W^{2,1}(0,1)$ with $x(0) = 0$, $x(1) = \alpha x(\eta)$ we have

$$\|x\|_\infty \leq C\|x''\|_1, \tag{17}$$

where $C = \min\{\frac{1}{1-\tau}, C_1\}$ is given by

$$C = \begin{cases} \max\{\frac{1+|\alpha|\eta}{1+|\alpha|}, \frac{|\alpha|(1-\eta)}{1+|\alpha|}\} & \text{for } \alpha \leq 0, \\ 1 & \text{for } 0 \leq \alpha < 1, \\ \min\{\frac{1-\eta}{1-\alpha\eta}, \max\{\frac{\alpha\eta}{\alpha-1}, \frac{\alpha(1-\eta)^2}{(\alpha-1)(1-\alpha\eta)}\}\} & \text{for } \alpha\eta < 1 \text{ and } \alpha > 1, \\ \min\{\frac{\alpha-1}{\alpha\eta-1}, \max\{\frac{\alpha\eta}{\alpha-1}, \frac{\alpha(1-\eta)}{(\alpha\eta-1)}\}\} & \text{for } \alpha\eta > 1 \text{ and } \alpha > 1. \end{cases}$$

The following theorem gives a better estimate than (17) for an $x(t) \in W^{2,1}(0,1)$ with $x(0) = 0$, $x(1) = \alpha x(\eta)$.

Theorem 5 *Let $\alpha \in \mathbb{R}$ and $\eta \in (0,1)$ with $\alpha \neq 1$, $\alpha\eta \neq 1$, be given. Then for $x(t) \in W^{2,1}(0,1)$ with $x(0) = 0$, $x(1) = \alpha x(\eta)$ we have*

$$\|x\|_\infty \leq M\|x''\|_1$$

where

$$M = \begin{cases} \max\{\frac{1+|\alpha|\eta}{1+|\alpha|}, \frac{|\alpha|(1-\eta)}{1+|\alpha|}\} & \text{if } \alpha \leq -1. \\ \frac{1-\alpha\eta}{1-\alpha} & \text{if } -1 \leq \alpha < 0, \\ 1 & \text{if } 0 \leq \alpha < 1, \\ \max\{\frac{\eta}{2}, \frac{\alpha(1-\eta)}{\alpha-1}, \frac{\alpha\eta(1-\eta)}{1-\alpha\eta}\} & \text{if } \alpha > 1 \text{ and } \alpha\eta < 1, \\ \max\{\frac{\eta}{2}, \frac{\alpha\eta-1}{\alpha-1}, \frac{\alpha\eta(1-\eta)}{\alpha\eta-1}\} & \text{if } \alpha > 1 \text{ and } \alpha\eta > 1\,. \end{cases}$$

Proof For $\alpha \leq 0$ we see from Theorem 2 and remark 4 that

$$M = \max\{\frac{1+|\alpha|\eta}{1+|\alpha|}, \frac{|\alpha|(1-\eta)}{1+|\alpha|}\}.$$

This implies, in particular, for $\alpha \leq -1$ that $M = \max\{\frac{1+|\alpha|\eta}{1+|\alpha|}, \frac{|\alpha|(1-\eta)}{1+|\alpha|}\}$. Note that for $-1 \leq \alpha < 0$,

$$\frac{1-\alpha\eta}{1-\alpha} = \frac{1+\eta|\alpha|}{1+|\alpha|} \geq \frac{|\alpha|(1+\eta)}{1+|\alpha|} > \frac{|\alpha|(1-\eta)}{1+|\alpha|}$$

and so we again see from Theorem 2 and Remark 4 that

$$M = \begin{cases} \frac{1-\alpha\eta}{1-\alpha} & \text{if } -1 \leq \alpha < 0 \\ 1 & \text{if } 0 \leq \alpha < 1\,. \end{cases}$$

Finally, we consider the case $\alpha > 1$. Let $x(\eta) = z$ so that $x(1) = \alpha z$. We may assume without loss of generality that $z \geq 0$, replacing $x(t)$ by $-x(t)$ if necessary. Suppose, now, $\|x\|_\infty = 1$ so that there exists a $c \in [0,1]$ such that either $x(c) = 1$ or $x(c) = -1$. We consider all possible cases of the location for c.

(i) Suppose that $c \in (0,\eta]$ and $x(c) = 1$. Then $x'(c) = 0$, $c \neq \eta$. Now, by mean value theorem there exist $\nu_1 \in [c,\eta], \nu_2 \in [\eta,1]$ such that

$$x'(\nu_1) = \frac{x(\eta) - x(c)}{\eta - c} = -\frac{1-z}{\eta - c}, \quad x'(\nu_2) = \frac{x(1) - x(\eta)}{1-\eta} = \frac{\alpha z - z}{1-\eta}.$$

We note that $x'(\nu_1) \leq 0$, $x'(\nu_2) \geq 0$ since $0 \leq z \leq 1$ and $\alpha > 1$. It follows that

$$\begin{aligned}\int_0^1 |x''(s)|ds \geq & |\int_c^{\nu_1} x''(s)ds| + |\int_{\nu_1}^{\nu_2} x''(s)ds| \\ =& 2|x'(\nu_1)| + x'(\nu_2) = 2\frac{1-z}{\eta - c} + \frac{\alpha z - z}{1-\eta} \\ \geq & \min_{c\in[0,\eta), z\in[0,\frac{1}{\alpha}]} \{2\frac{1-z}{\eta-c} + \frac{\alpha z - z}{1-\eta}\} \\ \geq & \min_{c\in[0,\eta)} \{\frac{2}{\eta - c}, \frac{2(\alpha-1)}{\alpha(\eta-c)} + \frac{\alpha - 1}{\alpha(1-\eta)}\} \\ \geq & \min\{\frac{2}{\eta}, \frac{\alpha-1}{\alpha(1-\eta)}\}.\end{aligned}$$

(ii) Let, now, $c \in (0,\eta]$, $x(c) = -1$. Then since $x'(c) = 0$, $c \neq \eta$, we again see from mean value theorem that there exist $\nu_3 \in [c,\eta], \nu_4 \in [\eta,1]$ such that

$$x'(\nu_3) = \frac{x(\eta) - x(c)}{\eta - c} = \frac{z+1}{\eta - c}, \quad x'(\nu_4) = \frac{x(1) - x(\eta)}{1-\eta} = \frac{\alpha z - z}{1-\eta}.$$

Again we note that $x'(\nu_3) > 0$, $x'(\nu_4) \geq 0$ since $0 \leq z \leq 1$ and $\alpha > 1$ and we have

$$\begin{aligned}\int_0^1 |x''(s)|ds \geq & |\int_c^{\nu_3} x''(s)ds| + |\int_{\nu_3}^{\nu_4} x''(s)ds| \\ =& x'(\nu_3) + |x'(\nu_4) - x'(\nu_3)| = \frac{1+z}{\eta - c} + |\frac{\alpha z - z}{1-\eta} - \frac{1+z}{\eta - c}|.\end{aligned} \tag{18}$$

Let $F(z,c) = \frac{1+z}{\eta-c} + |\frac{\alpha z - z}{1-\eta} - \frac{1+z}{\eta - c}|$. We need to estimate $\min_{c\in[0,\eta), z\in[0,\frac{1}{\alpha}]} F(z,c)$. We note that

$$F(0,c) = \frac{2}{\eta - c} \geq \frac{2}{\eta} \quad \text{for } c \in [0,\eta),$$

$$F(\frac{1}{\alpha}, c) = \frac{\alpha+1}{\alpha(\eta - c)} + |\frac{\alpha - 1}{\alpha(1-\eta)} - \frac{\alpha+1}{\alpha(\eta-c)}| \geq \frac{\alpha-1}{\alpha(1-\eta)} \quad \text{for } c \in [0,\eta).$$

Let z_0 be such that $\frac{\alpha z_0 - z_0}{1-\eta} - \frac{1+z_0}{\eta - c} = 0$ so that $z_0 = \frac{1-\eta}{\alpha\eta - 1 - c(\alpha-1)}$. It is easy to see that $z_0 \in [0, \frac{1}{\alpha}]$ if $\eta > \frac{\alpha+1}{2\alpha}$ and $c \in (0, \frac{2\alpha\eta - \alpha - 1}{\alpha - 1})$. In this case we get $F(z_0, c) =$

$\frac{\alpha-1}{\alpha\eta-1-c(\alpha-1)} \geq \frac{\alpha-1}{\alpha\eta-1}$. Accordingly we see that $F(z,c) \geq \min\{\frac{2}{\eta}, \frac{\alpha-1}{\alpha(1-\eta)}\}$ if $\alpha\eta \leq 1$ and $F(z,c) \geq \min\{\frac{2}{\eta}, \frac{\alpha-1}{\alpha(1-\eta)}, \frac{\alpha-1}{\alpha\eta-1}\}$ if $\alpha\eta > 1$. We thus have from (18) that

$$\begin{aligned}\int_0^1 |x''(s)|ds \geq& |\int_c^{\nu_3} x''(s)ds| + |\int_{\nu_3}^{\nu_4} x''(s)ds| = x'(\nu_3) + |x'(\nu_4) - x'(\nu_3)| \\ =& \frac{1+z}{\eta-c} + |\frac{\alpha z - z}{1-\eta} - \frac{1+z}{\eta-c}| \\ \geq& \begin{cases} \min\{\frac{2}{\eta}, \frac{\alpha-1}{\alpha(1-\eta)}\}, & \text{if } \alpha\eta \leq 1, \\ \min\{\frac{2}{\eta}, \frac{\alpha-1}{\alpha(1-\eta)}, \frac{\alpha-1}{\alpha\eta-1}\}, & \text{if } \alpha\eta > 1. \end{cases}\end{aligned}$$

(iii) Next, suppose that $c \in (\eta, 1)$, $x(c) = 1$. Again, $x'(c) = 0$ and we have from mean value theorem that there exist $\nu_5 \in [\eta, c], \nu_6 \in [c, 1]$ such that

$$x'(\nu_5) = \frac{x(c) - x(\eta)}{c-\eta} = \frac{1-z}{c-\eta}, \quad x'(\nu_6) = \frac{x(1) - x(c)}{1-c} = \frac{\alpha z - 1}{1-c}.$$

Note that $x'(\nu_5) \geq 0$, $x'(\nu_6) \leq 0$ since $x(1) = \alpha z \leq 1$. Accordingly, we obtain

$$\begin{aligned}\int_0^1 |x''(s)|ds \geq& |\int_0^{\nu_5} x''(s)ds| + |\int_{\nu_5}^{\nu_6} x''(s)ds| \\ =& x'(\nu_5) + |x'(\nu_6) - x'(\nu_5)| = 2x'(\nu_5) + |x'(\nu_6)| \\ =& 2\frac{1-z}{c-\eta} + \frac{1-\alpha z}{1-c} \geq \frac{2(\alpha-1)}{\alpha(1-\eta)}, \quad \text{since } 0 \leq z \leq \frac{1}{\alpha}.\end{aligned} \tag{19}$$

(iv) Next, suppose that $c \in (\eta, 1)$, $x(c) = -1$. Again, $x'(c) = 0$ and we have from mean value theorem that there exist $\nu_7 \in [\eta, c], \nu_8 \in [c, 1]$ such that

$$x'(\nu_7) = \frac{x(c) - x(\eta)}{c-\eta} = \frac{-1-z}{c-\eta}, \quad x'(\nu_8) = \frac{x(1) - x(c)}{1-c} = \frac{\alpha z + 1}{1-c}.$$

Note that $x'(\nu_7) \leq 0$, $x'(\nu_8) \geq 0$. Accordingly, we obtain

$$\begin{aligned}\int_0^1 |x''(s)|ds \geq& |\int_0^{\nu_7} x''(s)ds| + |\int_{\nu_7}^{\nu_8} x''(s)ds| \\ =& |x'(\nu_7)| + |x'(\nu_8) - x'(\nu_7)| = 2|x'(\nu_7)| + x'(\nu_8) \\ =& 2\frac{1+z}{c-\eta} + \frac{1+\alpha z}{1-c} \geq \frac{2}{c-\eta} + \frac{1}{1-c} \\ \geq& \frac{2}{1-\eta} \geq \frac{2(\alpha-1)}{\alpha(1-\eta)}.\end{aligned}$$

(v) Finally suppose that $c = 1$, so that $x(1) = 1 = \alpha z$. We then have that there exists a $\nu_9 \in (\eta, 1)$ such that

$$x'(\nu_9) = \frac{x(1) - x(\eta)}{1-\eta} = \frac{1 - \frac{1}{\alpha}}{1-\eta} = \frac{\alpha - 1}{\alpha(1-\eta)}.$$

Also, there exists a $\nu_{10} \in (0, \eta)$ such that

$$x'(\nu_{10}) = \frac{x(\eta) - x(0)}{\eta - 0} = \frac{1}{\alpha\eta}.$$

Thus

$$\begin{aligned} \int_0^1 |x''(s)|ds &\geq |\int_{\upsilon_{10}}^{\nu_9} x''(s)ds| = |x'(\nu_9) - x'(\nu_{10})| \\ &= |\frac{1 - \frac{1}{\alpha}}{1 - \eta} - \frac{1}{\alpha\eta}| = |\frac{\alpha\eta - 1}{\alpha\eta(1 - \eta)}|. \end{aligned}$$

We thus see from (i), (ii), (iii), (iv) and (v) that for $\alpha > 1$, $\|x\|_\infty \leq M\|x''\|_1$ with

$$M = \begin{cases} \max\{\frac{\eta}{2}, \frac{\alpha(1-\eta)}{\alpha-1}, \frac{\alpha\eta(1-\eta)}{1-\alpha\eta}\} & \text{if } \alpha\eta \leq 1, \\ \max\{\frac{\eta}{2}, \frac{\alpha\eta-1}{\alpha-1}, \frac{\alpha\eta(1-\eta)}{\alpha\eta-1}\} & \text{if } \alpha\eta > 1, \end{cases}$$

since for $\alpha > 1$, $\alpha\eta > 1$, $\frac{\alpha\eta(1-\eta)}{\alpha\eta-1} > \frac{\alpha(1-\eta)}{\alpha-1}$. This completes the present proof. □

Remark 6 Let $\alpha = 4$ and $\eta = \frac{1}{2}$. Let us consider the estimate

$$\|x\|_\infty \leq C\|x''\|_1, \tag{20}$$

for $x(t) \in W^{2,1}(0, 1)$ with $x(0) = 0$, $x(1) = 4x(\frac{1}{2})$. Now, the function

$$\varphi(t) = \begin{cases} 2t^3, & \text{for } t \in [0, \frac{1}{2}], \\ \frac{3t-1}{2}, & \text{for } t \in [\frac{1}{2}, 1], \end{cases} \tag{21}$$

is such that $\varphi(t) \in W^{2,1}(0, 1)$ with $\varphi(0) = 0$ and $\varphi(1) = 4\varphi(\frac{1}{2})$. Moreover, $\|\varphi\|_\infty = 1$ and $\|\varphi''\|_1 = \frac{3}{2}$. It follows that $C \geq \frac{2}{3}$ in (20). Now, Proposition 1 and Remark 3 give $C = 3$ in (20); while Theorem 2 and Remark 4 give $C = 2$ in (20); and Theorem 5 gives $C = 1$ in (20). This shows that Theorem 5 gives the best estimate $\|x\|_\infty \leq \|x''\|_1$ for $x(t) \in W^{2,1}(0, 1)$ with $x(0) = 0$, $x(1) = 4x(\frac{1}{2})$. However, the function $\varphi(t)$ defined in (21) indicates that it may be possible to improve C in (20). This question remains open at this time.

To explore this further we introduce the notion of *approximate best constant* in the following.

Definition $B \in \mathbb{R}$ is called "approximate best constant" if for every $\varepsilon > 0$ there exists an $\alpha \in \mathbb{R}$ and an $\eta \in (0, 1)$ such that (i) for every $x(t) \in W^{2,1}(0, 1)$ with $x(0) = 0$, $x(1) = \alpha x(\eta)$, $\|x\|_\infty \leq (B + \varepsilon)\|x''\|_1$; (ii) there exists a function $\phi(t) \in W^{2,1}(0, 1)$ with $\phi(0) = 0$, $\phi(1) = \alpha\phi(\eta)$, and $\|\phi\|_\infty > B\|\phi''\|_1$.

Theorem 7 *For every $k > 1$, $1 - \frac{1}{k}$ is an approximate best constant.*

Proof For each integer $n > 2$, consider the function $\phi_{kn}(t) \in W^{2,1}(0,1)$ defined by

$$\phi_{kn}(t) = \begin{cases} t^n, & \text{for } t \in [0, \frac{1}{k}], \\ \frac{nt}{k^{n-1}} - \frac{n-1}{k^n}, & \text{for } t \in [\frac{1}{k}, 1]. \end{cases}$$

It is easy to see that $\phi_{kn}(t) \in W^{2,1}(0,1)$, with $\phi_{kn}(0) = 0$, $\phi_{kn}(1) = \alpha_{kn}\phi_{kn}(\frac{1}{k})$, where $\alpha_{kn} = n(k-1)+1$, and

$$\|\phi''_{kn}\|_1 = \frac{n}{k^{n-1}}, \quad \|\phi_{kn}\|_\infty = \phi_{kn}(1) = \frac{n(k-1)+1}{k^n},$$

so that

$$\|\phi_{kn}\|_\infty = \frac{n(k-1)+1}{nk}\|\phi''_{kn}\|_1. \tag{22}$$

Now, since $\alpha_{kn} \cdot \frac{1}{k} = \frac{n(k-1)+1}{k} = n - \frac{n-1}{k} > 1$ for $n > 2$, we obtain using Theorem 5 the estimate

$$\begin{gathered} \|x\|_\infty \le \frac{n(k-1)+1}{k(n-1)}\|x''\|_1 \quad \text{for } x(t) \in W^{2,1}(0,1) \\ x(0) = 0, \quad x(1) = \alpha_{kn}x(\frac{1}{k}). \end{gathered} \tag{23}$$

Let us set $B_{kn} = \frac{n(k-1)+1}{nk} = 1 - \frac{1}{k} + \frac{1}{nk}$, $M_{kn} = \frac{n(k-1)+1}{k(n-1)} = 1 - \frac{1}{k} + \frac{1}{n-1}$. We notice that

$$M_{kn} - B_{kn} = \frac{1}{n-1} - \frac{1}{nk} = \frac{n(k-1)+1}{n(n-1)k} > 0,$$

so that $M_{kn} - B_{kn} > 0$. Also, we note that

$$\lim_{n\to\infty} B_{kn} = \lim_{n\to\infty} M_{kn} = 1 - \frac{1}{k}.$$

Let, now, $\varepsilon > 0$ be given. Choose, n_0 such that $M_{kn_0} < 1 - \frac{1}{k} + \varepsilon$. It, now, follows from (23) and (22) that

$$\begin{gathered} \|x\|_\infty \le (1 - \frac{1}{k} + \varepsilon)\|x''\|_1 \quad \text{for } x(t) \in W^{2,1}(0,1) \\ x(0) = 0, \quad x(1) = \alpha_{kn_0}x(\frac{1}{k}), \end{gathered}$$

and

$$\|\phi_{kn_0}\|_\infty = (1 - \frac{1}{k} + \frac{1}{n_0 k})\|\phi''_{kn}\|_1 > (1 - \frac{1}{k})\|\phi''_{kn}\|_1.$$

This completes the proof of the Theorem. □

Remark 8 We note that $\lim_{k\to\infty}(1 - \frac{1}{k}) = 1$. In view of this, it may be conjectured that 1 may be a best constant in the sense that there exists an $\alpha \in \mathbb{R}$ and an $\eta \in (0,1)$ such that for $x(t) \in W^{2,1}(0,1)$ with $x(0) = 0$, $x(1) = \alpha x(\eta)$ one has the estimate

$$\|x\|_\infty \le \|x''\|_1.$$

However, since $\lim_{k\to\infty} \alpha_{kn} = \infty$ and $\lim_{k\to\infty} \frac{1}{k} = 0$, it is not clear if such $\alpha \in \mathbb{R}$ and an $\eta \in (0,1)$ exist.

3 Existence theroems

We state below the existence theorems one obtains using the a priori estimates obtained above. We omit the proof of these theorems as they are similar to the corresponding theorems in [2].

Theorem 9 *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Caratheodory's conditions. Assume that there exist functions $p(t)$, $q(t)$, $r(t)$ in $L^1(0,1)$ such that*

$$|f(t,x_1,x_2)| \le p(t)\,|x_1| + q(t)\,|x_2| + r(t)$$

for a.e. $t \in [0,1]$ and all $(x_1,x_2) \in \mathbb{R}^2$. Let $a_i \in \mathbb{R}$, $\xi_i \in (0,1)$, $i = 1,2,\ldots,m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ with $\sum_{i=1}^{m-2} a_i\xi_i \ne 1$ and $\sum_{i=1}^{m-2} a_i \ne 1$, be given. Then the multi-point boundary-value problem

$$x''(t) = f(t,x(t),x'(t)) + e(t), \quad 0 < t < 1,$$
$$x(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i).$$

has at least one solution in $C^1[0,1]$ provided

$$C\,\|p(t)\|_1 + \frac{1}{1-\tau}\,\|q(t)\|_1 < 1,$$

where C is as given in Theorem 2 and τ as given in Proposition 1.

Theorem 10 *Let $f : [0,1] \times \mathbb{R}^2 \mapsto \mathbb{R}$ be a function satisfying Caratheodory's conditions. Assume that there exist functions $p(t)$, $q(t)$, $r(t)$ such that the functions $p(t)$, $q(t)$, $r(t)$ are in $L^1(0,1)$ and*

$$|f(t,x_1,x_2)| \le p(t)\,|x_1| + q(t)\,|x_2| + r(t)$$

for a.e. $t \in [0,1]$ and all $(x_1,x_2) \in \mathbb{R}^2$. Let $\alpha \in \mathbb{R}$, $\eta \in (0,1)$, $\alpha \ne 1$, and $\alpha\eta \ne 1$ be given. Then, the three-point boundary-value problem

$$x''(t) = f(t,x(t),x'(t)) + e(t), \quad 0 < t < 1,$$
$$x(0) = 0, \quad x(1) = \alpha x(\eta).$$

has at least one solution in $C^1[0,1]$ provided

$$M\,\|p(t)\|_1 + \frac{1}{1-\tau}\,\|q(t)\|_1 < 1.$$

where M is as given in Theorem 5 and τ as given in Proposition 1.

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