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Existence of many positive nonradial solutions for a superlinear Dirichlet problem on thin annuli *

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Abstract

We study the existence of many positive nonradial solutions of a superlinear Dirichlet problem in an annulus in $\mathbb{R}^N$. Our strategy consists of finding the minimizer of the energy functional restricted to the Nehrai manifold of a subspace of functions with symmetries. The minimizer is a global critical point and therefore is a desired solution. Then we show that the minimal energy solutions in different symmetric classes have mutually different energies. The same approach has been used to prove the existence of many sign-changing nonradial solutions (see [5]).

1 Introduction

In this article we study the existence of many positive nonradial solutions of the equation

$$\begin{aligned} \Delta u + f(u) = 0 \quad \text{in } \Omega_\epsilon \\ u > 0 \quad \text{in } \Omega_\epsilon \\ u = 0 \quad \text{on } \partial\Omega_\epsilon\,, \end{aligned} \tag{1}$$

where

$$\Omega_\epsilon = \{x \in \mathbb{R}^3 : 1-\epsilon < |x| < 1\}.$$

The non-linearity f is of class $C^1(\mathbb{R})$ and satisfies the following conditions:

(C1) $f(0) = 0$ and $f'(0) < \lambda_1$, where λ_1 is the smallest eigenvalue of $-\Delta$ with zero Dirichlet boundary condition in Ω_ϵ.

(C2) $f'(u) > \frac{f(u)}{u}$ for all $u \neq 0$.

(C3) (Superlinearity) $\lim_{|u|\to\infty} \frac{f(u)}{u} = \infty$.

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(C4) (Subcritical growth) There exist constants $C > 0$ and $p \in (1, 5)$ such that
$$|f'(u)| \le C(|u|^{p-1} + 1), \quad \forall u \in \mathbb{R}.$$

(C5) There exist constants $m \in (0, 1)$ and ρ such that for $|u| > \rho$,
$$uf(u) \ge \frac{2}{m} F(u) > 0,$$
where $F(u) = \int_0^u f(s)\, ds$.

This paper is motivated by the work of Coffman [6], Li [7] and Lin [8]. In 1984, Coffman showed that for $f(t) = -t + t^p$, where $p = 2N + 1, N = 2$, the above problem has many positive nonradial solutions. His result was extended by Li [7] to the N-dimensional case with $N \ge 4$ or $N = 2$ and for $p \in (1, \frac{N+2}{N-2})$, when the nonlinear term is $\lambda t + t^p$ for $\lambda \le 0$. Our main result (see Theorem 1.1 below) concerns the case $N = 3$. We will show that for $N = 3$ our problem has many distinct nonradial solutions. We follow the strategy used in [6, 7] and [8]. That is, we look for the minimizer of the energy functional restricted to the Nehari manifold of a subspace of functions with symmetries. The minimizer is a global critical point and therefore is a solution to (1). Then we show that the minimal energy solutions in different symmetric classes has mutually different energies.

We would like to point out here that during the preparation of this article we were unaware of the papers by Byeon [1] and Catrina and Wang [2], where the above problem has been solved. However, our approach is different from theirs.

Our main result is the following

Theorem 1.1 *Let conditions (C1) - (C5) be satisfied. Then, for each positive integer k there exists $\epsilon_1(k) > 0$ such that if $0 < \epsilon < \epsilon_1(k)$ then (1) has k distinct positive nonradial solutions.*

We note that our argument works for $N \ge 2$ and not only for the three dimensional case.

The approach used in proving Theorem 1.1 is similar to the one used in [7], i.e., we consider the functionals
$$J(u) = \int_{\Omega_\epsilon} \{\frac{1}{2}|\nabla u|^2 - F(u)\}dx$$
and
$$\gamma(u) = \int_{\Omega_\epsilon} \big(|\nabla u|^2 - uf(u)\big)\, dx$$
on $H_0^1(\Omega_\epsilon)$, and for $k \ge 1$, we consider the space of invariant functions
$$\begin{aligned} H(\epsilon, k) \;&:=\; \mathrm{Fix}(G_k \times \{id_{\mathbb{R}}, -id_{\mathbb{R}}\}) \\ &=\; \big\{v \in H_0^1(\Omega_\epsilon) : v(g(x_1, x_2), Tx_3) = v(x_1, x_2, x_3), \end{aligned}$$

$$
\begin{aligned}
& \quad \forall\ (g,T) \in G_k \times \{id_{\mathbb{R}}, -id_{\mathbb{R}}\}\big\} \\
= & \quad \big\{v \in H_0^1(\Omega_\epsilon) : v(x_1,x_2,x_3) = u(x_1,x_2,|x_3|) \text{ for some } u \\
& \quad \text{satisfying } u(g(x_1,x_2),|x_3|) = u(x_1,x_2,|x_3|)\ \forall g \in G_k\big\}
\end{aligned}
$$

where $G_k = \{g^i : 0 \le i \le k-1\}$ and $gz = e^{2\pi i/k}z$, $z \in \mathbb{C} \simeq \mathbb{R}^2$. Also, we consider the Nehari manifold

$$S(\epsilon,k) = \{v \in H(\epsilon,k)\backslash\{0\} : \gamma(v) = 0\}.$$

Similarly, we define the space of functions

$$
\begin{aligned}
H(\epsilon,\infty) \quad := & \quad \mathbf{O}(2) \times \{id_{\mathbb{R}}, -id_{\mathbb{R}}\} \\
= & \quad \big\{v \in H_0^1(\Omega_\epsilon) : v(g(x_1,x_2),Tx_3) = v(x_1,x_2,x_3), \\
& \quad \forall (g,T) \in \mathbf{O}(2) \times \{id_{\mathbb{R}}, -id_{\mathbb{R}}\}\big\} \\
= & \quad \big\{v \in H_0^1(\Omega_\epsilon) : v(x_1,x_2,x_3) = u(\sqrt{x_1^2+x_2^2}, |x_3|), \text{ for some } u\big\},
\end{aligned}
$$

and the manifold

$$S(\epsilon,\infty) = \{v \in H(\epsilon,\infty)\backslash\{0\} : \gamma(v) = 0\},$$

where $\mathbf{O}(2)$ denotes the space of 2×2 orthogonal matrices. Note that if $u \in H(\epsilon,\infty)$ then u is $\theta-$independent.

Associated with the above sets, we consider the numbers

$$j_k^\epsilon = \inf_{v \in S(\epsilon,k)} J(v) \quad \text{and} \quad j_\infty^\epsilon = \inf_{v \in S(\epsilon,\infty)} J(v).$$

We prove Theorem 1.1 by establishing the following lemmas:

Lemma 1.2 *For $k = 1,2,...,\infty$, j_k^ϵ is achieved by some $u_{\epsilon,k} \in S(\epsilon,k)$ and $u_{\epsilon,k}$ is a critical point of J on $H(\epsilon,k)$.*

Lemma 1.3 *Let $u_{\epsilon,k}$ be as in Lemma 1.2. Then $u_{\epsilon,k}$ is a critical point of J on $H_0^1(\Omega_\epsilon)$.*

Lemma 1.4 *Given a positive integer k, there exists $\epsilon_1(k)$ such that for $0 < \epsilon < \epsilon_1(k)$ we have*

$$j_k^\epsilon < j_\infty^\epsilon.$$

Lemma 1.5 *For $n = 2,3,4,...$, $k = 1,2,3,...$, $j_{kn}^\epsilon < j_\infty^\epsilon$ implies $j_k^\epsilon < j_{kn}^\epsilon$.*

This article is organized as follows: In Section 2, we prove Lemmas 1.2–1.5. In Section 3, we prove Theorem 1.1.

2 Proof of Lemmas 1.2–1.5

Proof of Lemma 1.2: The proof follows from [3] combined with the facts that the embeddings $H(\epsilon,k)\subset L^p(\Omega_\epsilon)$ and $H(\epsilon,\infty)\subset L^\infty(\Omega_\epsilon)$ are compact (See [9]) $\diamondsuit$

Lemma 1.3 is a result of the symmetric criticality principle: if $u_{\epsilon,k}$ is a critical point of J on $H(\epsilon,k)$, then $u_{\epsilon,k}$ is a critical point of J on $H_0^1(\Omega_\epsilon)$ (See [9]).

To prove Lemma 1.4 we need the following results.

Lemma 2.1 (Poincaré's inequality) *Let $\Omega\subset\mathbb{R}^N$ be a smooth bounded domain with diameter d. Then for $u\in H_0^1(\Omega)$ we have*

$$\int_\Omega u^2dx\le\frac{d^2}{N}\int_\Omega|\nabla u|^2dx\,.$$

Lemma 2.2 ([3]) *0 is a local minimum of J. If $u\in H(\epsilon,k)-\{0\}$, then there exists a unique $\alpha=\alpha(u)\in(0,\infty)$ such that $\alpha u\in S(\epsilon,k)$. Moreover, $J(\alpha u)=$* $\max_{\lambda>0}J(\lambda u)>0$.

Lemma 2.3 ([4]) *For $|v|>\rho$ and $s>2$ we have*

$$svf(sv)\ge Cs^{2/m}vf(v)$$

for some constant $C>0$.

Proof of Lemma 1.4: Let $k\ge1$ be an integer and $\epsilon>0$ to be chosen below. According to Lemma 1.2, there exist $u_{\epsilon,k}\in S(\epsilon,k)$ and $u_{\epsilon,\infty}\in S(\epsilon,\infty)$ such that $j_k^\epsilon=J(u_{\epsilon,k})$ and $j_\infty^\epsilon=J(u_{\epsilon,\infty})$. By Lemma 1.3, $u_{\epsilon,k}$ and $u_{\epsilon,\infty}$ are solutions to Problem 1. Let Ω_ϵ^k be the set of points $x=(x_1,x_2,x_3)\equiv(r,\theta,x_3)\in\Omega_\epsilon$ such that $\theta\in[0,2\pi/k]$.

Let j be a positive integer to be chosen independent of k and ϵ. Define

$$\omega(r,\theta,x_3)=\begin{cases}u_{\epsilon,\infty}(r,|x_3|)\sin(jk\theta) & \text{for }\theta\in[0,\pi/(jk)]\\ 0 & \text{for }\theta\in[\pi/(jk),2\pi/k]\,,\end{cases}$$

where $r=(x_1^2+x_2^2)^{1/2}$. Extend ω periodically to all of Ω_ϵ in the θ direction. Let $z\in H(\epsilon,k)$ be the resulting extension. Since $u_{\epsilon,\infty}>0$ then $z>0$. By the chain rule we have

$$\begin{aligned}|\nabla\omega(r,\theta,x_3)|^2 &= (u_{\epsilon,\infty})_r^2\sin^2(jk\theta)+\frac{1}{r^2}(u_{\epsilon,\infty})^2(jk)^2\cos^2(jk\theta)\\ &\quad+|\nabla_{x_3}u_{\epsilon,\infty}|^2\sin^2(jk\theta)\end{aligned}$$

if $\theta\in[0,\pi/(jk)]$, otherwise $\nabla\omega=0$. Thus,

$$|\nabla\omega(r,\theta,x_3)|^2\le(u_{\epsilon,\infty})_r^2+\frac{1}{r^2}(jk)^2(u_{\epsilon,\infty})^2+|\nabla_{x_3}u_{\epsilon,\infty}|^2.$$

By Lemma 2.1 we have

$$\int_{\Omega_\epsilon^{2jk}} u_{\epsilon,\infty}^2\, dx_1\, x_2\, dx_3 \le \frac{(\mathrm{diam}(\Omega_\epsilon^{2jk}))^2}{3} \int_{\Omega_\epsilon^{2jk}} |\nabla u_{\epsilon,\infty}|^2\, dx_1\, x_2\, dx_3.$$

Thus,

$$\begin{aligned}
&\int_{\Omega_\epsilon} |\nabla z|^2\, dx_1\, x_2\, dx_3 \\
&= k \int_{\Omega_\epsilon^{2jk}} |\nabla \omega|^2\, dx_1\, x_2\, dx_3 \\
&\le k \int_{\Omega_\epsilon^{2jk}} [(1 + \frac{16(jk)^2}{27}\epsilon^2)(u_{\epsilon,\infty})_r^2 + |\nabla_{x_3} u_{\epsilon,\infty}|^2]\, dx_1\, x_2\, dx_3 \\
&\le 2k \int_{\Omega_\epsilon^{2jk}} ((u_{\epsilon,\infty})_r^2 + |\nabla_{x_3} u_{\epsilon,\infty}|^2)\, dx_1\, x_2\, dx_3 \qquad (2)\\
&= 2k \int_{\Omega_\epsilon^{2jk}} |\nabla u_{\epsilon,\infty}|^2\, dx_1\, x_2\, dx_3
\end{aligned}$$

provided that $\epsilon < 3\sqrt{3}/(4jk)$. By Lemma 2.2 we can find $\alpha > 0$ such that $\gamma(\alpha z) = 0$. Let $D = \{(r,\theta,x_3) : u_{\epsilon,\infty}(r,|x_3|) > \rho, \theta \in [\frac{\pi}{4jk}, \frac{\pi}{2jk}]\}$. Suppose that $\alpha > \sqrt{2}$. Then for $(r,\theta,x_3) \in D$ we have $\alpha \sin(jk\theta) > 2$. This, the fact that $tf(t)$ is bounded from below, say by E, and Lemma 2.3 imply

$$\begin{aligned}
&\int_{\Omega_\epsilon} \alpha z f(\alpha z)\, dx_1\, x_2\, dx_3 \\
&= k \int_{\Omega_\epsilon^{2jk}} \alpha \sin(jk\theta) u_{\epsilon,\infty} f(\alpha \sin(jk\theta) u_{\epsilon,\infty})\, dx_1\, x_2\, dx_3 \\
&\ge k \left(E|\Omega_\epsilon^{2jk}| + C\alpha^{2/m} \int_D u_{\epsilon,\infty} f(u_{\epsilon,\infty})\, dx_1\, x_2\, dx_3 \right) \qquad (3)\\
&= k \left(E|\Omega_\epsilon^{2jk}| + C\alpha^{2/m} (\int_{u_{\epsilon,\infty} > \rho} u_{\epsilon,\infty} f(u_{\epsilon,\infty}) r\, dr\, dx_3)(\int_{\pi/(4jk)}^{\pi/(2jk)} d\theta) \right) \\
&= kE|\Omega_\epsilon^{2jk}| + k\frac{C}{4}\alpha^{2/m} (\int_{u_{\epsilon,\infty} > \rho} u_{\epsilon,\infty} f(u_{\epsilon,\infty}) r\, dr\, dx_3)(\int_0^{\pi/(jk)} d\theta),
\end{aligned}$$

where $|\Omega_\epsilon|$ denotes the volume of Ω_ϵ.

Now, let $\epsilon < 1/4$. Then $K = \inf_{u \in S(1/4,k)} J(u) \le \inf_{u \in S(\epsilon,k)} J(u)$. Let $u_{\epsilon,\infty}$ also denote the zero extension of $u_{\epsilon,\infty}$ to all of $\Omega_{1/4}$. Then $u_{\epsilon,\infty} \in S(1/4,k)$. Thus, $J(u_{\epsilon,\infty}) \ge K$ and consequently

$$\int_{\Omega_\epsilon} |\nabla u_{\epsilon,\infty}|^2\, dx_1\, x_2\, dx_3 \quad \ge \quad 2K + 2M|\Omega_\epsilon| \qquad (4)$$

where $M = \inf\{F(t) : t \in \mathbb{R}\}$.
If we choose ϵ so that $|\Omega_\epsilon| < K/(-2M)$ then (4) implies

$$\int_{\Omega_\epsilon} |\nabla u_{\epsilon,\infty}|^2 dx_1\, dx_2\, dx_3 \ge K \qquad (5)$$

and this leads to

$$\begin{aligned}
&\int_{u_{\epsilon,\infty}>\rho} u_{\epsilon,\infty} f(u_{\epsilon,\infty}) r\,dr\,dx_3 \\
&= \int_{u_{\epsilon,\infty}\geq 0} u_{\epsilon,\infty} f(u_{\epsilon,\infty}) r\,dr\,dx_3 - \int_{u_{\epsilon,\infty}\leq\rho} u_{\epsilon,\infty} f(u_{\epsilon,\infty}) r\,dr\,dx_3 \\
&= \Big[1 - \frac{\int_{u_{\epsilon,\infty}\leq\rho} u_{\epsilon,\infty} f(u_{\epsilon,\infty}) r\,dr\,dx_3}{\int_{u_{\epsilon,\infty}\geq 0} u_{\epsilon,\infty} f(u_{\epsilon,\infty}) r\,dr\,dx_3}\Big] \int_{u_{\epsilon,\infty}\geq 0} u_{\epsilon,\infty} f(u_{\epsilon,\infty}) r\,dr\,dx_3 \\
&= \Big[1 - \frac{2\pi\int_{u_{\epsilon,\infty}\leq\rho} u_{\epsilon,\infty} f(u_{\epsilon,\infty}) r\,dr\,dx_3}{\int_{\Omega_\epsilon} |\nabla u_{\epsilon,\infty}|^2\,dx_1\,x_2\,dx_3}\Big] \int_{u_{\epsilon,\infty}\geq 0} u_{\epsilon,\infty} f(u_{\epsilon,\infty}) r\,dr\,dx_3 \\
&\geq (1 - C\epsilon) \int_{u_{\epsilon,\infty}\geq 0} u_{\epsilon,\infty} f(u_{\epsilon,\infty}) r\,dr\,dx_3
\end{aligned}$$

where the constant C is independent of (ϵ, j, k). By choosing ϵ in such a way that $1 - C\epsilon > 1/2$ we conclude

$$\int_{u_{\epsilon,\infty}>\rho} u_{\epsilon,\infty} f(u_{\epsilon,\infty}) r\,dr\,dx_3 > \frac{1}{2} \int_{\Omega_\epsilon} u_{\epsilon,\infty} f(u_{\epsilon,\infty}) r\,dr\,dx_3.$$

This reduces (3) to

$$\int_{\Omega_\epsilon} \alpha z f(\alpha z)\,dx_1\,dx_2\,dx_3 \geq kE|\Omega_\epsilon^{2jk}| + kC\alpha^{2/m} \int_{\Omega_\epsilon^{2jk}} |\nabla u_{\epsilon,\infty}|^2\,dx_1\,dx_2\,dx_3. \quad (6)$$

On the other hand, using (2) we have

$$\begin{aligned}
\int_{\Omega_\epsilon} \alpha z f(\alpha z)\,dx_1\,x_2\,dx_3 &= \alpha^2 \int_{\Omega_\epsilon} |\nabla z|^2\,dx_1\,x_2\,dx_3 \\
&\leq 2k\alpha^2 \int_{\Omega_\epsilon^{2jk}} |\nabla u_{\epsilon,\infty}|^2\,dx_1\,x_2\,dx_3. \quad (7)
\end{aligned}$$

Combining (6) and (7) to obtain

$$(C\alpha^{2/m} - 2\alpha^2) \int_{\Omega_\epsilon^{2jk}} |\nabla u_{\epsilon,\infty}|^2\,dx_1\,x_2\,dx_3 \leq \frac{C\epsilon}{jk}$$

Hence,

$$\begin{aligned}
C\alpha^{2/m} - 2\alpha^2 &\leq \frac{C\epsilon/(jk)}{\int_{\Omega_\epsilon^{2jk}} |\nabla u_{\epsilon,\infty}|^2\,dx_1\,x_2\,dx_3} \\
&\leq \frac{C\epsilon}{\int_{\Omega_\epsilon} |\nabla u_{\epsilon,\infty}|^2\,dx_1\,x_2\,dx_3} \leq M_1\,,
\end{aligned}$$

where we have used (5). But the function $g(\alpha) = C\alpha^{2/m} - 2\alpha^2$ satisfies $g(0) = 0$ and $\lim_{\alpha\to\infty} g(\alpha) = +\infty$. Thus, if $g(\alpha) \leq M_1$ then there is a constant K' such that $\alpha \leq K'$. By letting

$$\alpha \leq \max\{\sqrt{2}, K'\} \equiv K'' \quad (8)$$

we conclude that α is bounded. Let $w = \alpha z \in S(\epsilon, k)$. Since $F(t)$ is bounded from below, say by M, then (8) and (2) imply

$$\begin{aligned}
J(w) &= \int_{\Omega_\epsilon} \left(\frac{|\nabla w|^2}{2} - F(w) \right) dx_1\, x_2\, dx_3 \\
&= \int_{\Omega_\epsilon} \frac{\alpha^2 |\nabla z|^2}{2}\, dx_1\, x_2\, dx_3 - \int_{\Omega_\epsilon} F(w)\, dx_1\, x_2\, dx_3 \\
&\le 2(K'')^2 k \int_{\Omega_\epsilon^{2jk}} \frac{|\nabla u_{\epsilon,\infty}|^2}{2}\, dx_1\, x_2\, dx_3 - kM|\Omega_\epsilon^{2jk}| \\
&\le 2\frac{(K'')^2}{j} \int_{\Omega_\epsilon} \frac{|u_{\epsilon,\infty}|^2}{2}\, dx_1\, x_2\, dx_3 + \frac{kC\epsilon}{jk} \\
&\le \frac{(K'')^2}{j} \int_{\Omega_\epsilon} \frac{|\nabla u_{\epsilon,\infty}|^2}{2}\, dx_1\, x_2\, dx_3 + \frac{1}{j} \int_{\Omega_\epsilon} \frac{|\nabla u_{\epsilon,\infty}|^2}{2}\, dx_1\, x_2\, dx_3 \\
&\le \frac{C}{j} \int_{\Omega_\epsilon} \frac{|\nabla u_{\epsilon,\infty}|^2}{2}\, dx_1\, x_2\, dx_3 \qquad (9)
\end{aligned}$$

where we have used (4) with ϵ chosen in such a way that $C\epsilon < K$ and $|\Omega_\epsilon| < K/(-2M)$. We claim that there exists a constant C such that

$$\int_{\Omega_\epsilon} \frac{|\nabla u_{\epsilon,\infty}|^2}{2}\, dx_1\, x_2\, dx_3 \le CJ(u_{\epsilon,\infty}).$$

Indeed, since $\gamma(u_{\epsilon,\infty}) = 0$ and by (C5) we have

$$\begin{aligned}
&\int_{\Omega_\epsilon} u_{\epsilon,\infty} f(u_{\epsilon,\infty})\, dx_1\, x_2\, dx_3 \\
&= \int_{\Omega_\epsilon} |\nabla u_{\epsilon,\infty}|^2\, dx_1\, x_2\, dx_3 \\
&= \int_{\Omega_\epsilon} (|\nabla u_{\epsilon,\infty}|^2 - 2F(u_{\epsilon,\infty}))\, dx_1\, x_2\, dx_3 + 2\int_{\Omega_\epsilon} F(u_{\epsilon,\infty})\, dx_1\, x_2\, dx_3 \\
&\le 2(J(u_{\epsilon,\infty}) + \frac{m}{2} \int_{\Omega_\epsilon} u_{\epsilon,\infty} f(u_{\epsilon,\infty})\, dx_1\, x_2\, dx_3 + C|\Omega_\epsilon|).
\end{aligned}$$

It follows that

$$(1-m) \int_{\Omega_\epsilon} u_{\epsilon,\infty} f(u_{\epsilon,\infty})\, dx_1\, x_2\, dx_3 \le 2(J(u_{\epsilon,\infty}) + C|\Omega_\epsilon|). \qquad (10)$$

On the other hand, using (4) and (C5) we have

$$\begin{aligned}
J(u_{\epsilon,\infty}) &= \int_{\Omega_\epsilon} (\frac{1}{2}|\nabla u_{\epsilon,\infty}|^2 - F(u_{\epsilon,\infty}))\, dx_1\, x_2\, dx_3 \\
&\ge (\frac{1-m}{2}) \int_{\Omega_\epsilon} u_{\epsilon,\infty} f(u_{\epsilon,\infty})\, dx_1\, x_2\, dx_3 - C'|\Omega_\epsilon| \\
&\ge (\frac{1-m}{2})K - C'|\Omega_\epsilon|.
\end{aligned}$$

By choosing ϵ in such a way that $C'|\Omega_\epsilon| < \frac{1}{4}(1-m)K$, we see that $J(u_{\epsilon,\infty}) > (\frac{1-m}{4})K$. Now, we choose ϵ such that $C|\Omega_\epsilon| < (\frac{1-m}{4})K$. Using this in (10) we obtain

$$\int_{\Omega_\epsilon} u_{\epsilon,\infty} f(u_{\epsilon,\infty})\, dx_1\, x_2\, dx_3 \le CJ(u_{\epsilon,\infty}).$$

Also, using this in (9) we obtain

$$J(w) \le \frac{C}{j} J(u_{\epsilon,\infty}).$$

Choosing j such that $j > 2C$ we obtain $J(w) < J(u_{\epsilon,\infty})$ and this concludes the present proof. ♢

To prove Lemma 1.5 we need

Lemma 2.4 *Let $v \in H_0^1(\Omega_\epsilon)$. Then the function*

$$P_v(\lambda) = \frac{\lambda v f(\lambda v)}{2} - F(\lambda v)$$

is increasing on $(0, \infty)$.

Proof. Differentiating $P - v$ with respect to λ we find

$$P_v'(\lambda) = \frac{\lambda v^2}{2}\left(f'(\lambda v) - \frac{f(\lambda v)}{\lambda v}\right) > 0\,,$$

where we have used (C2). This completes the proof ♢

Proof of Lemma 1.5: Fix k and n. Let $0 < \epsilon < \epsilon_1(kn)$. Lemma 1.2 guarantees the existence of a minimizer $u_{\epsilon,kn}$ of J on $S(\epsilon, kn)$. From Lemma 1.3 we see that $u_{\epsilon,kn}$ is a solution to (1). Furthermore, from Lemma 1.4 we know that $u_{\epsilon,kn}$ is nonradial. Now, by the regularity theory of elliptic equations we know that $u_{\epsilon,kn}$ is a C^2 function. Let $x = (r, \theta)$ be the polar coordinates of $x \in \mathbb{R}^2$ and write $u = u_{\epsilon,kn}(r, \theta, |x_3|)$. Then

$$\int_{\Omega_\epsilon} |\nabla u|^2\, dx_1\, x_2\, dx_3 = \int_{(r,|x_3|)} \int_0^{2\pi} (u_r^2 + \frac{1}{r^2} u_\theta^2 + |\nabla_{x_3} u|^2) r\, dr\, d\theta\, dx_3$$

and

$$\int_{\Omega_\epsilon} F(u)\, dx_1\, x_2\, dx_3 = \int_{(r,|x_3|)} \int_0^{2\pi} F(u) r\, dr\, d\theta\, dx_3\,.$$

Define the function

$$v(r, \theta, |x_3|) = u(r, \frac{\theta}{n}, |x_3|), \quad 0 \le \theta \le 2\pi.$$

Then

$$\begin{aligned} v(r, \theta + \frac{2\pi}{k}, |x_3|) &= u(r, \frac{\theta}{n} + \frac{2\pi}{kn}, |x_3|) \\ &= u(r, \frac{\theta}{n}, |x_3|) \\ &= v(r, \theta, |x_3|). \end{aligned}$$

It follows that $v \in H(\epsilon, k)$. On the other hand, it is easy to check that the following equalities hold

$$\begin{aligned} v_r(r,\theta,|x_3|) &= u_r(r,\frac{\theta}{n},|x_3|)\,,\\ v_\theta(r,\theta,|x_3|) &= \frac{1}{n}u_\theta(r,\frac{\theta}{n},|x_3|)\,,\\ \nabla_{x_3}v(r,\theta,|x_3|) &= \nabla_{x_3}u(r,\frac{\theta}{n},|x_3|). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\Omega_\epsilon} |\nabla v|^2\,dx_1\,x_2\,dx_3 &= k\int_{(r,|x_3|)}\int_0^{2\pi/k}(u_r^2(r,\frac{\theta}{n},|x_3|)+\frac{1}{r^2n^2}u_\theta^2(r,\frac{\theta}{n},|x_3|)\\ &\quad +|\nabla_{x_3}u(r,\frac{\theta}{n},|x_3|)|^2)r\,dr\,d\theta\,dx_3\\ &= kn\int_0^{\frac{2\pi}{nk}}(u_r^2(r,\theta,|x_3|)+\frac{1}{r^2n^4}u_\theta^2(r,\theta,|x_3|)\\ &\quad +|\nabla_{x_3}u(r,\theta,|x_3|)|^2)r\,dr\,d\theta\,dx_3\\ &= \int_{(r,|x_3|)}\int_0^{2\pi}(u_r^2(r,\theta,|x_3|)+\frac{1}{r^2n^4}u_\theta^2(r,\theta,|x_3|)\\ &\quad +|\nabla_{x_3}u(r,\theta,|x_3|)|^2)r\,dr\,d\theta\,dx_3. \end{aligned}$$

Also

$$\begin{aligned} \int_{\Omega_\epsilon} F(v)\,dx_1\,x_2\,dx_3 &= k\int_{(r,|x_3|)}\int_0^{2\pi/k}F(u(r,\frac{\theta}{n},|x_3|))r\,dr\,d\theta\,dx_3\\ &= \int_{(r,|x_3|)}\int_0^{2\pi}F(u(r,\theta,|x_3|))r\,dr\,d\theta\,dx_3\,. \end{aligned}$$

Since u does not belong to $S(\epsilon,\infty)$ we have

$$\int_{(r,|x_3|)}\int_0^{2\pi}u_\theta^2(r,\theta,|x_3|)r\,dr\,d\theta dx_3 > 0\,.$$

It follows that

$$\begin{aligned} \gamma(v) &= \int_{\Omega_\epsilon}(|\nabla v|^2 - vf(v))\,dx_1\,x_2\,dx_3\\ &= \int_{(r,|x_3|)}\int_0^{2\pi}(u_r^2(r,\theta,|x_3|)+\frac{1}{r^2n^4}u_\theta^2(r,\theta,|x_3|)\\ &\quad +|\nabla_{x_3}u(r,\theta,|x_3|)|^2 - uf(u))r\,dr\,d\theta\,dx_3\\ &< \int_{(r,|x_3|)}\int_0^{2\pi}(u_r^2(r,\theta,|x_3|)+\frac{1}{r^2}u_\theta^2(r,\theta,|x_3|)\\ &\quad +|\nabla_{x_3}u(r,\theta,|x_3|)|^2 - uf(u))r\,dr\,d\theta\,dx_3\\ &= \int_{\Omega_\epsilon}(|\nabla u|^2 - uf(u))\,dx_1\,x_2\,dx_3 = 0. \end{aligned}$$

This yields

$$\int_{\Omega_\epsilon} |\nabla v|^2 \, dx_1 \, x_2 \, dx_3 < \int_{\Omega_\epsilon} v f(v) \, dx_1 \, x_2 \, dx_3 \, . \tag{11}$$

Now, by Lemma 2.2 we can find $0 < \alpha < 1$ such that $\alpha v \in S(\epsilon, k)$. Let $w = \alpha v \in S(\epsilon, k)$. Using Lemma 2.4 and the definition of j_k^ϵ we have

$$\begin{aligned} j_k^\epsilon &\leq J(w) = J(\alpha v) = P_v(\alpha) \\ &< P_v(1) \\ &= \int_{\Omega_\epsilon} (\frac{1}{2} v f(v) - F(v)) \, dx_1 \, x_2 \, dx_3 \\ &= \int_{\Omega_\epsilon} (\frac{1}{2} u f(u) - F(u)) \, dx_1 \, x_2 \, dx_3 \\ &= J(u) = j_{kn}^\epsilon . \end{aligned}$$

Putting together all the arguments above we conclude a proof of the lemma ♢

3 Proof of Main Theorem

For any integer $k \geq 1$, according to Lemma 1.4 there exists $\epsilon_1(2^k)$ such that if $0 < \epsilon < \epsilon_1(2^k)$ then

$$j_{2^k}^\epsilon < j_\infty^\epsilon .$$

It follows from Lemma 1.5 that

$$j_2^\epsilon < j_{2^2}^\epsilon < ... < j_{2^k}^\epsilon < j_\infty^\epsilon . \tag{12}$$

According to Lemma 1.2, there exists $u_{\epsilon,i} \in S(\epsilon, i), i = 1, ..., k$, such that

$$j_{2^i}^\epsilon = J(u_{\epsilon,i}).$$

Moreover, by Lemma 1.3 $u_{\epsilon,i}$ is a solution of Problem 1, $i = 1, \ldots, k$. By (12), $\{u_{\epsilon,i}\}_{i=1}^k$, are nonrotationally equivalent and nonradial. This completes the proof of the theorem. ♢

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