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TRANSITION FROM HYPERBOLICITY TO ELLIPTICITY FOR A p-SYSTEM WITH VISCOSITY

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Abstract. In this article we consider a viscous regularization of a p-system with a Van der Waals pressure law, which presents both hyperbolic and elliptic zones. Even if the purely hyperbolic Van der Walls system is strongly ill-posed, we prove that the solutions of the regularized equation exist and experience a transition from ellipticity to hyperbolicity, i.e. solutions issued from initial data in the elliptic zone will enter the hyperbolic zone at some time $T > 0$, and viceversa.

1. Introduction

We study a viscous regularization of a one dimensional p-system with a Van der Waals pressure law,

$$\begin{aligned} &\partial_t u + \partial_x v = 0, \\ &\partial_t v + \partial_x (p(u)) = \varepsilon \partial_x^2 v, \end{aligned} \tag{1.1}$$

where $(u, v) \in \mathbb{R}^2$ depend on time $t \geq 0$ and $x \in \mathbb{R}$ and the artificial viscosity parameter $\varepsilon > 0$ is thought to be small; the Van der Waals pressure law is given by

$$p(u) = (u^2 - 1)u. \tag{1.2}$$

Throughout this article we refer to system (1.1) as a Van der Walls p-system with viscosity.

1.1. Van der Waals gases.

If $\varepsilon = 0$, system (1.1) corresponds to the compressible Euler equations in Lagrangian variables,

$$\begin{aligned} &\partial_t u + \partial_x v = 0, \\ &\partial_t v + \partial_x (p(u)) = 0, \end{aligned} \tag{1.3}$$

where $u > 0$ is the specific volume and $v \in \mathbb{R}$ is the velocity.

The pressure law is not assumed to be everywhere growing: indeed, because of the specific choice (1.2) it holds

$$p'(u) < 0 \quad \text{for } |u| < 1/\sqrt{3}. \tag{1.4}$$

Condition (1.4) implies in particular that for some specific volumes the first-order operator in (1.3) is *not* hyperbolic, so that system (1.3) presents both hyperbolic

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and elliptic zones; in particular, local-in-time existence holds true a priori only for analytical data [2]. However, system (1.3) admits the formal conservation law

$$E(u) = \int_{\mathbb{T}} \frac{1}{4}|u(x)|^4 - \frac{1}{2}|u(x)|^2 + \frac{1}{2}|v(x)|^2 \, dx \equiv \text{constant}. \tag{1.5}$$

Métivier [3] mentioned how this conservation law can be used for sequences of frequency-truncated solutions.

1.2. **Regularization.** In order not to restrict to analytical data and solutions, or to frequency-truncated equations, we may regularize (1.1). Of course, the viscous regularization done in (1.1) has the effect of modifying the conservation law; in this sense, our regularization is not energy-preserving.

By contrast, it allows to keep the real character of the solution (as opposite, for example, to a dispersive regularization that would instead maintain the conservation law but would give complex solutions), and our goal is to prove that the behavior of solutions here is qualitatively similar to the expected behavior in phase transition. More precisely, we want to show that system (1.1) displays some properties of systems describing Van der Walls gases (as, for instance, transition from hyperbolic to elliptic zones), making it a good approximation of the Euler equations and a good model system for the study of phase transitions.

1.3. **Main results.** To prove that solutions to (1.1) experience a phase transition, first, in section 2, we present a local in time existence result for time of the order $\mathcal{O}(\varepsilon)$, hence proportional to the viscosity parameter. As it was in [6], we expect that a local in time existence result for time of the order one will be achievable only if considering high frequencies initial data. However, such a result is beyond the scope of the paper.

Once the existence of a Sobolev solution (at least for small time), is proved, we turn to the real core of the paper: in section 3 we show that, if a solution to (1.1) exists for ulterior times, then it experiences a phase transition. More precisely, we show that, if for some time $t_0 \geq 0$ the solution belongs to the elliptic zone for almost every $x \in \mathbb{R}$ (the precise definition will be given later on in the paper), then there exists a time $t_1 > t_0$ such that $u(t_1, x)$ belongs to the hyperbolic zone for almost every $x \in \mathbb{R}$. What is more, we also show that, once in the hyperbolic zone, the solution will eventually leave it again for some time $t_2 > t_1$, meaning that $u(t_2, x)$ will belong to the elliptic zone again for almost every $x \in \mathbb{R}$. These results are summarized in Theorem 3.7, the main result of this paper.

The technique we use is a localization of our system around a certain point $x_0 \in \mathbb{R}$ such that $u(t_0, x_0)$ belongs to the elliptic (hence unstable) zone (see also [5]).

This two-phase dynamics is reminiscent of the phenomenon of *metastability*, in which the speed of convergence of solutions towards their stable asymptotic configuration depends on the viscosity parameter (see, among others [1, 4] and the references therein). These two phenomena, metastability and transition from hyperbolicity to ellipticity, have in common the fact that the solutions exhibit a certain stable behavior before they converge to the asymptotic limit in the case of a metastable behavior, or enter the elliptic (then unstable) zone in the case of phase transitions.

2. Existence results for times of order $\mathcal{O}(\varepsilon)$

We start our analysis by proving the following theorem, which gives an existence result of a Sobolev solution over time intervals of the order ε. We stress that this is exactly what one should expect, having introduced into a strongly ill posed system like (1.3) a regularizing term of the order $\mathcal{O}(\varepsilon)$.

Theorem 2.1. *Given $(u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$, there exists $T := T_\varepsilon > 0$ such that the initial value problem*

$$
\begin{aligned}
\partial_t u + \partial_x v &= 0, \\
\partial_t v + \partial_x p(u) &= \varepsilon \partial_x^2 v, \\
(u, v)(0, x) &= (u_0, v_0)(x),
\end{aligned} \tag{2.1}
$$

has a solution $(u, v) \in L^\infty([0, T], H^1(\mathbb{R}))$.

Proof. Starting from the system

$$
\begin{aligned}
\partial_t u + \partial_x v &= 0, \\
\partial_t v + \partial_x p(u) &= \varepsilon \partial_x^2 v,
\end{aligned}
$$

and by taking the L^2 scalar product with (u, v), we obtain

$$
\begin{aligned}
\frac{1}{2}\partial_t(|u|_{L^2}^2 + |v|_{L^2}^2) &= (\partial_x v, u)_{L^2} + (-\partial_x p(u), v)_{L^2} + (\varepsilon \partial_x^2 v, v)_{L^2} \\
&= (\partial_x v, u)_{L^2} + (-\partial_x p(u), v)_{L^2} - \varepsilon |\partial_x v|_{L^2}^2 \\
&\le (|u|_{L^2} + |p(u)|_{L^2})|\partial_x v|_{L^2} - \varepsilon |\partial_x v|_{L^2}^2 \\
&\le \frac{1}{4\varepsilon}(|u|_{L^2} + |p(u)|_{L^2})^2,
\end{aligned} \tag{2.2}
$$

where we used Young's inequality. In particular, because of the explicit form of $p(u)$, we have, by Sobolev's embedding

$$
|p(u)|_{L^2} \le |u|_{L^2}|u|_{L^\infty} \le |u|_{H^1}^2,
$$

and we thus need and estimate for $|\partial_x u|_{L^2}$. By differentiating the original system and by taking the L^2 scalar product with $(\partial_x u, \partial_x v)$, we obtain

$$
\begin{aligned}
\frac{1}{2}\partial_t(|\partial_x u|_{L^2}^2 + |\partial_x v|_{L^2}^2) &\le |\partial_x u|_{L^2}|\partial_x^2 v|_{L^2} + |\partial_x p(u)|_{L^2}|\partial_x^2 v|_{L^2} - \varepsilon|\partial_x^2 v|_{L^2}^2 \\
&\le \frac{1}{4\varepsilon}(|\partial_x u|_{L^2} + |\partial_x p(u)|_{L^2})^2,
\end{aligned} \tag{2.3}
$$

where we used again Young's inequality. We now observe that

$$
|\partial_x p(u)|_{L^2} = |p'(u)\partial_x u|_{L^2} = |u(u-1)\partial_x u|_{L^2} \le |u|_{L^\infty}^2 |\partial_x u|_{L^2} + |\partial_x u|_{L^2},
$$

so that, summing (2.2) and (2.3) we end up with

$$
\begin{aligned}
\frac{1}{2}\partial_t(|u|_{H^1}^2 + |v|_{H^1}^2) &\le \frac{C}{4\varepsilon}(|u|_{L^2}^2 + |p(u)|_{L^2}^2 + |u|_{L^\infty}^4 |\partial_x u|_{L^2}^2 + |\partial_x u|_{L^2}^2) \\
&\le \frac{C}{4\varepsilon}(|u|_{L^2}^2 + |p(u)|_{L^2}^2 + |u|_{L^\infty}^4(|u|_{L^2}^2 + |\partial_x u|_{L^2}^2) + |\partial_x u|_{L^2}^2) \\
&\lesssim \frac{C}{\varepsilon}(1 + |u|_{L^\infty}^4)|u, v|_{H^1}^2.
\end{aligned}
$$

Let us now set $y(t) := |u,v|^2_{H^1}$, that solves

$$\partial_t y \lesssim \frac{1}{\varepsilon}(1+y^2)y,$$

implying

$$y(t) \lesssim y(0) + \frac{1}{\varepsilon}\int_0^t (1+y^2(s))y(s)\,dt.$$

In particular, if $|y(t)| \leq 1$, then, by Gronwall,

$$\sup_t |y(t)| \lesssim |y(0)|\big(e^{t/\varepsilon} - 1\big).$$

The condition $|y(t)| \leq 1$ gives a constrain on the final time T_ε; indeed, it reads

$$|y(0)|e^{t/\varepsilon} \leq 1 + |y(0)| \implies t \lesssim \varepsilon \log\Big(\frac{1+|y(0)|}{|y(0)|}\Big),$$

and we have proven the existence of a solution $u \in L^\infty_t H^1_x$ up to times of order $\mathcal{O}(\varepsilon)$. □

Remark 2.2. Let us stress that, even if considering initial data with small (with respect to ε) amplitude, i.e. initial data of the form

$$u_0(x) = \varepsilon^\alpha\, a(x) \quad \text{and} \quad v_0(x) = \varepsilon^\alpha\, b(x),$$

for some $(a,b) \in H^1(\mathbb{R})$ and $\alpha > 0$, the existence time T_ε is of the order ε as well. As already mentioned, this is consistent with the strongly ill posed nature of the hyperbolic system (1.3): since we regularized it with a viscous term of " amplitude" ε, we thus expect to have a stable behavior only for short times. In particular, when $\varepsilon \to 0$, we recover the results of [2], where it is proved that one needs more regular data to prove a local-in-time existence result.

As mentioned in the introduction, we expect to be able to achieve an existence result for $\mathcal{O}(1)$ time intervals only if considering highly oscillation initial data, i.e. initial data of the form

$$u_0(x) = a(\frac{x}{\varepsilon^\alpha}) \quad \text{and} \quad v_0(x) = b(\frac{x}{\varepsilon^\alpha}). \tag{2.4}$$

However, the investigation of such a result is beyond the scope of the present paper.

3. Transition from ellipticity to hyperbolicity

The goal of this section is to prove the following Claim C:

> Solutions to (1.1) issued from initial data belonging to the elliptic zone will leave the elliptic zone for some time $\bar{t} > 0$

3.1. **Symbol.**

The first issue to be cleared is what we exactly mean when we say that a differential operator is either hyperbolic or elliptic, and consequently what means that a solution to (1.1) belongs to the elliptic (or to the hyperbolic) zone. This definition can be given through the concept of the symbol of a differential operator.

We start with the case $\varepsilon = 0$; the eigenvalues associated with the first order differential operator in (1.3) are given by $\lambda_\pm = \sqrt{p'(u)}$. In particular, if $p'(u)$ is negative, then the eigenvalues are not real, meaning that the operator is not hyperbolic. Motivated by this, we can thus give the following definition.

Definition 3.1. A solution u to (1.1) belongs to the elliptic zone at time $\bar{t} > 0$ if $p'(u(\bar{t}, x)) < 0$ for almost every $x \in \mathbb{R}$.

When considering the regularized version of (1.3), i.e. when considering system (1.1), we firstly argue that the solution to system (1.1) qualitatively behaves, at least for small times, as the solution of the constant coefficient system

$$\begin{aligned} \partial_t u + \partial_x v &= 0, \\ \partial_t v - t\partial_x u &= \varepsilon \partial_x^2 v, \end{aligned} \tag{3.1}$$

where, essentially, we made the approximation $p'(u) \sim -t + o(t^2)$ for $t \sim 0$.

When computing the symbol A associated to the equation (3.1) we have

$$A = \begin{pmatrix} 0 & i\xi \\ -it\xi & \varepsilon\xi^2 \end{pmatrix}.$$

In this case negativity of the symbol (or, more precisely, negativity of at least one of the eigenvalues) indicates ellipticity of the system, potentially corresponding to a growth (hence, to an instability) for the solution. We have

$$\lambda_{\pm}(\xi, t) = \varepsilon\xi^2 \pm \sqrt{\varepsilon^2\xi^4 + 4t\xi^2},$$

and $\lambda_+(\xi, t) > 0$ while $\lambda_-(\xi, t) < 0$ for all $t > 0$; we thus expect that, as soon as $t > 0$, the solution will start to grow.

Remark 3.2. If we consider characteristic frequencies that are highly oscillating, i.e. if considering $\xi \in \mathbb{Z}/\varepsilon$, the eigenvalues $\lambda_{\pm}$ reads

$$\lambda_{\pm}(\xi, t) = \frac{\xi^2}{\varepsilon}(1 \pm \sqrt{1 + \frac{4t}{\xi^2}}),$$

where we used the fact that relevant frequencies are now large; that is, $|\xi| \sim 1/\varepsilon$. While on the one side we still have $\lambda(\xi, t)_+ > 0$ for all $t > 0$, on the other hand, if we expand the square root for ξ large we have

$$\lambda_-(\xi, t) \sim \frac{\xi^2}{\varepsilon}(-\frac{2}{\xi^2} + \frac{t}{2\xi^4}).$$

In particular, λ_- becomes positive for $t > 4\xi^2$; hence, if considering initial data which are highly oscillating (as in (2.4)), we expect the solution to stop to grow for $t > 4\xi^2$, and we thus expect to be able to prove an existence results for time interval that do not depend on ε, as we already stressed in Remark 2.2.

When considering the original system (1.1), the symbol reads

$$A = \begin{pmatrix} 0 & i\xi \\ p'(u)i\xi & \varepsilon\xi^2 \end{pmatrix},$$

and we have

$$\lambda_{\pm}(\xi, t, x) = \varepsilon\xi^2 \pm \varepsilon\xi^2\sqrt{1 - \frac{4p'(u)}{\varepsilon^2\xi^2}}.$$

In particular, if $p'(u(0, x)) < 0$ (that is, if the initial datum belongs to the elliptic zone), then $\lambda_{\pm}(\xi, 0, x) \in \mathbb{R}$ and $\lambda_-(\xi, 0, x) < 0$ for all $x \in \mathbb{R}$.

Heuristically, what we expect for the dynamics of (1.1) is the following: starting with an initial datum u_0 in the elliptic zone (i.e. such that, following the definition 3.1, $p'(u_0(x)) < 0$ for almost every $x \in \mathbb{R}$), as soon as $t > 0$ we have a growth for the solution (dictated by the negative sign of λ_-). However, if u starts to grow,

$p'(u)$ grows as well, and we expect that such a growth will led to the existence of a time $\bar{t} > 0$ such that $p'(u(\bar{t}, x))$ is positive, thus leading to the proof of Claim C.

3.2. **Transition from the elliptic to the hyperbolic zone.** As already mentioned, our goal is to prove Claim C. To do so, the strategy we want to follow is a local analysis of system (1.1) around a certain point (t_0, x_0) to be chosen appropriately (see also [5]).

Choice of the initial datum. we choose the initial datum $u(0, x) = u_0(x)$ such that

$$p'(u_0(x)) < 0, \quad \text{for almost every } x \in \mathbb{R},$$

that is, we are starting with an initial configuration belonging to the elliptic zone.

Remark 3.3. We notice that, because of the explicit form of the pressure given in (1.2), a function u belongs to the elliptic zone at a certain time $\bar{t} \geq 0$ if $|u(\bar{t}, x)| < 1/\sqrt{3}$; hence, in this specific case (see Figure 1) the initial datum u_0 will be chosen so that

$$|u_0(x)| < 1/\sqrt{3} \quad \text{for almost every } x \in \mathbb{R}.$$

Localization. As to be as general as possible, we analyze the behavior of the solution to (1.1) at a fixed time $t_0 \geq 0$ where the solution $u(t_0, x)$ belong to the elliptic zone for almost every x (we stress that $t_0 = 0$ is only a subcase of this analysis). We thus localize around a point $x_0 \in \mathbb{R}$ such that $p'(u(t_0, x_0)) < 0$, that is $|u(t_0, x_0)| < 1/\sqrt{3}$ is in the elliptic zone, and we restrict our analysis in a certain neighborhood of x_0, named here $B_\delta(x_0)$, $\delta > 0$ (see Figure 1). In particular, because of the continuity of the pressure p there holds

$$p'(u(t_0, x)) < 0 \quad \forall\, x \in B_\delta(x_0).$$

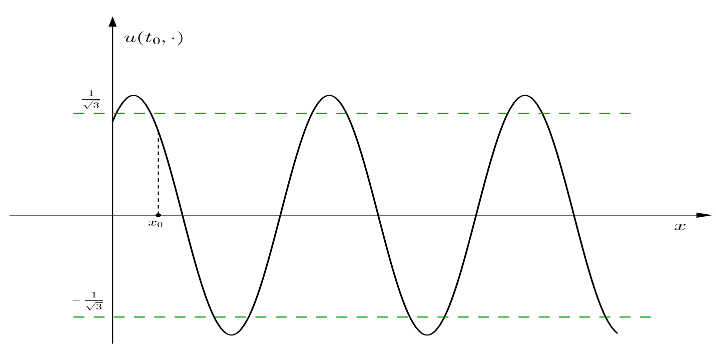

FIGURE 1. Choice of initial configuration when $p(u) = (u^2 - 1)u$: $|u(t_0, x)| < 1/\sqrt{3}$ for almost every $x \in \mathbb{R}$. The solution at time t_0 belongs to the elliptic zone. The localization is done around a generic point x_0 such that $u(t_0, x_0) < 1/\sqrt{3}$.

3.2.1. *The solution leaves the elliptic zone: a local analysis.* Going further, we choose x_0 such that the following assumption is satisfied (see again Figure 1)

(H0) $\partial_x u(t_0,x)<0$ and $\partial_x^2 u(t_0,x)<0$ for all $x\in B_\delta(x_0)$.

We stress that the case $\partial_x u(t_0,x),\partial_x^2 u(t_0,x)>0$ is completely symmetric.

Our goal is to prove that, as soon as time increases, the solution leaves the elliptic zone; more precisely, we want to prove that here exists a time $t_1>t_0$ such that $p'(u(t_1,x))>0$ for almost every $x\in\mathbb{R}$.

To this aim, let us consider the time interval $[t_0,t_0+\varepsilon T]$, with $T>0$ to be determined and $\varepsilon\ll 1$ fixed. By differentiating the first equation in (1.1) with respect to x, and by using the equation for the variable v we obtain

$$\partial_t(\partial_x u)+\frac{1}{\varepsilon}p'(u)\partial_x u=-\frac{1}{\varepsilon}\partial_t v.$$

In particular, if we consider $\partial_t v$ as a small source in the equation, locally around x_0 and for all $t\in[t_0,t_0+\varepsilon T]$ we have

$$\begin{aligned}\partial_x u(t,x_0)&=\exp\Big(-\int_{t_0}^t\frac{p'(u(s,x_0))}{\varepsilon}\,ds\Big))\partial_x u(t_0,x_0)\\&\quad-\frac{1}{\varepsilon}\int_{t_0}^t\exp\Big(-\int_{t_0}^s\frac{p'(u(s',x_0))}{\varepsilon}\,ds'\Big)\,\partial_t v(s,x_0)\,ds.\end{aligned}\tag{3.2}$$

Let us now make the following a priori assumption.

(H1) Let x_0 satisfy $\partial_t v(t,x_0)<0$ and $|\partial_t v(t,x_0)|\le h_1$ for some $h_1>0$ and for all $t\in[t_0,t_0+\varepsilon T]$.

Note that assumption (H1) is reasonable; indeed, from the equation for v we have, for all $x\in B_\delta(x_0)$

$$\partial_t v(t_0,x)=\varepsilon\partial_x^2 v(t_0,x)-p'(u(t_0,x))\partial_x u(t_0,x),$$

which is in fact negative for small ε and because of the choice of x_0. We will see later on that the assumption is in fact satisfied.

Because of assumption (H1) and recalling that $p'(u(t,x_0))<0$ and (3.2), we obtain

$$\begin{aligned}\partial_x u(t,x_0)&=\exp\Big(\int_{t_0}^t\frac{|p'(u(s,x_0))|}{\varepsilon}\,ds\Big)\partial_x u(t_0,x_0)\\&\quad-\frac{1}{\varepsilon}\int_{t_0}^t\exp\Big(\int_{t_0}^s\frac{|p'(u(s',x_0))|}{\varepsilon}\,ds'\Big)\,\partial_t v(s,x_0)\,ds,\end{aligned}$$

which implies

$$\begin{aligned}&|\partial_x u(t,x_0)|\\&\ge\exp\Big(\int_{t_0}^t\frac{|p'(u(s,x_0))|}{\varepsilon}\,ds\Big)|\partial_x u(t_0,x_0)|\\&\quad-\frac{1}{\varepsilon}\int_{t_0}^t\exp\Big(\int_{t_0}^s\frac{|p'(u(s',x_0))|}{\varepsilon}\,ds'\Big)\,|\partial_t v(s,x_0)|\,ds\\&\ge\exp\Big(\int_{t_0}^t\frac{|p'(u(s,x_0))|}{\varepsilon}\,ds\Big)\big[|\partial_x u(t_0,x_0)|-\frac{1}{\varepsilon}|\partial_t v(s,x_0)|(t-t_0)\big].\end{aligned}\tag{3.3}$$

To obtain some informations on $p'(u(s,x_0)$, we use its Taylor expansion around t_0. We have

$$\begin{aligned} p'(u(t,x_0)) &= p'(u(t_0,x_0)) + \partial_t p'(u(t,x_0))|_{t=t_0}(t-t_0) \\ &= p'(u(t_0,x_0)) + p''(u(t_0,x_0))\partial_t u(t_0,x_0)(t-t_0) \\ &= p'(u(t_0,x_0)) - p_0''\partial_x v(t_0,x_0)(t-t_0), \end{aligned} \tag{3.4}$$

with notation $p''(u(t_0,x_0)) = p_0'' < 0$.

We now require the following assumption on $\partial_x v(t,x_0)$.

(H2) Let x_0 satisfy $|\partial_x v(t,x_0)| < c_1$ for all $t \in [t_0, t_0+\varepsilon T]$ with c_1 such that

$$c_1|p_0''| \le |p'(u(t_0,x_0))|. \tag{3.5}$$

From (3.4) we have

$$p'(u(t,x_0)) - p'(u(t_0,x_0)) = -p_0''\partial_x v(t_0,x_0)(t-t_0),$$

which by assumption (H2) implies

$$\begin{aligned} |p'(u(t,x_0)) - p'(u(t_0,x_0))| &\le |p_0''||\partial_x v(t_0,x_0)|(t-t_0) \\ &\le |p'(u(t_0,x_0))|(t-t_0). \end{aligned}$$

Since $|p'(u(t,x_0)) - p'(u(t_0,x_0))| > |p'(u(t_0,x_0))| - |p'(u(t,x_0))|$ we end up with

$$|p'(u(t,x_0))| \ge (1-t+t_0)|p'(u(t_0,x_0))|.$$

In particular, from (3.3),

$$\begin{aligned} &|\partial_x u(t,x_0)| \\ &\ge \exp\Big(\frac{1-t+t_0}{\varepsilon}|p'(u(t_0,x_0))|(t-t_0)\Big)\Big(|\partial_x u(t_0,x_0)| - \frac{1}{\varepsilon}|\partial_t v(t,x_0)|(t-t_0)\Big), \end{aligned}$$

so that, setting $p_0' := p'(u(t_0,x_0))$, in $t = t_0 + \varepsilon T$ we have

$$|\partial_x u(t_0+\varepsilon T, x_0)| \ge e^{T|p_0'|}(|\partial_x u(t_0,x_0)| - h_1 T) \ge |\partial_x u(t_0,x_0)|, \tag{3.6}$$

where the first inequality in (3.6) holds provided $T < 1/\varepsilon$ is chosen small enough. We can summarize the previous statements in the following result.

Proposition 3.4. *Let (u,v) be the solution to* (1.1) *and let us fix $t_0 \ge 0$ such that*

$$p'(u(t_0,x)) < 0, \quad \textit{for almost every } x \in \mathbb{R}.$$

Let also fix $x_0 \in \mathbb{R}$ such that hypotheses (H0)–(H2) *are satisfied. Then*

$$\partial_t(\partial_x u(t,x)) > 0, \quad \textit{for all } x \in B_{\delta(x_0)} \textit{ and for all } t \in [t_0, t_0+\varepsilon T].$$

Essentially, Proposition 3.4 and in particular the lower bound (3.6) prove that the space derivative of the solution u is increasing in time, locally around x_0 (see Figure 2). Then the profile of the solution becomes steeper and steeper for $x \in B_\delta(x_0)$, and we expect that, because of such behavior, the solution will actually leave the elliptic zone, that is we expect to have

$$p'(u(t_0+\varepsilon T, x)) > 0 \quad \text{for almost every } x \in \mathbb{R},$$

provided T to be chosen appropriately.

To prove such a claim, let us see what happens to u locally around x_0 and for $t = t_0 + \varepsilon T$. We have

$$u(t_0+\varepsilon T, x) = u(t_0+\varepsilon T, x_0) + \partial_x u(t_0+\varepsilon T, x_0)(x-x_0)$$

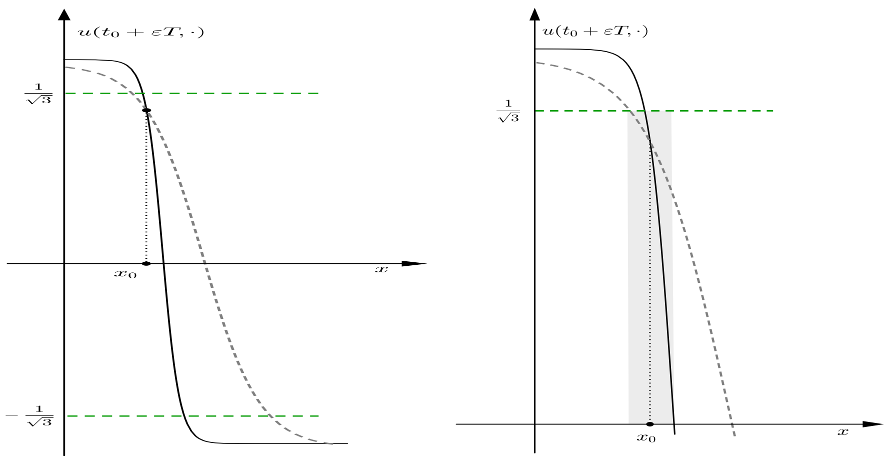

FIGURE 2. The dotted line is the profile of $u(t_0,\cdot)$; since (3.6) implies that the space derivative of u is increasing in time, we expect that, at least locally around x_0, there exists a time $T>0$ such that the profile of the solution for $t=t_0+\varepsilon T$ will be as depicted here.

$$+\int_0^1 (\partial_x^2 u)(t_0+\varepsilon T,(1-s)x_0+sx_0)(1-s)(x-x_0)^2\,ds,$$

Since $u(t_0+\varepsilon T,x_0)>0$, $\partial_x u(t_0+\varepsilon T,x)<0$ and $\partial_x^2 u(t_0+\varepsilon T,x)<0$ for $x\in B_\delta(x_0)$, while $1-s<0$ for $s\in(0,1)$, we have

$$\begin{aligned}|u(t_0+\varepsilon T,x)| &\geq |u(t_0+\varepsilon T,x_0)-\delta|\partial_x u(t_0+\varepsilon T,x_0)||\\ &\geq \delta|\partial_x u(t_0+\varepsilon T,x_0)|\\ &\geq \delta e^{T|p_0'|}(|\partial_x u(t_0,x_0)|-h_1T)\end{aligned}$$

for all $x\in B_\delta(x_0)$, where we used (3.6) to bound $|\partial_x u(t_0+\varepsilon T,x_0)|$. Since we want to prove that the solution leaves the elliptic zone, we choose $T>0$ such that

$$|u(t_0+\varepsilon T,x)|>(p')^{-1}(0),$$

that is, we choose $T>0$ in such a way the following inequality is satisfied

$$\delta e^{T|p_0'|}(|\partial_x u(t_0,x_0)|-h_1T)>(p')^{-1}(0). \tag{3.7}$$

Note that, in the case of a pressure law of the form (1.2), $(p')^{-1}(0)=1/\sqrt{3}$.

To solve (3.7) as a function of T we have to study the behavior of the function

$$f(T):=e^{T|p_0'|}(|\partial_x u_0|-h_1T),$$

where $|\partial_x u_0|:=|\partial_x u(t_0,x_0)|$, $|p_0'|$ and h_1 are positive constant. We have

- $f(0)=|\partial_x u_0|$,
- $\lim_{T\to+\infty} f(T)=-\infty$;
- $\lim_{x\to-\infty} f(T)=0$.

Moreover

$$T_{\max}=\frac{|\partial_x u_0|}{h_1}-\frac{1}{p_0'},\quad f(T_{\max})=\frac{h_1}{p_0'}e^{\frac{p_0'|\partial_x u_0|-h_1}{h_1}},$$

$$T_{\max} \to |\partial_x u_0|/h_1, \quad f(T_{\max}) \to +\infty \quad \text{as } p_0' \text{ increases.}$$

Hence, if $f(0) := |\partial_x u_0| > \frac{(p')^{-1}(0)}{\delta}$, there will always exists $T^* > 0$ such that (3.7) is satisfied for all $T < T^*$ (see Figure 3).

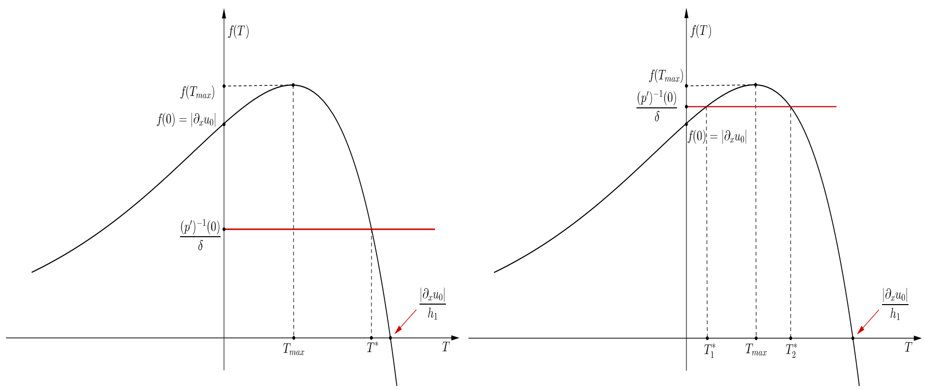

FIGURE 3. Function $f(T)$ when $T_{\max} > 0$: when $|\partial_x u_0| > (p')^{-1}(0)/\delta$ and (3.7) is satisfied for all $T < T^*$ (left), when $|\partial_x u_0| < (p')^{-1}(0)/\delta$ and (3.7) is satisfied for all $T_1^* < T < T_2^*$ (right).

Remark 3.5. We observe that, if $|\partial_x u_0| < \frac{(p')^{-1}(0)}{\delta}$, we can prove that there exist $T_1^* > 0$ and $T_2^* > 0$ such that (3.7) is satisfied for all $T_1^* < T < T_2^*$ (see Figure 3).

We also notice that $T_{\max} > 0$ if and only if $|\partial_x u_0| > h_1/|p_0'|$; however, if $T_{\max}$ is negative (Figure 4), $f(T_{\max})$ will still be positive but in this case we have to require condition

$$|\partial_x u_0| > \frac{(p')^{-1}(0)}{\delta}$$

to be satisfied (otherwise, we would have no positive solutions to the inequality (3.7), as shown in Figure 4).

To summarize, what we have proven so far is the following:

- We start the analysis at time $t_0 \geq 0$, where the solution belongs the elliptic zone (to simplify, one may think that $t_0 = 0$ and the initial datum satisfies $p'(u(0,x)) < 0$ for almost every $x \in \mathbb{R}$.)
- We localize around a certain point x_0 such that **H0-1-2** hold and we see that, at least locally around x_0, the derivative of u is growing in time, that is there exists $T > 0$ such that
$$|\partial_x u(t_0 + \varepsilon T, x)| > |\partial_x u(t_0, x_0)|, \quad \text{for all } x \in B_\delta(x_0). \tag{3.8}$$
- We use property (3.8) to prove that
$$|p'(u(t_0 + \varepsilon T, x))| > 0, \quad \text{for all } x \in B_\delta(x_0),$$
provided T to be chosen less than T^*. In particular, the solution at time $t = t_0 + \varepsilon T$ belongs to the hyperbolic zone, at least locally around x_0.

We have a *transition from the elliptic to the hyperbolic zone* during the time interval $[t_0, t_0 + \varepsilon T]$. In particular, since x_0 can be chosen arbitrarily, we expect the profile of u to evolve in the time interval $[t_0, t_0 + \varepsilon T]$ as depicted in Figure 5.

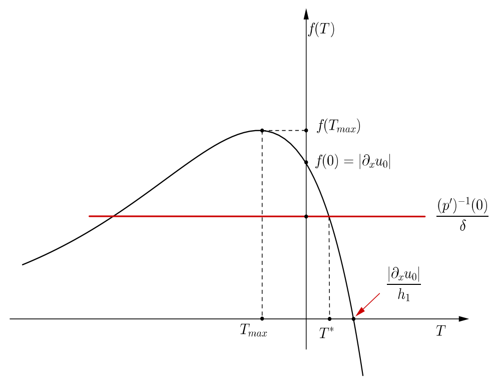

FIGURE 4. Function $f(T)$ when $T_{\max} < 0$. Note that T^* is positive if and only if $|\partial_x u_0| > (p')^{-1}(0)/\delta$.

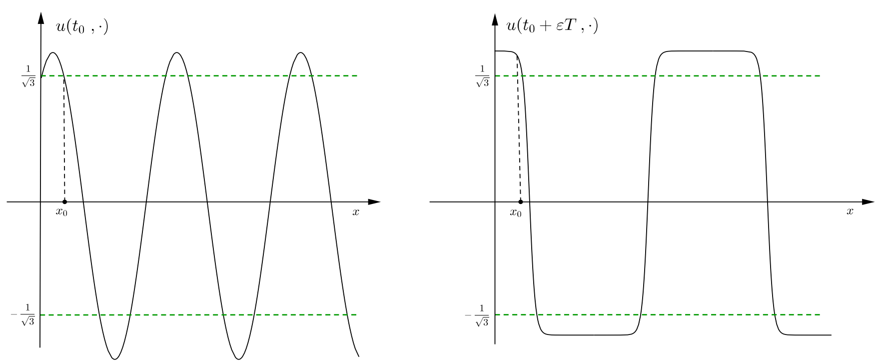

FIGURE 5. Transition from the elliptic to the hyperbolic zones: At time t_0, the solution belongs to the elliptic zone for almost every x (left). On the time interval $[t_0, t_0 + \varepsilon T]$, locally around x_0, the first derivative of the solution increases so that the profile becomes steeper. Since $x_0 \in \mathbb{R}$ can be chosen arbitrarily, at time $t = t_0 + \varepsilon T$ the solution is outside the elliptic zone (right); that is $|u(t_0 + \varepsilon T, x)| > \frac{1}{\sqrt{3}}$ for almost every x.

3.3. **Comments on the assumptions.** Concerning assumption (H0), the choice of a localization around a point x_0 where the solution is decreasing and concave can be replaced, without loss of generality, with one of the following assumptions (see Figure 6):

(a) $\partial_x u(t_0, x) < 0$ and $\partial_x^2 u(t_0, x) > 0$ for all $x \in B_\delta(x_0)$;
(b) $\partial_x u(t_0, x) > 0$ and $\partial_x^2 u(t_0, x) > 0$ for all $x \in B_\delta(x_0)$;
(c) $\partial_x u(t_0, x) > 0$ and $\partial_x^2 u(t_0, x) < 0$ for all $x \in B_\delta(x_0)$.

For example, in case (a), when trying to bound from below the solution u at time $t_0 + \varepsilon T$ we would have, as before,

$$u(t_0 + \varepsilon T, x) = u(t_0 + \varepsilon T, x_0) + \partial_x u(t_0 + \varepsilon T, x_0)(x - x_0)$$

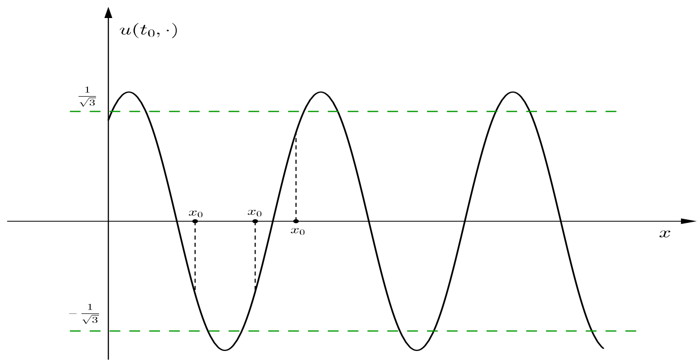

FIGURE 6. Choice of the point x_0 in the cases (a), (b), (c).

$$+\int_0^1 (\partial_x^2 u)(t_0+\varepsilon T,(1-s)x_0+sx_0)(1-s)(x-x_0)^2\,ds,$$

but this time $u(t_0+\varepsilon T,x_0)<0$, $\partial_x u(t_0+\varepsilon T,x)<0$ and $\partial_x^2 u(t_0+\varepsilon T,x)>0$ for $x\in B_\delta(x_0)$, while $1-s<0$ for $s\in(0,1)$. It follows directly that

$$|u(t_0+\varepsilon T,x)|\ge \delta|\partial_x u(t_0+\varepsilon T,x_0)\ge \delta e^{Tp_0'}(|\partial_x u(t_0,x_0)|-h_1T).$$

We also notice that, since the first derivative of the solution is growing in time, for $t\in[t_0,t_0+\varepsilon T]$ we have

$$\partial_x^2 v=-\partial_t(\partial_x u)<0,$$

so that

$$\partial_t v=\varepsilon\partial_x^2 v-p'(u)\partial_x u<0,$$

and the a priori assumption (H1) is satisfied.

Finally, concerning assumption (H2), condition (3.5) in the case $p(u)=u(u^2-1)$ is always satisfied, since it reads

$$u(t_0,x_0)>-3c_1+\sqrt{9c_1^2-3},$$

and the right hand side in the previous inequality is always negative for any choice of c_1, while $u(t_0,x_0)>0$.

3.4. **Transition from the hyperbolic to the elliptic zone.** We now start again our analysis in $t_1:=t_0+\varepsilon T$; the shape of the solution is as depicted in Figure 7, and we choose such profile as the new initial configuration.

Because of the choice of T, we have $u(t_1,x)>(p')^{-1}(0)$ for all $x\in B_\delta(x_0)$, implying

$$p'(u(t_1,x))>0 \quad \text{for all } x\in B_\delta(x_0),$$

that is, the solution belongs to the hyperbolic zone. By reasoning as before, in the time interval $[t_1,t_1+\varepsilon T_1]$ (with $T_1>0$ to be determined) we have

$$\partial_x u(t,x_0)=\exp\Big(-\int_{t_1}^t \frac{p'(u(s,x_0))}{\varepsilon}\,ds\Big)\partial_x u(t_1,x_0)$$

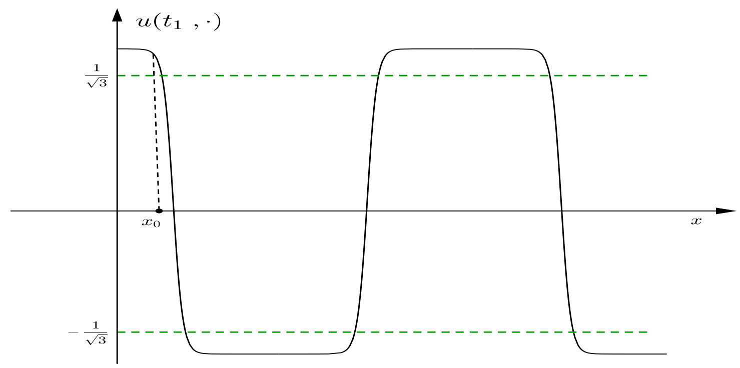

Figure 7. Solution u for $t = t_1$. The solution now belongs to the hyperbolic zone, i.e., $p'(u(t_1, x)) > 0$ for almost every $x \in R$. We recall that $p(u) = (u^2 - 1)u$.

$$- \frac{1}{\varepsilon} \int_{t_1}^{t} \exp(- \int_{t_1}^{s} \frac{p'(u(s', x_0))}{\varepsilon}\, ds')\, \partial_t v(s, x_0)\, ds.$$

This time the a priori assumption needed is

(H3) Let x_0 satisfy $\partial_t v(t, x_0) > 0$ and $|\partial_t v(t, x_0)| \le h_1'$ for some $h_1' > 0$ and for all $t \in [t_1, t_1 + \varepsilon T_1]$.

From this assumption we have

$$\begin{aligned}
|\partial_x u(t, x_0)| &\le \exp\Big(- \int_{t_1}^{t} \frac{p'(u(s, x_0))}{\varepsilon}\, ds\Big) |\partial_x u(t_1, x_0)| \\
&\quad + \frac{1}{\varepsilon} \int_{t_1}^{t} \exp\Big(- \int_{t_1}^{s} \frac{p'(u(s', x_0))}{\varepsilon}\, ds'\Big)\, |\partial_t v(s, x_0)|\, ds \\
&\le e^{-\frac{1}{\varepsilon} p_1'(t - t_1)} \Big(|\partial_x u(t_1, x_0)| + \frac{1}{\varepsilon} |\partial_t v(t_1, x_0)|(t - t_1)\Big),
\end{aligned}$$

with notation $p_1' := p'(u(t_1, x_0)) > 0$. Then, for $t = t_1 + \varepsilon T_1$ we have

$$|\partial_x u(t_1 + \varepsilon T_1, x_0)| \le e^{-T_1 p_1'} (|\partial_x u(t_1, x_0)| + h_1' T_1);$$

that is, the first derivative of the solution is decreasing locally around x_0 in the time interval $[t_1, t_1 + \varepsilon T_1]$. We thus expect the solution to evolve again into a configuration as the one depicted in Figure 1. Indeed, for all $x \in B_\delta(x_0)$,

$$\begin{aligned}
|u(t_1 + \varepsilon T_1, x)| &\le |u(t_1 + \varepsilon T_1, x_0)| + |\partial_x u(t_1 + \varepsilon T_1, x_0)|(x - x_0) \\
&\le |u(t_1 + \varepsilon T_1, x_0)| + \delta e^{-T_1 p_1'} (|\partial_x u(t_1, x_0)| + h_1' T_1),
\end{aligned}$$

meaning that u is decreasing and eventually, for an appropriate choice of T_1, it will be such that

$$|u(t_1 + \varepsilon T_1, x)| < (p')^{-1}(0), \quad \forall x \in B_\delta(x_0).$$

Precisely, T_1 will be chosen in such a way that

$$\delta e^{-T_1 p_1'} (|\partial_x u(t_1, x_0)| + h_1' T_1) < (p')^{-1}(0).$$

With such a choice of T_1, at time $t_2 := t_1 + \varepsilon T_1$ the profile $u(t_2, x)$ belongs to the elliptic zone, for almost every $x \in \mathbb{R}$ (again, because of the arbitrariness of x_0).

Remark 3.6. Again, note that from the equation for v, for all $x \in B_\delta(x_0)$, we have

$$\partial_t v = \varepsilon \partial_x^2 v - p'(u)\partial_x u \quad \text{and} \quad \varepsilon \partial_x^2 v = -\partial_t(\partial_x u)\,.$$

In particular, here $\partial_x u$ is decreasing in time, $p'(u) > 0$ and $\partial_x u < 0$, implying $\partial_t v > 0$ for all $t \in [t_1 + \varepsilon T_1]$. The a priori assumption (H3) is thus satisfied.

The strategy proposed here can be iterated again; we consider as the new initial configuration the profile of the solution at time $t_2 = t_1 + \varepsilon T_1$ and we restart the whole argument again. We can summarize the previous statements in the following result, showing the phase transition (from elliptic to hyperbolic zones and viceversa) occurring for a Van der Walls p-system with viscosity.

Theorem 3.7. *Let u be the solution to the following p-system with viscosity,*

$$\begin{gathered}\partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x(p(u)) = \varepsilon \partial_x^2 v,\end{gathered}$$

where the pressure $p(u)$ is given by the Van der Waals law $p(u) = (u^2 - 1)u$. If there exists $t_0 \geq 0$ such that

$$p'(u(t_0, x)) < 0,$$

for almost every $x \in \mathbb{R}$ (that is, $u(t_0, x)$ belongs to the elliptic zone), then there exists a time $T_0 > 0$ such that

$$p'(u(t_0 + \varepsilon T_0), x) > 0,$$

for almost every $x \in \mathbb{R}$, meaning that $u(t_0 + \varepsilon T, x)$ belongs to the hyperbolic zone.

More generally, there exists a sequence of times $T_n > 0$, with $n \geq 0$, such that if defined for all $i \in \mathbb{N}_0$ by

$$\bar{T}_{i+1} := t_0 + \varepsilon \sum_{n=0}^{i} T_n,$$

then, for almost every $x \in \mathbb{R}$ and for $k \in \mathbb{N}_0$, we have

$$p'(u(\bar{T}_{i+1}, x)) \begin{cases} > 0 & \text{for } i = 2k, \\ < 0 & \text{for } i = 2k+1. \end{cases}$$

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