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LOW MACH NUMBER LIMIT OF COMPRESSIBLE NEMATIC LIQUID CRYSTAL FLOWS WITH WELL-PREPARED INITIAL DATA IN A 3D BOUNDED DOMAIN

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ABSTRACT. In this article, we consider the low Mach number limit of the compressible nematic liquid crystal flows in a 3D bounded domain. We establish the uniform estimates with respect to the Mach number for the strong solutions with large initial data in a short time interval. Consequently, we obtain the convergence of the compressible nematic liquid crystal system to the incompressible nematic liquid crystals system as the Mach number tends to zero.

1. INTRODUCTION

In this article, we establish the uniform estimates of strong solutions with respect to the Mach number in a bounded domain $\Omega \subset \mathbb{R}^3$ to the compressible nematic liquid crystal flows [8].

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \frac{1}{\epsilon^2}\nabla P(\rho) - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u = -\nabla d \cdot \Delta d, \tag{1.2}$$

$$d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \quad |d| = 1, \tag{1.3}$$

where the unknowns ρ, u and d stand for the density, velocity, and the macroscopic of the nematic liquid crystal orientation field, respectively. The pressure $P(\rho)$ is a C^1 function satisfying $P'(\cdot) > 0$ and $P'(0) = 0$, such as the well-known $\gamma-$law $P(\rho) = a\rho^\gamma (\gamma > 1)$ which satisfies the assumptions. The parameter $\epsilon > 0$ is the scaled Mach number. The physical constants μ and λ denote the shear viscosity and bulk viscosity of the flow and satisfy

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0.$$

In fluid mechanics, the Mach number is an important physical quantity to determine whether the fluid is compressible or incompressible. If the Mach number is small, the fluid should behave asymptotically like an incompressible one, provided velocity and viscosity are small. As a result, the low Mach number limit problem has attracted much attention in recent years. When d is a constant vector field,

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the system (1.1)-(1.3) becomes the compressible Navier-Stokes system, of which the low Mach number limit problem has obtained a great number of results in the past decades. The readers may refer to [5, 10, 11, 12, 14], for instance, and the references therein for details.

Furthermore, a lot of progress on the low Mach number limit for the compressible nematic liquid crystal equations have been made. In [4], the authors concerned the low Mach number limit of system (1.1)-(1.3) with periodic boundary conditions. In [2], Bie, Bo, Wang and Yao obtained global existence and the low Mach number limit for compressible flow of liquid crystals in critical spaces. Particularly, for the bounded domain case, the low Mach number limit of weak solutions to the compressible flow of liquid crystals was proved in [13], and Yang [15] firstly studied the low Mach number limit of the strong solution to system (1.1)-(1.3) provided the initial data small enough. Motivated by the articles mentioned above, in this paper, we intend to establish the low Mach number limit of the strong solution for the system (1.1)-(1.3) with the lager initial data in a short time interval. The main difficulty comparing to the periodic case [4] and the whole space case [2] is the uniform high-norm estimates with respect to the Mach number and a time interval independent of the Mach number. In a bounded domain, after integrating by parts for the high-order derivatives, we have to estimate the boundary term which we will skillfully apply the slip conditions to control.

The low Mach number fluid can be regarded as a perturbation near the background isentropic fluid, where the density is usually set to be constant. Hence, we introduce the density variation by σ^ϵ as follows,

$$\rho^\epsilon = 1 + \epsilon\sigma^\epsilon,$$

and we will take $P'(1) = 1$. Then the non-dimensional system (1.1)-(1.3) can be rewritten as the form

$$\sigma_t^\epsilon + \operatorname{div}(\sigma^\epsilon u^\epsilon) + \frac{1}{\epsilon}\operatorname{div} u^\epsilon = 0, \tag{1.4}$$

$$\rho^\epsilon(u_t^\epsilon + u^\epsilon\cdot\nabla u^\epsilon) + \frac{1}{\epsilon}P'(1+\epsilon\sigma^\epsilon)\nabla\sigma^\epsilon - \mu\Delta u^\epsilon - (\lambda+\mu)\nabla\operatorname{div} u^\epsilon = -\nabla d^\epsilon\cdot\Delta d^\epsilon, \tag{1.5}$$

$$d_t^\epsilon + u^\epsilon\cdot\nabla d^\epsilon = (\Delta d^\epsilon + |\nabla d^\epsilon|^2 d^\epsilon), \quad |d^\epsilon| = 1. \tag{1.6}$$

System (1.4)-(1.6) is supplemented with the initial and boundary value conditions,

$$(\sigma^\epsilon, u^\epsilon, d^\epsilon)(\cdot,0) = (\sigma_0^\epsilon, u_0^\epsilon, d_0^\epsilon)(\cdot) \quad \text{in } \Omega, \tag{1.7}$$

$$u^\epsilon\cdot n = 0, \quad \operatorname{curl} u^\epsilon \times n = 0, \quad \frac{\partial d^\epsilon}{\partial n} = 0, \quad \text{on } \partial\Omega, \tag{1.8}$$

where n is the unit outer normal vector to the smooth boundary $\partial\Omega$.

Firstly, the local existence results for problem (1.4)-(1.8) can be established in a similar way as in [8].

Proposition 1.1 ((Local solution)). *Let $\Omega\subset\mathbb{R}^3$ be a bounded, simply connected domain with smooth boundary $\partial\Omega$. Assume the initial data $(\sigma_0^\epsilon, u_0^\epsilon, d_0^\epsilon)$ satisfy the condition*

$$\begin{gathered}(\partial_t^k\sigma^\epsilon(0), \quad \partial_t^k u^\epsilon(0)) \in H^{2-k}(\Omega), \quad \partial_t^k d^\epsilon(0)\in H^{3-k}(\Omega), \quad k=0,1,2,\\ \int_\Omega \sigma_0 dx = 0, \quad 1+\epsilon\sigma_0^\epsilon \ge m,\end{gathered} \tag{1.9}$$

for some constant $m > 0$. Moreover, under the compatibility conditions

$$
\begin{gathered}
\partial_t^k u^\epsilon(0)\cdot n = 0, \quad n\times \operatorname{curl} u_0^\epsilon = n\times \operatorname{curl} u_t^\epsilon(0) = 0, \quad \text{on } \partial\Omega,\ k=0,1,\\
\partial_t^k \frac{\partial d^\epsilon(0)}{\partial n} = 0 \quad \text{on } \partial\Omega,\ k=0,1.
\end{gathered} \tag{1.10}
$$

There exists a constant $T^\epsilon > 0$ such that the initial boundary value problem (1.4)-(1.8) *has a unique solution $(\sigma^\epsilon, u^\epsilon, d^\epsilon)$ satisfying*

$$
\begin{gathered}
1+\epsilon\sigma^\epsilon > 0 \quad \text{in } \Omega\times(0,T^\epsilon),\\
\partial_t^k \sigma^\epsilon \in C([0,T^\epsilon], H^{2-k}),\\
\partial_t^k u^\epsilon \in C([0,T^\epsilon], H^{2-k})\cap L^2(0,T^\epsilon; H^{3-k}),\\
\partial_t^k d^\epsilon \in C([0,T^\epsilon], H^{3-k})\cap L^2(0,T^\epsilon; H^{4-k}), \quad k=0,1,2.
\end{gathered}
$$

To simplify the statement, we used $\sigma_t^\epsilon(0)$ to denote the quantity $\sigma_t^\epsilon|_{t=0}$ which can be obtained from (1.4). The other quantities are defined in a similar way. For simplicity, we denote

$$
\begin{aligned}
M^\epsilon(t) = \sup_{0\le s\le t} \Big\{ &\|(\sigma^\epsilon, u^\epsilon, \nabla d^\epsilon)(\cdot,s)\|_{H^2} + \|(\sigma_s^\epsilon, u_s^\epsilon, \nabla d_s^\epsilon)(\cdot,s)\|_{H^1} + \Big\|\frac{1}{1+\epsilon\sigma(\cdot,s)}\Big\|_{L^\infty}\\
&+ \epsilon\|(\sigma_{ss}^\epsilon, u_{ss}^\epsilon, \nabla d_{ss}^\epsilon)(\cdot,s)\|_{L^2}\Big\} + \Big\{\int_0^t \Big(\|u^\epsilon\|_{H^3}^2 + \|u_s^\epsilon\|_{H^2}\\
&+ \|\epsilon(\sigma_{ss}^\epsilon, u_{ss}^\epsilon, \nabla d_{ss}^\epsilon)\|_{H^1}\Big) ds\Big\}^{1/2}.
\end{aligned}
$$

Then, we state the main results in this article as follows.

Theorem 1.2. *Assume that $(\sigma^\epsilon, u^\epsilon, d^\epsilon)$ is the solution obtained in Proposition 1.1, and the initial datum $(\sigma_0^\epsilon, u_0^\epsilon, d_0^\epsilon)$ further satisfies*

$$
\|(\sigma_0^\epsilon, u_0^\epsilon, \nabla d_0^\epsilon)\|_{H^2} + \|(\sigma_t^\epsilon, u_t^\epsilon, \nabla d_t^\epsilon)(0)\|_{H^1} + \epsilon\|(\sigma_{tt}^\epsilon, u_{tt}^\epsilon, \nabla d_{tt}^\epsilon)(0)\|_{L^2} \le D_0.
$$

Then there exist two positive constants T_0 and D such that $(\sigma^\epsilon, u^\epsilon, d^\epsilon)$ satisfies the uniform estimates

$$
M^\epsilon(T_0) \le D, \tag{1.11}
$$

where D_0, T_0 and D are constants independent of $\epsilon\in(0,1)$.

Based on the above uniform estimates, by applying the Arzelà-Ascolis theorem, we can prove the following convergence result in a standard way.

Theorem 1.3. *Let $(\sigma^\epsilon, u^\epsilon, d^\epsilon)$ be the solution obtained in Theorem 1.2, and the initial data $(\sigma_0^\epsilon, u_0^\epsilon, d_0^\epsilon)$ further satisfies that*

$$
\begin{gathered}
(u_0^\epsilon, \nabla d_0^\epsilon) \to (u_0, \nabla d_0) \quad \text{strongly in } H^s \text{ for all } 0\le s<2 \text{ as } \epsilon\to 0,\\
\epsilon\sigma_0^\epsilon \to 0 \quad \text{strongly in } H^s \text{ for all } 0\le s<1 \text{ as } \epsilon\to 0,
\end{gathered} \tag{1.12}
$$

Then $(\rho^\epsilon, u^\epsilon, \nabla d^\epsilon) \to (1, u, \nabla d)$ strongly in $C([0,T_0]; H^1)$ as the Mach number $\epsilon\to 0$, and there exists a function $\pi(x,t)$ such that (u,π,d) satisfies the following classical incompressible nematic crystal equations

$$
\begin{gathered}
u_t + u\cdot\nabla u + \nabla\pi - \mu\Delta u = -\nabla d\cdot\Delta d,\\
\operatorname{div} u = 0,\\
d_t + u\cdot\nabla d = \Delta d + |\nabla d|^2 d, \quad |d| = 1,
\end{gathered} \tag{1.13}
$$

with the initial and boundary conditions

$$
\begin{gathered}
(u, d)|_{t=0} = (u_0, d_0), \quad \text{in } \Omega \\
u \cdot n = 0, \quad \operatorname{curl} u \times n = 0, \quad \frac{\partial d}{\partial n} = 0, \quad \text{on } \partial\Omega.
\end{gathered} \tag{1.14}
$$

2. Proof of Theorem 1.2

We use the methods applied in [6, 5, 7]. According to similar arguments to those in [5, 6], we know that to prove Theorem 1.2 it is suffices to prove that

$$
M^\epsilon(T_0) \le C_0(M_0^\epsilon) \exp(t^{1/4} C(M^\epsilon(t))), \tag{2.1}
$$

for all $t \in [0, T^\epsilon]$ and for some given nondecreasing continuous functions $C_0(\cdot)$ and $C(\cdot)$.

For the sake of simplicity, we will drop the superscript ϵ of $\sigma^\epsilon, u^\epsilon, d^\epsilon$ and so on. Moreover, in the following, we will write $M^\epsilon(t)$ and M_0^ϵ as M and M_0, respectively. The symbol C denotes a generic constant and its value may change from line to line.

Firstly, we list some lemmas which will be used throughout this paper.

Lemma 2.1 ([9]). *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and outward normal n. For any $u \in H^1(\Omega)$ with $u \cdot n = 0$ or $u \times n = 0$ on $\partial\Omega$, there exists a positive constant C independent of u such that*

$$
\|u\|_{L^2(\Omega)} \le C(\|\operatorname{div} u\|_{L^2(\Omega)} + \|\operatorname{curl} u\|_{L^2(\Omega)}), \tag{2.2}
$$

where the vorticity $\operatorname{curl} u = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)^T$.

Lemma 2.2 ([14]). *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and outward normal n. Then, for any $u \in H^1(\Omega)$, $s \ge 1$, there exists a constant $C > 0$ independent of u, such that*

$$
\|u\|_{H^s(\Omega)} \le C(\|\operatorname{div} u\|_{H^{s-1}(\Omega)} + \|\operatorname{curl} u\|_{H^{s-1}(\Omega)} + \|u \times n\|_{H^{s-\frac{1}{2}}(\partial\Omega)} + \|u\|_{H^{s-1}(\Omega)}). \tag{2.3}
$$

Lemma 2.3 ([3]). *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and outward normal n. Then, for any $u \in H^1(\Omega)$, $s \ge 1$, there exists a constant $C > 0$ independent of u, such that*

$$
\begin{aligned}
\|u\|_{H^s(\Omega)} &\le C(\|\operatorname{div} u\|_{H^{s-1}(\Omega)} + \|\operatorname{curl} u\|_{H^{s-1}(\Omega)} + \|u \cdot n\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \\
&\quad + \|u\|_{H^{s-1}(\Omega)}).
\end{aligned} \tag{2.4}
$$

From Lemmas 2.1, 2.2 and 2.3, we have

$$
\|\operatorname{curl} u\|_{H^2} \le C(\|\Delta \operatorname{curl} u\|_{L^2} + \|u\|_{H^2}), \tag{2.5}
$$

for $u \cdot n = 0$ and $\operatorname{curl} u \times n = 0$ on $\partial\Omega$. In fact, the latter one gives (see [1, 7])

$$
\operatorname{curl} \operatorname{curl} u \cdot n = 0 \quad \text{on } \partial\Omega.
$$

Firstly, we know that ρ and its derivatives always appear as a coefficient of u and its derivatives. Thus, for simplicity, we use the standard energy method in [5, 12] to obtain

$$
\|\rho(\cdot, t)\|_{H^2} + \|\rho_t(\cdot, t)\|_{H^1} + \|\rho_{tt}(\cdot, t)\|_{L^2} + \|\frac{1}{\rho}(\cdot, t)\|_{L^\infty} \le C_0(M_0)(\sqrt{t} C(M)). \tag{2.6}
$$

Now we use the method in [5, 6, 15] to prove a priori estimates on σ, u and d. Multiplying (1.4)-(1.5) by σ and u, respectively, and integrating over $\Omega \times (0,t)$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\|(\sigma, \sqrt{\rho}u)\|_{L^2}^2 + \int_0^t \|(\sqrt{\mu}\operatorname{curl} u, \sqrt{\lambda+2\mu}\operatorname{div} u)\|_{L^2}^2 ds \\
&= -\frac{1}{2}\int_0^t \int_\Omega \sigma^2 \operatorname{div} u \,dx\,ds + \int_0^t \int_\Omega \frac{P'(1) - P'(1+\epsilon\sigma)}{\epsilon} u \nabla\sigma \,dx\,ds \\
&\quad - \int_0^t \int_\Omega (u\cdot\nabla)d \cdot \Delta d \,dx\,ds + \frac{1}{2}\|(\sigma_0, \sqrt{\rho_0}u_0)\|_{L^2}^2 \\
&\le C_0(M_0) + C\int_0^t \|\nabla\sigma\|_{L^2}^2 \|\nabla u\|^2 ds + C\int_0^t \|u\|_{L^6}\|\sigma\|_{L^3}\|\nabla\sigma\|_{L^2} ds \\
&\quad + C\int_0^t \|u\|_{L^6}\|\nabla d\|_{L^3}\|\Delta d\|_{L^2} ds \\
&\le C_0(M_0)\exp(tC(M)),
\end{aligned}
\tag{2.7}
$$

where we have used

$$
-\Delta u = -\nabla \operatorname{div} u + \operatorname{curl}\operatorname{curl} u. \tag{2.8}
$$

Multiplying (1.5) by $\nabla \operatorname{div} u$ and integrating the result over $\Omega \times (0,t)$, we find

$$
\begin{aligned}
&(\lambda+2\mu)\int_0^t \|\nabla \operatorname{div} u\|_{L^2}^2 ds - \frac{1}{\epsilon}\int_0^t \int_\Omega \nabla \operatorname{div} u \cdot \nabla\sigma \,dx\,ds \\
&= \int_0^t \int_\Omega (\rho u_t + \rho u\cdot\nabla u + \nabla d\cdot\Delta d)\nabla \operatorname{div} u \,dx\,ds \\
&+ \int_0^t \int_\Omega \frac{P'(1+\epsilon\sigma) - P'(1)}{\epsilon}\nabla\sigma\cdot\nabla \operatorname{div} u \,dx\,ds \\
&= -\frac{1}{2}\int_\Omega \rho(\operatorname{div} u)^2 dx + \frac{1}{2}\int_\Omega \rho(\operatorname{div} u_0)^2 dx + \frac{1}{2}\int_0^t \int_\Omega \rho_t(\operatorname{div} u)^2 \,dx\,ds \\
&\quad - \int_0^t \int_\Omega \nabla\rho\cdot u_t \operatorname{div} u \,dx\,ds + \int_0^t \int_\Omega (\rho u\cdot\nabla u + \nabla\cdot\Delta d)\nabla \operatorname{div} u \,dx\,ds \\
&\quad + \int_0^t \int_\Omega \frac{P'(1+\epsilon\sigma) - P'(1)}{\epsilon}\nabla\sigma\cdot\nabla \operatorname{div} u \,dx\,ds.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
&\int_\Omega \rho(\operatorname{div} u)^2 dx + \int_0^t \|\nabla \operatorname{div} u\|_{L^2}^2 ds - \frac{1}{\epsilon}\int_0^t \int_\Omega \nabla \operatorname{div} u\cdot\nabla\sigma \,dx\,ds \\
&\le C_0(M_0) + \int_0^t (\|\nabla u\|_{L^2}\|u\|_{H^2} + \|\nabla d\|_{H^2}\|\Delta d\|_{L^2} \\
&\quad + \|\sigma\|_{H^2}\|\nabla\sigma\|_{L^2})\|\nabla \operatorname{div} u\|_{L^2} ds \\
&\le C_0(M_0)\exp(tC(M)).
\end{aligned}
\tag{2.9}
$$

To eliminate the singular term in (2.9), we take ∇ to (1.4) and multiply the result by $\nabla\sigma$ to find

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla\sigma|^2 dx + \frac{1}{\epsilon}\int_0^t\int_\Omega \nabla\sigma\cdot\nabla\operatorname{div} u\,dx\,ds \\
&= \frac{1}{2}\int_\Omega |\nabla\sigma_0|^2 dx - \int_0^t\int_\Omega \nabla\operatorname{div}(\sigma u)\cdot\nabla\sigma dx \\
&\le C_0(M_0)\exp(tC(M)).
\end{aligned} \tag{2.10}$$

Summing (2.9) and (2.10), we obtain

$$\|(\operatorname{div} u,\nabla\sigma)\|_{L^2}^2 + \int_0^t \|\nabla\operatorname{div} u\|_{L^2}^2 ds \le C_0(M_0)\exp(tC(M)). \tag{2.11}$$

Denote $\omega = \operatorname{curl} u$. Taking curl to (1.4), we have

$$\rho\partial_t\omega + \rho u\cdot\nabla\omega - \mu\Delta\omega = f, \tag{2.12}$$

where $f = \nabla\rho\times\partial_t u + \nabla(\rho u_i)\times\partial_i u - \nabla\Delta d_j\times\nabla d_j$. Multiplying (2.12) by ω, we obtain

$$\|\operatorname{curl} u\|_{L^2}^2 + \int_0^t\int_\Omega |\operatorname{curl}\operatorname{curl} u|^2\,dx\,ds \le C_0(M_0)\exp(\sqrt{t}C(M)). \tag{2.13}$$

From Lemma 2.3 and the boundary condition $\frac{\partial d}{\partial n} = 0$ on $\partial\Omega$, we know that

$$\|\nabla d\|_{H^1} \le C(\|\operatorname{div}\nabla d\|_{L^2} + \|\operatorname{curl}\nabla d\|_{L^2}) = C\|\Delta d\|_{L^2} \tag{2.14}$$

Applying ∇ to (1.6), we have

$$\nabla d_t - \nabla\Delta d = \nabla(|\nabla d|^2 d) - \nabla(u\cdot\nabla d). \tag{2.15}$$

Multiplying (2.15) by ∇d_t and integrating over $\Omega\times(0,t)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\Delta d|^2 dx + \int_0^t\int_\Omega |\nabla d_t|^2\,dx\,ds \\
&= \frac{1}{2}\int_\Omega |\Delta d_0|^2 dx + \int_0^t\int_\Omega (\nabla(|\nabla d|^2 d) - \nabla(u\cdot\nabla d))\cdot\nabla d_t\,dx\,ds \\
&\le \int_0^t\int_\Omega (|\nabla d|^3 + |\nabla d||\nabla^2 d| + |\nabla u||\nabla d| + |u||\nabla^2 d|)\nabla d_t\,dx\,ds \\
&\le \int_0^t \|d\|_{H^3}^2(\|\nabla d\|_{L^2} + \|\nabla^2 d\|_{L^2} + \|\nabla u\|_{L^2})\|\nabla d_t\|_{L^2} ds \\
&\quad + \int_0^t \|u\|_{H^2}\|\nabla^2 d\|_{L^2}\|\nabla d_t\|_{L^2} ds \\
&\le C_0(M_0)\exp(tC(M)).
\end{aligned} \tag{2.16}$$

Combining (2.14) with (2.16), we obtain

$$\|\nabla d\|_{H^1}^2 + \int_0^t\int_\Omega |\nabla d_t|^2\,dx\,ds \le C_0(M_0)\exp(tC(M)). \tag{2.17}$$

Multiplying (2.12) by $\partial_t\omega - \Delta\omega$, we obtain

$$
\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}\int_\Omega |\operatorname{curl}\operatorname{curl} u|^2 dx + \int_\Omega (\mu|\Delta\omega|^2 + \rho|\omega_t|^2) dx \\
&= \int_\Omega \rho\omega_t\Delta\omega dx - \int_\Omega \rho(u\cdot\nabla)\omega(\omega_t - \Delta\omega) dx + \int_\Omega f(\omega_t - \Delta\omega) dx \\
&= I_1 + I_2 + I_3,
\end{aligned}
\tag{2.18}
$$

where by using (2.8), we have

$$
\begin{aligned}
-\mu\int_\Omega \Delta\omega\cdot\omega_t dx =&\mu\int_\Omega \operatorname{curl}\operatorname{curl}\omega\cdot\omega_t dx \\
=&\mu\int_\Omega \operatorname{curl}\omega\cdot\operatorname{curl}\omega_t dx + \int_{\partial\Omega}(\omega_t\times n)\operatorname{curl}\omega dS \\
=&\frac{\mu}{2}\frac{d}{dt}\int_\Omega |\operatorname{curl}\omega|^2 dx.
\end{aligned}
$$

Then, we estimate I_1, I_2 and I_3 as follows.

$$
\begin{aligned}
I_1 =& -\int_\Omega \rho\omega_t\operatorname{curl}\operatorname{curl}\omega dx \\
=& -\int_\Omega \rho\operatorname{curl}\omega\operatorname{curl}\omega_t dx - \int_\Omega \operatorname{curl}\omega\cdot(\nabla\rho\times\omega_t) dx \\
=& -\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|\operatorname{curl}\omega|^2 dx + C\|\rho\|_{L^\infty}\|\operatorname{curl}\omega\|_{L^2}\|\nabla\operatorname{curl}\omega\|_{L^2} \\
&+ \|\nabla\rho\|_{L^6}\|\operatorname{curl}\omega\|_{L^3}\|\omega_t\|_{L^2} \\
\le& -\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|\operatorname{curl}\omega|^2 dx + C\|\rho\|_{H^2}\|u\|_{H^2}^2\|u\|_{H^3} \\
&+ \|\rho\|_{H^2}\|u_t\|_{H^1}\|u\|_{H^2}^{1/2}\|u\|_{H^3}^{1/2},
\end{aligned}
$$

$$
\begin{aligned}
|I_2| \le& C\|\rho\|_{L^\infty}\|\nabla\omega\|_{L^2}(\|\omega_t\|_{L^2} + \|\Delta\omega\|_{L^2}) \\
\le& C\|\rho\|_{H^2}\|u\|_{H^2}(\|u\|_t\|_{H^1} + \|u\|_{H^3})
\end{aligned}
$$

and

$$
|I_3| \le C\|f\|_{L^2}(\|u_t\|_{H^1} + \|u\|_{H^3}),
$$

where

$$
\begin{aligned}
\|f\| \le& C\|(|\nabla\rho||\partial_t u|, |\nabla(\rho u)||\nabla u|, |\nabla^3 d||\nabla d|)\|_{L^2} \\
\le& C\|\rho\|_{H^2}\|\partial_t u\|_{L^2}^{1/2}\|\partial_t u\|_{H^1}^{1/2} + C\|\rho\|_{H^2}\|u\|_{H^1}^{1/2}\|u\|_{H^2}^{3/2} + C\|d\|_{H^3}^2 \\
\le& C(M).
\end{aligned}
$$

Substituting the above estimates into (2.18) and integrating over $(0,t)$, we obtain

$$
\begin{aligned}
&\|\operatorname{curl}\operatorname{curl} u\|_{L^2}^2 + \int_0^t\int_\Omega (|\Delta\operatorname{curl} u|^2 + |\operatorname{curl} u_t|^2)\,dx\,ds \\
&\le C_0(M_0)\exp(\sqrt{t}C(M)).
\end{aligned}
\tag{2.19}
$$

Applying ∂_t to (1.4) and (1.5), respectively, we obtain

$$
\sigma_{tt} + \frac{1}{\epsilon}\operatorname{div} u_t = -\operatorname{div}(\sigma u)_t,
\tag{2.20}
$$

$$\begin{aligned}&\rho u_{tt}+\rho u\cdot\nabla u_t-\mu\Delta u_t-(\lambda+\mu)\nabla\operatorname{div}u_t\\&=-\rho_t u_t-(\rho u)_t\cdot\nabla u-\frac{1}{\epsilon}(P'(1+\epsilon\sigma)\nabla\sigma)_t-\nabla d_t\cdot\Delta d-\nabla d\cdot\Delta d_t.\end{aligned}\tag{2.21}$$

Multiplying (2.21) by $-\nabla\operatorname{div}u$, we have

$$\begin{aligned}&\frac{\lambda+2\mu}{2}\int_\Omega|\nabla\operatorname{div}u|^2dx-\frac{P'(1)}{\epsilon}\int_0^t\int_\Omega\nabla\sigma_t\nabla\operatorname{div}u\,dx\,ds\\&=\frac{\lambda+2\mu}{2}\int_\Omega|\nabla\operatorname{div}u_0|^2dx\\&\quad+\int_0^t\int_\Omega\Big(\frac{P'(1+\epsilon\sigma)-P'(1)}{\epsilon}\nabla\sigma\Big)_t\nabla\operatorname{div}u\,dx\,ds\\&\quad+\int_0^t\int_\Omega(\rho u_{tt}+\rho u\cdot\nabla u_t+\rho_t u_t-(\rho u)_t\cdot\nabla u)\nabla\operatorname{div}u\,dx\,ds\\&\quad+\int_0^t\int_\Omega(\nabla d_t\cdot\Delta d+\nabla d\cdot\Delta d_t)\nabla\operatorname{div}u\,dx\,ds\\&=\frac{\lambda+2\mu}{2}\int_\Omega|\nabla\operatorname{div}u_0|^2dx+I_4+I_5+I_6.\end{aligned}\tag{2.22}$$

We estimate I_4,I_5 and I_6 as follows.

$$\begin{gathered}|I_4|\le C\int_0^t\|\sigma\|_{H^2}\|\nabla\operatorname{div}u\|_{L^2}(\|\sigma_t\|_{L^3}+\|\nabla\sigma_t\|_{L^2})ds\le tC(M),\\|I_5|\le C\int_0^t\|\rho\|_{H^2}\|u_{tt}\|_{L^2}\|u\|_{H^2}ds+tC(M)\le tC(M),\\|I_6|\le\int_0^t\|d\|_{H^2}\|\nabla\operatorname{div}u\|_{L^2}(\|\nabla d_t\|_{L^3}+\|\Delta d_t\|_{L^2})ds\le tC(M).\end{gathered}$$

To eliminate the singular term, we apply ∇ to (1.4) and multiply the result by $\nabla\sigma_t$ to obtain

$$\begin{aligned}&\int_0^t\|\nabla\sigma_t\|_{L^2}^2ds+\frac{1}{\epsilon}\int_0^t\int_\Omega\nabla\sigma_t\nabla\operatorname{div}u\,dx\,ds\\&=-\int_0^t\int_\Omega\nabla\operatorname{div}(\sigma u)\cdot\nabla\sigma_t\,dx\,ds\le tC(M).\end{aligned}\tag{2.23}$$

Summing (2.22) and (2.23), we have

$$\int_\Omega|\nabla\operatorname{div}u|^2dx+\int_0^t\|\nabla\sigma_t\|_{L^2}^2ds\le C_0(M_0)\exp(tC(M)).\tag{2.24}$$

Applying ∂_i to (1.5) and multiplying the result by $\partial_i\nabla\operatorname{div}u$, we have

$$\begin{aligned}&\int_0^t\int_\Omega|\partial_i\nabla\operatorname{div}u|^2\,dx\,ds-\frac{1}{\epsilon}\int_0^t\int_\Omega\partial_i\nabla\sigma\cdot\partial_i\nabla\operatorname{div}u\,dx\,ds\\&\le\int_0^t\int_\Omega(|\nabla(\rho u_t+\rho u\cdot\nabla u)|^2+|\nabla(\sigma\nabla\sigma)|^2+|\nabla(\nabla d\cdot\Delta d)|^2)\,dx\,ds\\&\le tC(M).\end{aligned}\tag{2.25}$$

To eliminate the singular term, taking $\partial_i \nabla$ to (1.4) and multiplying the result by $\partial_i \nabla \sigma$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |\partial_i \nabla \sigma|^2 dx + \frac{1}{\epsilon}\int_0^t \int_\Omega \partial_i \nabla \operatorname{div} u \cdot \partial_i \nabla \sigma \, dx\, ds \\
&= \frac{1}{2}\int_\Omega |\partial_i \nabla \sigma_0|^2 dx + \int_0^t \int_\Omega \partial_i \nabla(\sigma \operatorname{div} u + \nabla \sigma \cdot u)\partial_i \nabla \sigma \, dx\, ds \\
&\le C_0(M_0) + \int_0^t \|u\|_{H^3}\|\sigma\|_{H^2}^2 ds \\
&\le C_0(M_0)\exp(\sqrt{t}C(M)).
\end{aligned} \tag{2.26}
$$

Summing (2.25) with (2.26), we obtain

$$
\|\nabla^2 \sigma\|_{L^2}^2 + \int_0^t \int_\Omega |\nabla^2 \operatorname{div} u|^2 \, dx\, ds \le C_0(M_0)\exp(\sqrt{t}C(M)). \tag{2.27}
$$

To obtain a priori estimate on $\|d\|_{L_t^\infty(H^3)}$, we use elliptic regularity theory, (2.17), (2.19) and (2.24).

$$
\begin{aligned}
&\|\nabla d\|_{H^2} \\
&\le C\|\nabla d_t\|_{L^2} + C\|\nabla u \cdot \nabla d\|_{L^2} + C\|u \cdot \nabla^2 d\|_{L^2} + C\||\nabla d|^3\|_{L^2} + \||\nabla d||\nabla^2 d|\|_{L^2} \\
&\le C\|\nabla d_t\|_{L^2} + C\|\nabla u\|_{L^3}\|\nabla d\|_{L^6} + C\|u\|_{L^6}\|\nabla^2 d\|_{L^3} + C\|\nabla d\|_{L^6}\|\nabla^2 d\|_{L^3} \\
&\le C\|\nabla d_t\|_{L^2} + \frac{1}{2}\|\nabla d\|_{H^2} + C_0(M_0)\exp(tC(M)),
\end{aligned}
$$

where we used Nirenberg's interpolation inequality and Young inequality. Then, we conclude that

$$
\|\nabla d\|_{H^2} \le C\|\nabla d_t\|_{L^2} + C_0(M_0)\exp(tC(M)). \tag{2.28}
$$

Hence, to obtain the estimate on $\|\nabla d\|_{L_t^\infty(H^2)}$, it is sufficient to estimate $\|\nabla d_t\|_{L_t^\infty(L^2)}$. Taking ∂_t to (1.6), we obtain

$$
d_{tt} + (u \cdot \nabla d)_t = \Delta d_t + (|\nabla d|^2 d)_t. \tag{2.29}
$$

Multiplying (2.29) by $-\Delta d_t$ and integrating over $\Omega \times (0,t)$, we have

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla d_t|^2 dx + \int_0^t \int_\Omega |\Delta d_t|^2 \, dx\, dt \\
&= \frac{1}{2}\int_\Omega |\nabla d_t(0)|^2 dx + \int_0^t \int_\Omega \Big(u_t \cdot \nabla d + u \cdot \nabla d_t \\
&\quad - |\nabla d|^2 d_t - d\partial_t |\nabla d|^2 \Big) \Delta d_t \, dx\, ds \\
&\le C_0(M_0) + C\int_0^t (\|\nabla d\|_{L^\infty}\|u_t\|_{L^2} + \|u\|_{L^\infty}\|\nabla d_t\|_{L^2})\|\Delta d_t\|_{L^2} ds \\
&\quad + C\int_0^t (\|\nabla d\|_{L^\infty}^2 \|d_t\|_{L^2} + \|\nabla d\|_{L^\infty}\|\nabla d_t\|_{L^2})\|\Delta d_t\|_{L^2} ds \\
&\le C_0(M_0)\exp(tC(M)).
\end{aligned} \tag{2.30}
$$

Substituting (2.30) into (2.28), we obtain

$$
\|\nabla d\|_{H^2} \le C_0(M_0)\exp(tC(M)). \tag{2.31}
$$

Then, by using calculations similar to those in [5], we can obtain the basic a priori estimates for σ_t, u_t. Multiplying (2.20), (2.21) by σ_t and u_t, respectively and integrating over $\Omega \times (0,t)$, we obtain

$$(\|\sigma_t\|_{L^2}^2 + \|u_t\|_{L^2}^2) + \int_0^t \|(\operatorname{curl} u_t, \operatorname{div} u_t)\|_{L^2}^2 ds \le C_0(M_0)\exp(tC(M)). \tag{2.32}$$

Multiplying (2.20), (2.21) by $-\Delta\sigma_t$ and $-\nabla \operatorname{div} u_t$, respectively, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla\sigma_t|^2 dx + \frac{1}{\epsilon}\int_0^t\int_\Omega \nabla \operatorname{div} u_t \cdot \nabla\sigma_t \,dx\,ds \\
&= \frac{1}{2}\int_\Omega |\nabla\sigma_t(0)|^2 dx + \int_0^t\int_\Omega \operatorname{div}(\sigma_t u + \sigma u_t)\Delta\sigma_t \,dx\,ds \\
&= \frac{1}{2}\int_\Omega |\nabla\sigma_t(0)|^2 dx + I_7,
\end{aligned} \tag{2.33}$$

where

$$\begin{aligned}
I_7 &= \int_0^t\int_\Omega u\cdot\nabla\sigma_t\Delta\sigma_t \,dx\,ds - \int_0^t\int_\Omega \nabla(\sigma_t \operatorname{div} u + u_t\nabla\sigma + \sigma \operatorname{div} u_t)\,dx\,ds \\
&= -\int_0^t\int_\Omega \partial_j u_i\partial_i\sigma_t\partial_j\sigma_t \,dx\,ds + \frac{1}{2}\int_0^t\int_\Omega \operatorname{div} u|\nabla\sigma_t|^2 \,dx\,ds \\
&\quad - \int_0^t\int_\Omega \nabla(\sigma_t \operatorname{div} u + u_t\nabla\sigma + \sigma \operatorname{div} u_t)\,dx\,ds \\
&\le tC(M) + C(M)\int_0^t \|u\|_{H^3} ds + C(M)\int_0^t \|u_t\|_{H^2} ds \\
&\le \sqrt{t}C(M),
\end{aligned} \tag{2.34}$$

and

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \rho(\operatorname{div} u_t)^2 dx + (\lambda+2\mu)\int_0^t\int_\Omega |\nabla \operatorname{div} u_t|^2 dx \\
&- \frac{P'(1)}{\epsilon}\int_0^t\int_\Omega \nabla\sigma_t\cdot\nabla \operatorname{div} u_t \,dx\,ds \\
&= \frac{1}{2}\int_\Omega \rho_0(\operatorname{div} u_t(0))^2 dx + \int_0^t\int_\Omega \Big(\frac{P'(1+\epsilon\sigma)-P'(1)}{\epsilon}\nabla\sigma\Big)_t \nabla \operatorname{div} u_t \,dx\,ds \\
&\quad + \int_0^t\int_\Omega (\frac{\epsilon}{2}\sigma_t(\operatorname{div} u_t)^2 - \epsilon u_{tt}\cdot\nabla\sigma \operatorname{div} u_t)\,dx\,ds \\
&\quad - \int_0^t\int_\Omega (\rho_t u_t + (\rho u\cdot\nabla u)_t)\nabla \operatorname{div} u_t \,dx\,ds \\
&\quad + \int_0^t\int_\Omega (\nabla d_t\cdot\Delta d + \nabla d\cdot\Delta d_t)\nabla \operatorname{div} u_t \,dx\,ds \\
&\le C_0(M_0) + \sqrt{t}C(M).
\end{aligned} \tag{2.35}$$

Summing (2.33), (2.34) and (2.35), we obtain

$$\begin{aligned}
&\int_\Omega (|\nabla\sigma_t|^2 + (\operatorname{div} u_t)^2)dx + \int_0^t\int_\Omega |\nabla \operatorname{div} u_t|^2 \,dx\,ds \\
&\le C_0(M_0)\exp(\sqrt{t}C(M)).
\end{aligned} \tag{2.36}$$

To complete the estimate of $\|u_t\|_{L_t^\infty(H^1)}$, we apply ∂_t to (2.12) to obtain

$$\begin{aligned}
&\rho_t\omega_t + \rho\omega_{tt} + (\rho u)_t\cdot\nabla\omega + \rho u\cdot\nabla\omega_t - \mu\Delta\omega_t \\
&= \nabla\rho_t\times u_t + \nabla\rho\times u_{tt} + \nabla\Delta(d_j)_t\times\nabla d_j \\
&\quad + \nabla\Delta d_j\times\nabla(d_j)_t + \nabla(\rho u_i)_t\times\partial_i u + \nabla(\rho u_i)\times\partial_i u_t.
\end{aligned}\tag{2.37}$$

Multiplying (2.37) by ω_t in $L^2(\Omega\times(0,t))$, we deduce that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \rho|\omega_t|^2dx + \mu\int_0^t\int_\Omega |\mathrm{curl}\omega_t|^2\,dx\,ds \\
&= \frac{1}{2}\int_\Omega \rho|\omega_t(0)|^2dx + \int_0^t\int_\Omega \Big(\frac{\epsilon}{2}\sigma_t|\omega_t|^2 - \epsilon\sigma_t\omega_t - (\rho u)_t\cdot\nabla\omega - \rho u\cdot\nabla\omega_t\Big)\omega_t\,dx\,ds \\
&\quad + \epsilon\int_0^t\int_\Omega (\nabla\sigma_t\times u_t + \nabla\sigma\times u_{tt})\omega_t\,dx\,ds + \int_0^t\int_\Omega \nabla\Delta(d_j)_t\times\nabla d_k\omega_t\,dx\,ds \\
&\quad + \int_0^t\int_\Omega (\nabla\Delta d_j\times\nabla(d_j)_t + \nabla(\rho u_i)_t\times\partial_i u + \nabla(\rho u_i)\times\partial_i u_t)\omega_t\,dx\,ds \\
&= C_0(M_0) + I_8 + I_9 + I_{10} + I_{11},
\end{aligned}\tag{2.38}$$

where, by using (2.32) and integrating by parts, we have

$$\begin{aligned}
-\mu\int_0^t\int_\Omega \Delta\omega\cdot\omega_t\,dx\,ds &= \mu\int_0^t\int_\Omega \mathrm{curl}\,\mathrm{curl}\,\omega_t\cdot\omega_t\,dx\,ds \\
&= \mu\int_0^t\int_\Omega |\,\mathrm{curl}\,\omega_t|^2\,dx\,ds + \mu\int_0^t\int_{\partial\Omega} \mathrm{curl}\,\omega_t\cdot(\omega_t\times n)dS \\
&= \mu\int_0^t\int_\Omega |\mathrm{curl}\omega_t|^2\,dx\,ds.
\end{aligned}$$

We estimate I_i $(i = 8, 9, 10, 11)$ as follows.

$$\begin{aligned}
I_{10} &= \int_0^t\int_\Omega \nabla\Delta(d_j)_t\cdot(\nabla d_j\times\omega_t)\,dx\,ds \\
&= -\int_0^t\int_\Omega \Delta(d_j)_t\,\mathrm{div}(\nabla d_j\times\omega_t)\,dx\,ds + \int_0^t\int_{\partial\Omega} \Delta(d_j)_t\nabla(d_j)_t\cdot(\omega_t\times n)dS \\
&= \int_0^t\int_\Omega \Delta(d_j)_t\nabla d_j\cdot\mathrm{curl}\,\omega_t\,dx\,ds \le \sqrt{t}C(M).
\end{aligned}$$

With calculations similar to those in [5], we have

$$|J_8| + |J_9| + |J_{11}| \le \sqrt{t}C(M).$$

Substituting the above estimates into (2.38), we obtain

$$\int_\Omega \rho|\,\mathrm{curl}\,u_t|^2dx + \int_0^t\int_\Omega |\mathrm{curlcurl}u_t|^2\,dx\,ds \le C_0(M_0)\exp C(\sqrt{t}C(M)).\tag{2.39}$$

Now, we have a priori estimate on $\|\nabla d_t\|_{L_t^\infty(H^1)}$. Multiplying (2.29) by $-\Delta d_{tt}$ and integrating over $\Omega\times(0,t)$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |\Delta d_t|^2 dx + \int_0^t\int_\Omega |\nabla d_{tt}|^2\,dx\,ds\\
&= \frac{1}{2}\int_\Omega |\Delta d_t(0)|^2 dx + \int_0^t\int_\Omega [(u\cdot\nabla d)_t - (|\nabla d|^2 d)_t]\Delta d_{tt}\,dx\,ds\\
&\le C_0(M_0) + C\int_0^t (\|\nabla d\|_{H^2}\|u_t\|_{L^2} + \|u\|_{H^2}\|\nabla d_t\|_{L^2}\\
&\quad + \|\nabla d\|_{H^2}\|\nabla d_t\|_{L^2})\|\Delta d_{tt}\|_{L^2} ds\\
&\le C_0(M_0)\exp(\sqrt{t}C(M)).
\end{aligned}
\tag{2.40}
$$

By the same reasoning as for (2.14), we conclude that

$$
\|\nabla d_t\|_{H^1}^2 + \int_0^t\int_\Omega |\nabla d_{tt}|^2\,dx\,ds \le C_0(M_0)\exp(\sqrt{t}C(M)). \tag{2.41}
$$

Finally, we only need to estimate $\epsilon\sigma_{tt}, \epsilon u_{tt}, \epsilon\nabla d_{tt}$ to close the energy estimates. Multiplying ∂_{tt} (1.4), ∂_{tt}(1.5), ∂_{tt}(1.6) by $\epsilon^2\sigma_{tt}$, $\epsilon^2 u_{tt}$ and $\epsilon^2\Delta d_{tt}$, respectively, and integrating over $\Omega\times(0,t)$, we derive that

$$
\epsilon\|(\sigma_{tt}, u_{tt}, \nabla d_{tt})\|_{L^2}^2 + \epsilon\int_0^t \|(u_{tt}, \nabla d_{tt})\|_{H^1}^2 ds \le C_0(M_0)\exp(t^{1/4}C(M)). \tag{2.42}
$$

Collecting the estimates obtained in (2.7), (2.11), (2.13), (2.17), (2.19), (2.24), (2.27), (2.31), (2.32), (2.36), (2.39), (2.41), and (2.42), we have

$$
\begin{aligned}
&\|(\sigma,u)\|_{L^2} + \|(\operatorname{div} u, \operatorname{curl} u, \operatorname{curl}\operatorname{curl} u, \nabla\operatorname{div} u)\|_{L^2} + \|(\nabla\sigma,\nabla d)\|_{H^1} + \|\nabla d\|_{H^2}\\
&+ \|(\sigma_t,u_t)\|_{L^2} + \|(\nabla\sigma_t, \operatorname{div} u_t, \operatorname{curl} u_t)\|_{L^2} + \|\nabla d_t\|_{H^1} + \epsilon\|(\sigma_{tt}, u_{tt}, \nabla d_{tt})\|_{L^2}\\
&+ \|(\operatorname{div} u, \operatorname{curl} u, \operatorname{curl}\operatorname{curl} u)\|_{L_t^2(L^2)} + \|(\nabla^2\operatorname{div} u, \Delta\operatorname{curl} u)\|_{L_t^2(L^2)}\\
&+ \|(\operatorname{div} u_t, \operatorname{curl} u_t, \operatorname{curl}\operatorname{curl} u_t, \nabla\operatorname{div} u_t)\|_{L_t^2(L^2)} + \epsilon\|(\sigma_{tt}, u_{tt}, \nabla d_{tt})\|_{L_t^2(H^1)}\\
&\le C_0(M_0)\exp(t^{1/4}C(M)).
\end{aligned}
\tag{2.43}
$$

Thus, (2.1) holds. this completes the proof of Theorem 1.2.

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