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EXISTENCE OF INFINITELY MANY SMALL SOLUTIONS FOR SUBLINEAR FRACTIONAL KIRCHHOFF-SCHRÖDINGER-POISSON SYSTEMS

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Abstract. We study the Kirchhoff-Schrödinger-Poisson system

$$
\begin{gathered}
m([u]_\alpha^2)(-\Delta)^\alpha u + V(x)u + k(x)\phi u = f(x,u), \quad x \in \mathbb{R}^3, \\
(-\Delta)^\beta \phi = k(x)u^2, \quad x \in \mathbb{R}^3,
\end{gathered}
$$

where $[\cdot]_\alpha$ denotes the Gagliardo semi-norm, $(-\Delta)^\alpha$ denotes the fractional Laplacian operator with $\alpha, \beta \in (0,1]$, $4\alpha + 2\beta \geq 3$ and $m : [0,+\infty) \to [0,+\infty)$ is a Kirchhoff function satisfying suitable assumptions. The functions $V(x)$ and $k(x)$ are nonnegative and the nonlinear term $f(x,s)$ satisfies certain local conditions. By using a variational approach, we use a Kajikiya's version of the symmetric mountain pass lemma and Moser iteration method to prove the existence of infinitely many small solutions.

1. Introduction

In recent years, systems of the form

$$
\begin{gathered}
-\Delta u + V(x)u + \phi u = f(x,u), \quad x \in \mathbb{R}^3, \\
-\Delta \phi = u^2, \quad x \in \mathbb{R}^3,
\end{gathered}
\tag{1.1}
$$

have been widely studied by many researchers. In system (1.1), the first equation is a nonlinear Schrödinger equation in which the potential ϕ satisfies a nonlinear Poisson equation. In this context, it is well known the study of existence of solutions for system (1.1) by using variational methods, under suitable conditions. For instance, we refer the readers to [1, 4, 6, 19, 21] and the references given there. Particularly, we call attention to the work by Bao [4], where it was studied the existence of infinitely many small solutions for (1.1) with sign-changing potential $V(x)$ and without require any global growth condition on the nonlinearity $f(x,s)$.

We mention that a great attention has been focused on the study of problems involving fractional Sobolev spaces and corresponding nonlocal equations, both from a pure mathematical point of view and their concrete applications. In fact, fractional Schrödinger equations naturally arise in many different contexts, such as, obstacle problems, flame propagation, minimal surfaces, conservation laws, financial market, optimization, crystal dislocation, phase transition and water waves. The

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literature is quite large, here we just refer the reader to the important works [13, 9] and references therein.

There are some works concerned with the existence of solutions for the following class of nonlinear fractional Schrödinger-Poisson systems,

$$\begin{gathered}(-\Delta)^{\alpha} u+V(x)u+k(x)\phi u=f(x,u), \quad x\in\mathbb{R}^3,\\ (-\Delta)^{\beta}\phi=k(x)u^2, \quad x\in\mathbb{R}^3,\end{gathered} \tag{1.2}$$

where $\alpha,\beta\in(0,1]$. For instance, Liu [17] studied the case when $\alpha,\beta\in(0,1)$, $V(x)\equiv 1$, $f(x,u)=|u|^{p-1}u$, $k(x)=V(|x|)$ and $1<p<(3+2\alpha)/(3-2\alpha)$. The author obtained the existence of infinitely many nonradial positive solutions for (1.2), based on Lyapunov-Schmidt reduction. By considering a general nonlinear term, Li [16], studied the case when $k(x), V(x)\equiv 1$ and $\alpha,\beta\in(0,1]$ with $4\alpha+2\beta>3$. The author has obtained the existence of non-trivial solutions based on the perturbation method and the mountain pass theorem, supposing that $f(x,s)$ is a subcritical nonlinearity satisfying an Ambrosetti-Rabinowitz type condition, precisely, there exists $\mu>4$ such that

$$0<\mu F(x,s):=\mu\int_0^s f(x,\tau)dx\le f(x,s)s, \quad \text{for all } (x,s)\in\mathbb{R}^N\times\mathbb{R}. \tag{1.3}$$

In a similar fashion, Duarte et al. [12] studied (1.2) under more general conditions, where it is assumed a positive potential $V(x)$ is bounded away from zero, and a general autonomous nonlinearity with 4-superlinear growth, namely $\inf_{x\in\mathbb{R}^3}V(x)>0$,

$$\lim_{s\to\infty}\frac{F(s)}{s^4}=\infty \text{ and the function } s\mapsto\frac{f(s)}{|s|^3} \text{ is increasing for } |s|\neq 0. \tag{1.4}$$

For more works in this direction, we refer the readers to [25, 22, 27]. To the best of our knowledge, there are few works concerned with the class of fractional Schrödinger-Poisson equations (1.2) in the presence of Kirchhoff term with general $\alpha\in(0,1]$. Here we cite [2], where the author used a minimax type argument to prove the existence of a non-trivial solution for a fractional Kirchhoff-Schrödinger-Poisson system in $\mathbb{R}^3$ involving a Berestycki-Lions type nonlinearity.

Motivated by the above discussion, we study the existence of infinitely many small solutions for the following class of fractional Kirchhoff-Schrödinger-Poisson equations

$$\begin{gathered}m([u]_\alpha^2)(-\Delta)^{\alpha} u+V(x)u+k(x)\phi u=f(x,u), \quad x\in\mathbb{R}^3,\\ (-\Delta)^{\beta}\phi=k(x)u^2, \quad x\in\mathbb{R}^3,\end{gathered} \tag{1.5}$$

where $\alpha,\beta\in(0,1]$ such that $4\alpha+2\beta\ge 3$ and $(-\Delta)^\alpha$ denotes the fractional Laplacian operator which can be represented by the singular integral

$$(-\Delta)^\alpha u(x)=C(\alpha)\,\mathrm{P.V.}\int_{\mathbb{R}^3}\frac{u(x)-u(y)}{|x-y|^{3+2\alpha}}\,\mathrm{d}y,$$

for u sufficiently smooth (see [13]). Henceforth, we omit the normalization constant $C(\alpha)$. The term

$$[u]_\alpha=\Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2\alpha}}\,\mathrm{d}x\mathrm{d}y\Big)^{1/2}$$

is the so-called *Gagliardo semi-norm* of the function u. In Section 2 we give more details about the fractional setting. In the present paper, $k(x)$ and $V(x)$ are nonnegative functions, where the potential $V(x)$ is locally integrable. In addition, we assume the following hypotheses:

(H1) $k \in L^r(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$ such that

$$\begin{gathered} r > r_* := \frac{6}{4\alpha + 2\beta - 3}, \quad \text{if } 4\alpha + 2\beta > 3, \\ r = r_* = \infty, \quad \text{if } 4\alpha + 2\beta = 3. \end{gathered}$$

(H2) There exists $\delta_0 > 0$ such that for the level set $\mathcal{G}_{\delta_0} := \{x \in \mathbb{R}^3 : V(x) < \delta_0\}$, we have $0 < |\mathcal{G}_{\delta_0}| < \infty$, where $|\cdot|$ denotes the Lebesgue measure.

(H3) For each $\delta > 0$ and level set $\mathcal{G}_\delta := \{x \in \mathbb{R}^3 : V(x) < \delta\}$, we have $0 \le |\mathcal{G}_\delta| < \infty$.

(H4) $m(t) \ge m_0 > 0$, for all $t \in [0, +\infty)$.

(H5) There exist constants $a_1, a_2 > 0$ and $t_0 > 0$ such that for some $\sigma \ge 0$

$$M(t) := \int_0^t m(\tau)\,\mathrm{d}\tau \le a_1 t + \frac{a_2}{2} t^{\sigma+2}, \quad \text{for all } t \le t_0.$$

(H6) $f \in C(\mathbb{R}^3 \times [-\delta_1, \delta_1], \mathbb{R})$ for some $\delta_1 > 0$ and there exist $\nu \in (1, 2)$, $\mu \in (3/(2\alpha), 2/(2-\nu))$ and a nonnegative function $\xi \in L^\mu(\mathbb{R}^3)$ such that

$$|f(x,s)| \le \nu \xi(x) |s|^{\nu-1}, \quad \text{for all } (x,s) \in \mathbb{R}^3 \times [-\delta_1, \delta_1].$$

(H7) There exist $x_0 \in \mathbb{R}^3$ and a constant $r_0 > 0$ such that

$$\begin{gathered} \liminf_{s \to 0} \Big(\inf_{x \in B_{r_0}(x_0)} \frac{F(x,s)}{s^2} \Big) > -\infty, \\ \limsup_{s \to 0} \Big(\inf_{x \in B_{r_0}(x_0)} \frac{F(x,s)}{s^2} \Big) = +\infty, \end{gathered}$$

where $F(x,s) := \int_0^s f(x,\tau)\,\mathrm{d}\tau$.

(H8) There exists $\delta_2 > 0$ such that $f(x,-s) = -f(x,s)$, for all $(x,s) \in \mathbb{R}^3 \times [-\delta_2, \delta_2]$.

From the nature of the problem, it is well known that system (1.5) can be reduced to a nonlinear Schrödinger equation with an additional nonlocal term (see Section 2). This new term has forth order homogeneity and it is usual to apply variational arguments for nonlinearities which behave like $|s|^{p-2}s$, for $4 < p < 2^*_\alpha := 6/(3-2\alpha)$ by considering hypothesis (1.3) or (1.4) (see [11, 12, 16, 25] and the references therein). In order to get the strictly inequality $4 < 2^*_\alpha$, it is necessary to impose the lower bound $\alpha > 3/4$ in the fractional Laplacian operator. Differently from this case, and similar ones, our assumptions (H6)-(H8) allow the fractional parameter α to vary in $(0, 1]$ submitted only to condition (H1) (see [12]).

Another interesting feature of our assumptions is that the function $\xi(x)$ in (H6) may not be bounded (see Remark 1.2 (iii) below). Thus, the nonlinear term $f(x,s)$ may not be uniformly bounded in x. For this reason, unlike [4], we consider general nonnegative potentials. However, we mention that our arguments also permit to consider sign-changing potentials provided that $\xi(x)$ is bounded. In fact, in this case, under (H2) and(H3), we can assume $\inf_{x\in\mathbb{R}^3} V(x) > -\infty$ and $V_0 > 0$ such that $\tilde{V}(x) = V(x) + V_0 > 0$, in order to apply our approach to the equivalent problem

$$m([u]_\alpha^2)(-\Delta)^\alpha u + \tilde{V}(x)u + k(x)\phi u = f(x,u) + V_0 u, \quad x \in \mathbb{R}^3,$$

$$(-\Delta)^\beta \phi = k(x)u^2, \quad x \in \mathbb{R}^3.$$

In this new framework, it is possible to follow the arguments contained in the proof of [4, Theorem 1.1] and [28, Lemma 3.3], to get a suitable L^∞-estimate, which is an important part of our main result. To do this, it is crucial the use of a cut-off type argument and the boundedness of $\xi(x)$ to conclude that the truncated nonlinear term $f_h(x,s) := (f(x,s) + V_0 s)h(s)$ is uniformly bounded. The main result of this work can now be stated as follows.

Theorem 1.1. *Suppose* (H1)–(H8) *hold. Then, system* (1.5) *has infinitely many non-trivial solutions* $(u_n, \phi_n)_{n\in\mathbb{N}}$ *such that*

$$\frac{1}{2}M([u_n]_\alpha^2) + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u_n^2 \,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} k(x)\phi_{u_n} u_n^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} F(x,u_n)\,\mathrm{d}x \le 0.$$

Moreover, $u_n \to 0$ *as* $n \to +\infty$.

We mention that our result extends some papers in the literature, since we are considering a general class of fractional Kirchhoff-Schrödinger-Poisson systems. Precisely, we deal with a class of potentials $V(x)$ under assumptions which induce compactness of the corresponding Sobolev embedding, the nonnegative term $k(x)$ is bounded or belongs to a suitable Lebesgue space and we are assuming that the nonlinear term $f(x,s)$ satisfies only local conditions. To prove the existence of infinitely many small solutions to ystem (1.5), we use a Kajikiya's version of the symmetric mountain pass lemma (see [14]). One shall also notice that the novelty of our result also provides a regularity type result for ystem (1.5), showing that the solutions have a priori L^∞-bound (see Lemma 4.1), which is crucial to obtain more regularity for solutions of elliptic problems involving the fractional Laplacian (see [8]). For this purpose, we use the α-harmonic extension jointly with a Moser iteration method. To the best of our knowledge, there seems to be no similar results in the current literature for the class of equations studied here, even in the local case $\alpha = \beta = 1$.

Remark 1.2. Now we give some remarks and examples of functions which satisfy our assumptions:

(i) It is important to mention that the potential considered here may null in nonempty interior sets of $\mathbb{R}^3$. This class of potentials is somehow inspired by [24, 5], where it first appeared for the local case. Examples of potentials which satisfy (H2) and (H3) are given by $V_1(x) = |x| + 1/|x| - 2$ and $V_2(x) = |x|$, if $|x| > 1$, and $V_2(x) = 0$, if $|x| \le 1$. We emphasize that our arguments are general and thus, it allow many other classes of nonnegative potentials whose may go to infinity as $|x| \to \infty$ (see the local case [23]).

(ii) A typical example of $m : [0,+\infty) \to [0,+\infty)$ verifying (H4) and (H5) is given by $m(t) = m_0 + a_2 t$, $a_2 \ge 0$, which is the one considered in the classical Kirchhoff equation, see [15]. More generally, the following function

$$m(t) = m_0 + a_2 t + \sum_{i=1}^{k} b_i t^{d_i},$$

with $b_i \ge 0$ and $d_i \in (0,1)$ for all $i \in \{1,2,\dots,k\}$ satisfies assumptions (H4) and (H5).

(iii) One can see that the following function satisfies conditions (H6)–(H8). More precisely, consider

$$F(x,s) = \begin{cases} \xi(x)|s|^{\theta}\sin^2(|s|^{\varepsilon}), & \text{if } x=(x_1,x_2,x_3)\in\mathbb{R}^3 \text{ and } 0<|s|\le 1,\\ 0, & \text{if } s=0, \end{cases}$$

a primitive of the function $f(x,s)$, where $\xi(x)=|x|^{-d}$ for $0<d<3/\mu$ if $|x|\le 1$, and $\xi(x)=0$, if $|x|>1$. We take $\varepsilon>0$ small enough, $\theta\in(1+\varepsilon,2)$, $\delta=1$ and $\nu=\theta-\varepsilon$. Notice that $\xi\in L^{\mu}(\mathbb{R}^3)$, for $\mu\in(3/2\alpha,2/(2-\nu))$.

The remainding of the paper is organized as follows: In the forthcoming section we present some preliminary results and we set up the variational framework to our problem. In Section 3, we prove the existence of a sequence of solutions for the modified problem associated to (1.5). In Section 4, we introduce the α-harmonic extension and we apply Moser iteration method in order to prove that our sequence of solutions converges to zero in L^{∞}-norm. Throughout this paper, the symbols C, C_1, C_2, ... represent several (possibly different) positive constants.

2. Preliminary results

In this Section we collect some basic results of fractional Sobolev spaces and we introduce the variational framework of system (1.5). For $0<\alpha<1$, the fractional Sobolev space is defined as

$$H^{\alpha}(\mathbb{R}^3) := \Big\{u\in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y<+\infty\Big\}.$$

For $u,v\in H^{\alpha}(\mathbb{R}^3)$, we define

$$(u,v)_{\alpha} := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y.$$

It is well known that $H^{\alpha}(\mathbb{R}^3)$ is a Hilbert space when endowed with the standard inner product

$$\langle u,v\rangle = (u,v)_{\alpha} + \int_{\mathbb{R}^3} uv\,\mathrm{d}x,$$

and the correspondent induced norm

$$\|u\|_{H^{\alpha}(\mathbb{R}^3)} = \big([u]_{\alpha}^2 + \|u\|_{L^2(\mathbb{R}^3)}^2\big)^{1/2}.$$

To introduce a variational approach to our problem we define the suitable subspace of $H^{\alpha}(\mathbb{R}^3)$,

$$E := \Big\{u\in H^{\alpha}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2<+\infty\Big\}.$$

In view of assumptions (H2) and (H3) it is not hard to check that E is a Hilbert space when endowed with the inner product

$$\langle u,v\rangle_E = (u,v)_{\alpha} + \int_{\mathbb{R}^3} V(x)uv\,\mathrm{d}x,$$

and the corresponding induced norm $\|u\|^2=\langle u,v\rangle_E$ (see Proposition 2.2). For $u\in E$ and a subset $\Omega\subset\mathbb{R}^3$ we denote

$$\|u\|_{\Omega}^2 := \int_{\Omega}\int_{\Omega}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega} V(x)u^2\,\mathrm{d}x.$$

For any $\beta \in (0,1)$, we recall the homogeneous fractional Sobolev space $\mathcal{D}^{\beta,2}(\mathbb{R}^3)$ as

$$\mathcal{D}^{\beta,2}(\mathbb{R}^3) = \{u \in L^{2^*_\beta}(\mathbb{R}^3) : [u]_\beta < +\infty\},$$

which is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{\beta,2}(\mathbb{R}^3)} = \Big(\int_{\mathbb{R}^3} |(-\Delta)^{\beta/2} u|^2 \,\mathrm{d}x\Big)^{1/2}.$$

We recall that $2^*_\beta := 6/(3-2\beta)$ is the critical Sobolev exponent for $H^\beta(\mathbb{R}^3)$.

Lemma 2.1. *For any $\beta \in (0,1)$, the space $\mathcal{D}^{\beta,2}(\mathbb{R}^3)$ is continuously embedded into $L^{2^*_\beta}(\mathbb{R}^3)$; that is, there exists $S_\beta > 0$ such that*

$$\Big(\int_{\mathbb{R}^3} |u|^{2^*_\beta} \,\mathrm{d}x\Big)^{2/2^*_\beta} \le S_\beta \int_{\mathbb{R}^3} |(-\Delta)^{\beta/2} u|^2 \,\mathrm{d}x, \quad \text{for all } u \in \mathcal{D}^{\beta,2}(\mathbb{R}^3).$$

For a more information about fractional Sobolev spaces we refer the readers to [13]. Next we prove an embedding result involving our space of functions.

Proposition 2.2. *If* (H2) *holds, then E is continuously embeeded into $H^\alpha(\mathbb{R}^3)$. In addition, if* (H3) *holds, then E is compactly embedded into $L^p(\mathbb{R}^3)$, for $p \in [2, 2^*_\alpha)$.*

Proof. We start by proving that $E \hookrightarrow H^\alpha(\mathbb{R}^3)$, i.e., there exists $C > 0$ such that $\|u\|^2_{H^\alpha(\mathbb{R}^3)} \le C\|u\|^2$, for all $u \in E$. To do that, we use Hölder inequality and Lemma 2.1 to see that

$$\int_{\mathcal{G}_{\delta_0}} u^2 \,\mathrm{d}x \le |\mathcal{G}_{\delta_0}|^{\frac{2\alpha}{N}} S_\alpha [u]^2_\alpha, \quad \text{for all } u \in H^\alpha(\mathbb{R}^3).$$

By using this estimate, assumption (H2) and the fact that $V(x)$ is nonnegative, we have that

$$\begin{aligned}
\|u\|^2 &\ge \frac{1}{2}[u]^2_\alpha + \frac{1}{2}|\mathcal{G}_{\delta_0}|^{-\frac{2\alpha}{N}} S_\alpha^{-1} \int_{\mathcal{G}_{\delta_0}} u^2 \,\mathrm{d}x + \delta_0 \int_{\mathbb{R}^3\setminus\mathcal{G}_{\delta_0}} u^2 \,\mathrm{d}x + \int_{\mathcal{G}_{\delta_0}} V(x) u^2 \,\mathrm{d}x \\
&\ge C\|u\|^2_{H^\alpha(\mathbb{R}^3)},
\end{aligned}$$

where $C = \min\{1/2, (1/2)|\mathcal{G}_{\delta_0}|^{-2\alpha/N} S_\alpha^{-1}, \delta_0\}$, which implies the continuous embedding.

Now we prove the compact embedding $E \hookrightarrow\!\!\!\!\rightarrow L^p(\mathbb{R}^3)$, for $2 \le p < 2^*_\alpha$. Let $(u_n)_{n\in\mathbb{N}} \subset E$ be such that $u_n \rightharpoonup u$ weakly in E. In view of an interpolation inequality, it suffices to show that $u_n \to u$ strongly in $L^2(\mathbb{R}^3)$, up to subsequence. To prove this fact, we claim that for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{\mathbb{R}^3\setminus B_R} u_n^2 \,\mathrm{d}x < \varepsilon, \quad \text{for all } n \in \mathbb{N} \text{ (uniformly in } n), \tag{2.1}$$

where B_R denotes the open ball with radius R centered at zero. In fact, let us consider $p \in (1, 3/(3-2\alpha))$ and constants $M, \mathcal{C} > 0$ satisfying

$$\frac{1}{M}\sup_{n\in\mathbb{N}} \|u_n\|^2 < \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{u\in E\setminus\{0\}} \Big[\frac{\|u\|^2_{L^{2p}(\mathbb{R}^3)}}{\|u\|^2}\Big] \le \mathcal{C}.$$

On the other hand, note that there exits $R > 0$ such that

$$|\{x \in \mathbb{R}^3 \setminus B_R : V(x) < M\}| \le \Big[\frac{\varepsilon}{2\mathcal{C}\sup_{n\in\mathbb{N}}\|u_n\|^2}\Big]^{p'}, \quad \text{where } \frac{1}{p} + \frac{1}{p'} = 1.$$

If $A=\{x\in\mathbb{R}^3\setminus B_R : V(x)\geq M\}$ and $B=\{x\in\mathbb{R}^3\setminus B_R : V(x)<M\}$, then one has

$$\begin{gathered}\int_A u_n^2\,\mathrm{d}x \leq \frac{1}{M}\int_A V(x)u_n^2\,\mathrm{d}x \leq \frac{1}{M}\sup_{n\in\mathbb{N}}\|u_n\|^2<\frac{\varepsilon}{2},\\ \int_B u_n^2\,\mathrm{d}x \leq |B|^{\frac{1}{p'}}\|u_n\|^2_{L^{2p}(B_R)} \leq \mathcal{C}|B|^{\frac{1}{p'}}\sup_{n\in\mathbb{N}}\|u_n\|^2<\frac{\varepsilon}{2},\end{gathered}$$

which proves (2.1). Let $\theta=\lim_{n\to\infty}\|u_n\|^2_{L^2(\mathbb{R}^3)}$. By the semicontinuity of the norm we have $\|u\|^2_{L^2(\mathbb{R}^3)}\leq\theta$. On the other hand, using (2.1) and the fact that $u_n\to u$ strongly in $L^2(B_R)$, we see that

$$\begin{aligned}\|u\|^2_{L^2(\mathbb{R}^3)} &= \|u\|^2_{L^2(B_R)}+\|u\|^2_{L^2(\mathbb{R}^3\setminus B_R)}\\ &\geq \lim_{n\to\infty}\big[\|u_n\|^2_{L^2(\mathbb{R}^3)}-\|u_n\|_{L^2(\mathbb{R}^3\setminus B_R)}\big]\\ &\geq \theta-\varepsilon.\end{aligned}$$

Therefore, $\|u\|^2_{L^2(\mathbb{R}^3)}\geq\theta$, which implies that $\|u_n\|^2_{L^2(\mathbb{R}^3)}\to\|u\|^2_{L^2(\mathbb{R}^3)}$, up to a subsequence. □

For any $u\in H^\alpha(\mathbb{R}^3)$, let $L_u:\mathcal{D}^{\beta,2}(\mathbb{R}^3)\to\mathbb{R}$ be the linear functional defined by

$$L_u(v)=\int_{\mathbb{R}^3}k(x)u^2v\,\mathrm{d}x.$$

By using (H1), Lemma 2.1 and Hölder inequality we deduce that

$$|L_u(v)|\leq\begin{cases}\|k(x)\|_{L^\infty(\mathbb{R}^3)}\|u\|^2_{L^{l_\infty}(\mathbb{R}^3)}\|v\|_{L^{2^*_\beta}(\mathbb{R}^3)}, & \text{if } k\in L^\infty(\mathbb{R}^3),\\ \|k(x)\|_{L^r(\mathbb{R}^3)}\|u\|^2_{L^{l_r}(\mathbb{R}^3)}\|v\|_{L^{2^*_\beta}(\mathbb{R}^3)}, & \text{if } k\in L^r(\mathbb{R}^3),\end{cases}\tag{2.2}$$

for all $v\in\mathcal{D}^{\beta,2}(\mathbb{R}^3)$, where $l_\infty=2\cdot2^*_\alpha/(2^*_\alpha-1)$ and $l_r:=12r/((3+2\alpha)r-6)$. Condition $4\alpha+3\beta\geq3$ implies that $2\leq l_\infty,l_r\leq2^*_\alpha$. It follows from (2.2) that L_u is continuous. Thus, in light of Lax-Milgram Theorem, there exists a unique $\phi_u\in\mathcal{D}^{\beta,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3}(-\Delta)^{\beta/2}\phi_u(-\Delta)^{\beta/2}v\,\mathrm{d}x=\int_{\mathbb{R}^3}k(x)u^2v\,\mathrm{d}x,\quad\text{for all } v\in\mathcal{D}^{\beta,2}(\mathbb{R}^3),\tag{2.3}$$

that is, ϕ_u is a weak solution of the problem

$$(-\Delta)^\beta\phi_u=k(x)u^2,\quad x\in\mathbb{R}^3.$$

It is well known that the following representation formula holds

$$\phi_u(x)=c_\beta\int_{\mathbb{R}^3}\frac{k(y)u^2(y)}{|x-y|^{3-2\beta}}\,\mathrm{d}y,\quad\text{for all} x\in\mathbb{R}^3,$$

which is called β-Riesz potential, where

$$c_\beta=\frac{\Gamma(3-2\beta)}{\pi^{3/2}2^{2\beta}\Gamma(\beta)}.$$

Since we only required local assumptions on the nonlinear term $f(x,s)$, we use a cut-off argument similar to the one introduced in [14]. Let us consider $0<r<(1/2)\min\{\delta_1,\delta_2,1\}$. We define an even function $h\in C^\infty(\mathbb{R},\mathbb{R}^+)$ such that $0\leq h(t)\leq1$, $h(t)=1$ for $|t|\leq r$, $h(t)=0$ for $|t|\geq2r$ and h is decreasing in

$[r, 2r]$. Let $f_h(x,u) = f(x,u)h(u)$ and $F_h(x,u) = \int_0^u f_h(x,t)\,dt$. We introduce the modified problem

$$\begin{gathered} m([u]_\alpha^2)(-\Delta)^\alpha u + V(x)u + k(x)\phi u = f_h(x,u), \quad x \in \mathbb{R}^3, \\ (-\Delta)^\beta \phi = k(x)u^2, \quad x \in \mathbb{R}^3. \end{gathered} \tag{2.4}$$

Replacing ϕ by ϕ_u in the first equation of (2.4), we obtain the fractional Kirchhoff-Schrödinger equation

$$m([u]_\alpha^2)(-\Delta)^\alpha u + V(x)u + k(x)\phi_u u = f_h(x,u), \quad x \in \mathbb{R}^3. \tag{2.5}$$

Problem (2.5) admits a variational formulation and its solutions are the critical points of the energy functional

$$I_h(u) = \frac{1}{2}M([u]_\alpha^2) + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3} k(x)\phi_u u^2\,dx - \int_{\mathbb{R}^3} F_h(x,u)\,dx.$$

It follows from (H6) that

$$|f_h(x,s)| \le \nu\xi(x)|s|^{\nu-1}, \quad \text{for all } (x,s) \in \mathbb{R}^3 \times \mathbb{R}, \tag{2.6}$$

which implies that

$$|F_h(x,s)| \le \xi(x)|s|^\nu, \quad \text{for all } (x,s) \in \mathbb{R}^3 \times \mathbb{R}. \tag{2.7}$$

Let us define $\mu^* := \nu\mu/(\mu-1)$. Since $\nu \in (1,2)$ and $\mu \in (3/(2\alpha), 2/(2-\nu))$ we have that $\mu^* \in (2, 2_\alpha^*)$. Hence, for any $u \in E$, it follows from (2.7), Hölder inequality and Sobolev embedding that

$$\int_{\mathbb{R}^3} |F_h(x,u)|\,dx \le \|\xi\|_{L^\mu(\mathbb{R}^3)} \|u\|^\nu_{L^{\mu^*}(\mathbb{R}^3)} \le C(\mu^*,\nu)\|\xi\|_{L^\mu(\mathbb{R}^3)}\|u\|^\nu < +\infty.$$

Therefore, I_h is well defined.

Definition 2.3. We say that $(u,\phi_u) \in H^\alpha(\mathbb{R}^3) \times \mathcal{D}^{\beta,2}(\mathbb{R}^3)$ is a solution of (2.4) if u is a weak solution of (2.5); that is,

$$m([u]_\alpha^2)(u,v)_\alpha + \int_{\mathbb{R}^3} V(x)uv\,dx + \int_{\mathbb{R}^3} k(x)\phi_u uv\,dx = \int_{\mathbb{R}^3} f_h(x,u)v\,dx,$$

for all $v \in E$.

Note that if u is a critical point of the functional I_h and $\|u\|_{L^\infty(\mathbb{R}^3)} \le r$, then u is a solution of (1.5).

3. Kajikiya symmetric mountain pass lemma

Let X be a Banach space and Γ be the family of sets $A \subset X \setminus \{0\}$ which are closed in X and symmetric with respect to the origin, i.e. $x \in A$ implies $-x \in A$. For $A \in \Gamma$, the genus $\gamma(A)$ is defined as

$$\gamma(A) = \inf\big\{N \in \mathbb{N} : \exists\psi \in C(A, \mathbb{R}^N \setminus \{0\}) \text{ with } \psi(-z) = -\psi(z), \text{ for all } z \in A\big\}.$$

If there is no mapping as above for any $N \in \mathbb{N}$, then $\gamma(A) = +\infty$. Here we summarize the properties of genus whose will be used in the proof of Theorem 1.1. A detailed proof can be found in [20].

Proposition 3.1. *Let $A, B \in \Gamma \subset X\setminus\{0\}$. Then, the following properties hold:*

(a) *If there is an odd homeomorphism from A to B, then $\gamma(A) = \gamma(B)$.*
(b) *If $\mathbb{S}^{N-1}$ is the unit sphere in $\mathbb{R}^N$, then $\gamma(\mathbb{S}^{N-1}) = N$.*

Definition 3.2. Let X be a Banach space, $(u_n)_{n\in\mathbb{N}} \subset X$ be a sequence and $J : X \to \mathbb{R}$ be a C^1 functional. We say that $(u_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence at level $c \in \mathbb{R}$, if

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0. \tag{3.1}$$

We say that J satisfies the Palais-Smale condition at level $c \in \mathbb{R}$, whenever any Palais-Smale sequence at level $c \in \mathbb{R}$ admits a convergent subsequence.

To prove the existence of infinitely many solutions for system (1.5), we use the following version of the symmetric mountain pass lemma which is due to Kajikiya, see [14].

Theorem 3.3. *Let X be an infinite dimensional Banach space, Γ_k be the family of closed symmetric subsets $A \subset X$ such that $0 \notin A$ and the genus $\gamma(A) \geq k$, $J \in C^1(X)$ be an even functional such that $J(0) = 0$ and*

(H9) *J is bounded from below and satisfies the Palais-Smale condition;*
(H10) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u\in A_k} J(u) < 0$.*

Then, J admits a sequence of critical points $(u_n)_{n\in\mathbb{N}}$ such that $J(u_n) \leq 0$, $u_n \neq 0$ and $\lim_{n\to+\infty} u_n = 0$.

In the following, we prove that Palais-Smale sequences for I_h satisfy the properties (H9) and (H10) required in Theorem 3.3.

Proposition 3.4. *I_h is bounded from below.*

Proof. For any $u \in E$ we introduce the set $\Omega_u := \{x \in \mathbb{R}^3 : |u(x)| \leq 1\}$. By the definition of h we have

$$\int_{\mathbb{R}^3} F_h(x,u)\,\mathrm{d}x = \int_{\Omega_u} F_h(x,u)\,\mathrm{d}x.$$

Hence, in view of (H4), (2.3) and (2.7) it follows that

$$I_h(u) \geq \frac{m_0}{2}[u]_\alpha^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2\,\mathrm{d}x - \int_{\Omega_u} \xi(x)|u|^\nu\,\mathrm{d}x.$$

Thus, by using Hölder inequality and Sobolev embedding we obtain

$$I_h(u) \geq \min\big\{\frac{m_0}{2}, \frac{1}{2}\big\}\|u\|_{\Omega_u}^2 - C(\mu^*,\nu)\|\xi\|_{L^\mu(\mathbb{R}^3)}\|u\|_{\Omega_u}^\nu. \tag{3.2}$$

Since $\nu \in (1,2)$ we conclude that I_h is bounded from below. □

Lemma 3.5. *If $(u_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence for I_h, then $(u_n)_{n\in\mathbb{N}}$ is bounded in E.*

Proof. It follows from (3.1) and (3.2) that

$$C \geq I_h(u_n) \geq \min\big\{\frac{m_0}{2}, \frac{1}{2}\big\}\|u_n\|_{\Omega_{u_n}}^2 - C(\mu^*,\nu)\|\xi\|_{L^\mu(\mathbb{R}^3)}\|u_n\|_{\Omega_{u_n}}^\nu,$$

where $\Omega_{u_n} := \{x \in \mathbb{R}^3 : |u_n(x)| \leq 1\}$. Since $\nu \in (1,2)$ we conclude that $\|u_n\|_{\Omega_{u_n}} \leq C$, where C does not depends on $n \in \mathbb{N}$. Moreover, by using (2.7) we deduce that

$$\begin{aligned}
&\frac{1}{2}\Big[M([u_n]_\alpha^2) + \int_{\mathbb{R}^3} V(x)u_n^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} k(x)\phi_{u_n}u_n^2\,\mathrm{d}x\Big] \\
&\leq I_h(u_n) + C(\mu^*,\nu)\|\xi\|_{L^\mu(\mathbb{R}^3)}\|u_n\|_{\Omega_{u_n}}^\nu.
\end{aligned}$$

Since $\|u_n\|_{\Omega_{u_n}} \leq C$ and $I_h(u_n) \leq C$ we have

$$\frac{1}{2}\Big[M([u_n]_\alpha^2) + \int_{\mathbb{R}^3} V(x)u_n^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} k(x)\phi_{u_n}u_n^2 \,\mathrm{d}x\Big] \leq C,$$

where C does not depends on $n \in \mathbb{N}$. The above boundedness together with (H4) implies that

$$C \geq \frac{1}{2}\Big[M([u_n]_\alpha^2) + \int_{\mathbb{R}^3} V(x)u_n^2 \,\mathrm{d}x\Big] \geq \min\big\{\frac{m_0}{2}, \frac{1}{2}\big\}\|u_n\|^2,$$

which implies that $(u_n)_{n\in\mathbb{N}}$ is bounded in E. □

In view of Proposition 2.2 and Lemma 3.5 we may assume, up to a subsequence, that

$$u_n \rightharpoonup u \quad \text{weakly in } E;$$
$$u_n \to u \quad \text{strongly in } L^p(\mathbb{R}^3), \text{ for } p \in [2, 2_\alpha^*);$$
$$u_n(x) \to u(x) \quad \text{almost everywhere in } \mathbb{R}^3.$$

By using generalized Hölder inequality we deduce the following convergences:

$$\Big|\int_{\mathbb{R}^3} k(x)\phi_{u_n}u_n(u_n-u)\,\mathrm{d}x\Big| \leq \begin{cases} \|k\|_{L^\infty}\|\phi_{u_n}\|_{L^{2_\beta^*}}\|u_n\|_{L^{l_\infty}}\|u_n-u\|_{L^{l_\infty}} \to 0, \\ \|k\|_{L^r}\|\phi_{u_n}\|_{L^{2_\beta^*}}\|u_n\|_{L^{l_r}}\|u_n-u\|_{L^{l_r}} \to 0, \end{cases}$$

as $n \to +\infty$. Thus, we conclude that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3} k(x)(\phi_{u_n}u_n - \phi_u u)(u_n-u)\,\mathrm{d}x = 0. \tag{3.3}$$

Moreover, by using (2.6) and generalized Hölder inequality we obtain the estimate

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^3} (f_h(x,u_n) - f_h(x,u))(u_n-u)\,\mathrm{d}x\Big| \\
&\leq \nu(\|u_n\|_{L^{\mu^*}(\mathbb{R}^3)}^{\nu-1} + \|u\|_{L^{\mu^*}(\mathbb{R}^3)}^{\nu-1})\|\xi\|_{L^\mu(\mathbb{R}^3)}\|u_n-u\|_{L^{\mu^*}(\mathbb{R}^3)},
\end{aligned}$$

which together with the fact that $\mu^* \in (2, 2_\alpha^*)$ and Proposition 2.2 implies that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3} (f_h(x,u_n) - f_h(x,u))(u_n-u)\,\mathrm{d}x = 0. \tag{3.4}$$

Proposition 3.6. *I_h satisfies the Palais-Smale condition.*

Proof. It follows from (3.1) and the weak convergence that

$$\langle I'(u_n) - I'(u), u_n - u\rangle = o_n(1), \tag{3.5}$$

where o_n denotes the standard "little o notation". On the other hand we have

$$\begin{aligned}
&\langle I'(u_n) - I'(u), u_n - u\rangle \\
&= m([u_n]_\alpha^2)(u_n, u_n-u)_\alpha - m([u]_\alpha^2)(u, u_n-u)_\alpha + \int_{\mathbb{R}^3} V(x)(u_n-u)^2\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} k(x)(\phi_{u_n}u_n - \phi_u u)(u_n-u)\,\mathrm{d}x - \int_{\mathbb{R}^3} (f_h(x,u_n) - f_h(x,u))(u_n-u)\,\mathrm{d}x,
\end{aligned}$$

which together with (3.3) and (3.4) implies that

$$\begin{aligned}
\langle I'(u_n) - I'(u), u_n - u\rangle &= m([u_n]_\alpha^2)(u_n, u_n-u)_\alpha - m([u]_\alpha^2)(u, u_n-u)_\alpha \\
&\quad + \int_{\mathbb{R}^3} V(x)(u_n-u)^2\,\mathrm{d}x + o_n(1).
\end{aligned} \tag{3.6}$$

Notice that

$$\begin{aligned} & m([u_n]_\alpha^2)(u_n, u_n - u)_\alpha - m([u]_\alpha^2)(u, u_n - u)_\alpha \\ & = m([u_n]_\alpha^2)[u_n - u]_\alpha^2 + \big(m([u_n]_\alpha^2) - m([u]_\alpha^2)\big)(u, u_n - u)_\alpha . \end{aligned} \tag{3.7}$$

Since $(u_n)_{n\in\mathbb{N}}$ is bounded in E and m is a continuous function, there exists $A \geq 0$ such that $m([u_n]_\alpha^2) \to m(A)$. In particular, $(m([u_n]_\alpha^2))_{n\in\mathbb{N}}$ is bounded. Thus, by weak convergence one has

$$\big(m([u_n]_\alpha^2) - m([u]_\alpha^2)\big)(u, u_n - u)_\alpha = o_n(1). \tag{3.8}$$

It follows from (H4), (3.7) and (3.8) that

$$m([u_n]_\alpha^2)(u_n, u_n - u)_\alpha - m([u]_\alpha^2)(u, u_n - u)_\alpha \geq m_0[u_n - u]_\alpha^2 + o_n(1). \tag{3.9}$$

Combining (3.5), (3.6) and (3.9) we conclude that

$$o_n(1) \geq \min\{m_0, 1\}\|u_n - u\|^2 + o_n(1).$$

Therefore, $u_n \to u$ strongly in E which completes the proof. □

Lemma 3.7. *There exists a sequence of non-trivial critical points $(u_n)_{n\in\mathbb{N}}$ for I_h.*

Proof. The idea is essentially due to [14, Theorem 2] but for the reader's convenience we provide the proof here. For simplicity, we assume that $x_0 = 0$ in (H7), that is, there exists a constant $r_0 > 0$ such that

$$\liminf_{s\to 0}\Big(\inf_{x\in B_{r_0}} \frac{F(x,s)}{s^2}\Big) > -\infty \quad \text{and} \quad \limsup_{s\to 0}\Big(\inf_{x\in B_{r_0}} \frac{F(x,s)}{s^2}\Big) = +\infty.$$

In the following we denote

$$\mathcal{C} := \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : -\frac{r_0}{2} \leq x_i \leq \frac{r_0}{2}, \text{ where } 1 \leq i \leq 3\big\}.$$

By (H7), there exist constants $\vartheta, \epsilon > 0$ and two sequences of positive numbers $\vartheta_n \to 0$ and $M_n \to +\infty$ as $n \to +\infty$ such that

$$F(x, u) \geq -\epsilon u^2, \quad \text{for all } x \in \mathcal{C} \text{ and } |u| \leq \vartheta, \tag{3.10}$$

$$\frac{F(x, \delta_n)}{\vartheta_n^2} \geq M_n, \quad \text{for all } x \in \mathcal{C} \text{ and } n \in \mathbb{N}. \tag{3.11}$$

Fix $k \in \mathbb{N}$ arbitrarily and let $p \in \mathbb{N}$ be the smallest integer satisfying $p^3 \geq k$. We divide $\mathcal{C}$ equally into p^3 cubes by planes parallel to each face of $\mathcal{C}$ and we denote them by $\mathcal{C}_i$, with $1 \leq i \leq p^3$. Thus, the edge of each $\mathcal{C}_i$ has the length of $a = r_0/p$. For each $1 \leq i \leq k$, we make a cube $\tilde{\mathcal{C}}_i \subset \mathcal{C}_i$ such that $\tilde{\mathcal{C}}_i$ has the same center as that of $\mathcal{C}_i$, the faces of $\tilde{\mathcal{C}}_i$ and $\mathcal{C}_i$ are parallel and the edge of $\tilde{\mathcal{C}}_i$ has the length of $a/2$. Now, we define a continuous function $\rho : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{gathered} \rho(t) = 0 \quad \text{for } t \in \mathbb{R} \setminus [-\frac{a}{2}, \frac{a}{2}], \\ \rho(t) = 1 \quad \text{for } t \in [-\frac{a}{4}, \frac{a}{4}], \\ 0 \leq \rho(t) \leq 1 \quad \text{for } t \in (-\frac{a}{2}, -\frac{a}{4}) \cup (\frac{a}{4}, \frac{a}{2}). \end{gathered}$$

Define $\eta_1 : \mathbb{R}^3 \to \mathbb{R}$ such that $\eta_1(x) = \rho(x_1)\rho(x_2)\rho(x_3)$. For each $1 \leq i \leq k$, let y_i be the center of $\tilde{\mathcal{C}}_i$ and set $\eta_{1_i}(x) = \eta_1(x - y_i)$ for all $x \in \mathbb{R}^3$. It is easy to check that, for each $1 \le i \le k$, $0 \le \eta_{1_i}(x) \le 1$ for all $x \in \mathbb{R}^3$, $\operatorname{supp} \eta_{1_i} \subset \mathcal{C}_i$ and $\eta_{1_i}(x) = 1$ if $x \in \tilde{\mathcal{C}}_i$. Set

$$\mathcal{V}_k = \big\{(t_1, \dots, t_k) \in \mathbb{R}^k : \max_{1 \le i \le k} |t_i| = 1\big\},$$
$$\mathcal{W}_k = \big\{ \sum_{i=1}^k t_i \eta_{1_i} : (t_1, \dots, t_k) \in \mathcal{V}_k\big\}.$$

Since V_k is the surface of the k-dimensional cube, it is homeomorphic to the sphere $\mathbb{S}^{k-1}$ by an odd mapping. By Proposition 3.1 we have $\gamma(V_k) = k$. If we define the mapping $\zeta : \mathcal{V}_k \to \mathcal{W}_k$ by

$$\zeta(t_1, \dots, t_k) = \sum_{i=1}^k t_i \eta_{1_i},$$

then ζ is an odd homeomorphism between $\mathcal{V}_k$ and $\mathcal{W}_k$, which implies that $\gamma(V_k) = \gamma(W_k)$. Since $\mathcal{W}_k$ is compact, there exists a constant $C_k > 0$ such that $\|u\| \le C_k$ for all $u \in \mathcal{W}_k$. Thus, using (H5) and (2.2), for any $\beta \in (0, \min\{\vartheta, t_0/C_k\})$ and $u = \sum_{i=1}^k t_i \eta_{1_i} \in \mathcal{W}_k$ we have

$$\begin{aligned} I_h(\beta u) &= \frac{1}{2} M([\beta u]_\alpha^2) + \frac{\beta^2}{2} \int_{\mathbb{R}^3} V(x) u^2 \,\mathrm{d}x \\ &\quad + \frac{\beta^2}{4} \int_{\mathbb{R}^3} k(x) \phi_u u^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} F_h\Big(x, \beta \sum_{i=1}^k t_i \eta_{1_i}\Big) \,\mathrm{d}x \\ &\le \beta^2 C_1 \|u\|^2 + \frac{a_2}{2} \beta^{\sigma+2} \|u\|^{\sigma+2} \\ &\quad + \beta^2 C_2 \|\phi_u\|_{\mathcal{D}^{\beta,2}(\mathbb{R}^3)} \|u\|^2 - \sum_{i=1}^k \int_{\mathcal{C}_i} F_h(x, \beta t_i \eta_{1_i}) \,\mathrm{d}x. \end{aligned} \tag{3.12}$$

On the other hand, by the definition of $\mathcal{V}_k$, there exists some integer $1 \le i_u \le k$ such that $|t_{i_u}| = 1$. Then

$$\begin{aligned} &\sum_{i=1}^k \int_{\mathcal{C}_i} F_h(x, \beta t_i \eta_{1_i}) \,\mathrm{d}x \\ &= \int_{\tilde{\mathcal{C}}_{i_u}} F_h(x, \beta t_i \eta_{1_i}) \,\mathrm{d}x + \int_{\mathcal{C}_{i_u} \setminus \tilde{\mathcal{C}}_{i_u}} F_h(x, \beta t_i \eta_{1_i}) \,\mathrm{d}x + \sum_{i \ne i_u} \int_{\mathcal{C}_i} F_h(x, \beta t_i \eta_{1_i}) \,\mathrm{d}x. \end{aligned}$$

Observe that by (3.10),

$$\int_{\mathcal{C}_{i_u} \setminus \tilde{\mathcal{C}}_{i_u}} F_h(x, \beta t_i \eta_{1_i}) \,\mathrm{d}x + \sum_{i \ne i_u} \int_{\mathcal{C}_i} F_h(x, \beta t_i \eta_{1_i}) \,\mathrm{d}x \ge -\epsilon r_0^3 \beta^2, \tag{3.13}$$

where we used that the volume of $\mathcal{C}$ is r_0^3. We have $|\vartheta_n t_{i_u} \eta_{1_{i_u}}(x)| = \vartheta_n$ for all $x \in \tilde{\mathcal{C}_{i_u}}$ and the volume of $\tilde{\mathcal{C}_{i_u}}$ is $a^3/8$. Since $\vartheta_n \to 0$, we assume that there exists $n_0 \in \mathbb{N}$ such that $\vartheta_n < \min\{\vartheta, t_0/C_k\}$ for all $n \ge n_0$. Thus, using (3.11), (3.12) and (3.13) with $\beta = \vartheta_n$ we obtain

$$\begin{aligned} I(u_n) &\leq C_3\vartheta_n^2 \left(\|u\|^2 + \vartheta_n^\sigma \|u\|^{\sigma+2} + \|\phi_u\|_{\mathcal{D}^{\beta,2}(\mathbb{R}^3)}\|u\|^2 + \epsilon r_0^3\vartheta_n^2\right) \\ &\quad - \int_{\tilde{\mathcal{C}}_{i_u}} F_h(x, \vartheta_n t_i \eta_{1_i})\,\mathrm{d}x \\ &\leq C_3\vartheta_n^2 \Big(\|u\|^2 + \vartheta_n^\sigma \|u\|^{\sigma+2} + \|\phi_u\|_{\mathcal{D}^{\beta,2}(\mathbb{R}^3)}\|u\|^2 + \epsilon r_0^3\vartheta_n^2 - \frac{a^3 M_n}{8}\Big), \end{aligned} \tag{3.14}$$

where $u_n = \vartheta_n u$. Since $u \in W_k$, one has

$$\|\phi_u\|_{\mathcal{D}^{\beta,2}(\mathbb{R}^3)} \leq C_4\|u\| \leq C_4 C_k.$$

Since $\vartheta_n \to 0$ and $M_n \to +\infty$ as $n \to +\infty$, we can choose $n \in \mathbb{N}$ large enough such that

$$C_k^2 + \vartheta_n^\sigma C_k^{\sigma+2} + C_4 C_k^3 + \epsilon r_0^3 \vartheta_n^2 - \frac{a^3 M_n}{8} < 0.$$

This implies that the right-hand side of (3.14) is negative. To complete the proof, we define

$$A_k = \{\vartheta_{n_0} u : u \in \mathcal{W}_k\}.$$

Thus $\gamma(A_k) = \gamma(\mathcal{W}_k) = k$ and $\sup_{u\in A_k} I_h(u, \phi_u) < 0$. Thus, all the conditions of Theorem 3.3 are satisfied. Therefore, there exists a sequence of non-trivial critical points $(u_n)_{n\in\mathbb{N}}$ for I_h. $\square$

4. Moser iteration method

In this section, we focus our analysis for the case $0 < \alpha < 1$, since the local case $\alpha = 1$ can be treated similarly as [4, 28]. For the reader's convenience, before we prove our regularity result, we introduce some preliminary concepts about the α-harmonic extension (see [10]). We point out that our arguments are local and, for this reason, we are able to apply this technique to transform our nonlocal problem into a local one.

For $0 < \alpha < 1$ we define the space X^α as the completion of $C_0^\infty(\mathbb{R}^4_+)$ with respect to the norm

$$\|w\|_{X^\alpha} = \Big[\frac{1}{\kappa_\alpha}\int_{\mathbb{R}^4_+} y^{1-2\alpha}|\nabla w|^2\,\mathrm{d}x\,\mathrm{d}y\Big]^{1/2},$$

where $\kappa_\alpha = (2^{1-2\alpha}\Gamma(1-\alpha))/\Gamma(\alpha)$ and Γ is the well known gamma function. By [26], the space X^α is well defined and there is a continuous trace operator $\mathrm{Tr} : X^\alpha \to \mathcal{D}^{\alpha,2}(\mathbb{R}^3)$; that is, there exists $C > 0$ such that $\|\mathrm{Tr}(w)\|_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} \leq C\|w\|_{X^\alpha}$, for all $w \in X^\alpha$. When $w \in C(\overline{\mathbb{R}^4_+})$, we have $\mathrm{Tr}(w)(x) = w(x,0)$, and because of that we also use the notation $w(\cdot,0) = \mathrm{Tr}(w)$. It is also worth to call attention that considering the continuous Sobolev embedding $\mathcal{D}^{\alpha,2}(\mathbb{R}^3) \hookrightarrow L^{2^*_\alpha}(\mathbb{R}^3)$, we obtain that $\|w(\cdot,0)\|_{2\alpha^*} \leq C\|w\|_{X^\alpha}$, for all $w \in X^\alpha$.

Given $u \in \mathcal{D}^{\alpha,2}(\mathbb{R}^3)$, we call $w = E_\alpha(u)$ the α-harmonic extension of u, the unique solution of the minimization problem

$$\min\Big\{\frac{1}{\kappa_\alpha}\int_{\mathbb{R}^4_+} y^{1-2\alpha}|\nabla w|^2\,\mathrm{d}x\mathrm{d}y : w \in X^\alpha \text{ and } w(\cdot,0) = u \text{ on } \mathbb{R}^3\Big\}.$$

We have that E_α is a well defined operator acting on $\mathcal{D}^{\alpha,2}(\mathbb{R}^3)$ into X^α. Moreover, by [7, Lemma A.2], E_α is an isometry, precisely $\|E_\alpha(u)\|_{X^\alpha} = \|u\|_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)}$, for all

$u \in \mathcal{D}^{\alpha,2}(\mathbb{R}^3)$. We also have that E_α satisfies

$$\begin{gathered}\operatorname{div}(y^{1-2\alpha}\nabla w) = 0 \quad \text{in } \mathbb{R}^4_+,\\ -\frac{1}{\kappa_\alpha}\lim_{y\to 0^+} y^{1-2\alpha} w_y(x,y) = (-\Delta)^\alpha u(x) \quad \text{in } \mathbb{R}^3,\end{gathered}$$

in the weak sense, more precisely

$$\frac{1}{\kappa_\alpha}\int_{\mathbb{R}^4_+} y^{1-2\alpha}\langle \nabla E_\alpha(u), \nabla\psi\rangle \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^3} (-\Delta)^{\alpha/2} u (-\Delta)^{\alpha/2}\psi(\cdot,0)\,\mathrm{d}x,$$

for all $\psi \in X^\alpha$. Consequently we see that u is a weak solution for (2.5) if, and only if, $w = E_\alpha(u)$ is a weak solution for the problem

$$\begin{gathered}\operatorname{div}(y^{1-2\alpha}\nabla w) = 0 \quad \text{in } \mathbb{R}^4_+,\\ -\frac{1}{\kappa_\alpha}\lim_{y\to 0^+} y^{1-2\alpha} w_y(x,y) = g(x,u(x)) \quad \text{in } \mathbb{R}^3,\end{gathered} \tag{4.1}$$

where $g(x,u) = f_h(x,u) - V(x)u - k(x)\phi_u u$; that is,

$$\frac{1}{\kappa_\alpha}\int_{\mathbb{R}^4_+} y^{1-2\alpha}\langle \nabla E_\alpha(u), \nabla\psi\rangle \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^3} (f_h(x,u) - V(x)u - k(x)\phi_u u)\psi(\cdot,0)\,\mathrm{d}x,$$

for all $\psi \in X^\alpha$. In the following lemma, we show that a sequence of critical points of Problem (4.1) converges to zero in the L^∞-norm. Our proof is based on the Moser iteration method, a delicate estimate which take into account the α-harmonic extension and a suitable interpolation of Lebesgue spaces.

Lemma 4.1. *Let $(u_n)_{n\in\mathbb{N}}$ be a critical point sequence of I_h satisfying $u_n \to 0$ in E, as $n \to +\infty$. Then, $\|u_n\|_{L^\infty(\mathbb{R}^3)} \to 0$ as $n \to +\infty$.*

Proof. We first recall that w is a weak solution to (4.1) if w satisfies the equality

$$\frac{m([w(\cdot,0)]_\alpha^2)}{\kappa_\alpha}\int_{\mathbb{R}^4_+} y^{1-2\alpha}\langle \nabla w, \nabla\psi\rangle\,\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^3} g(x,w(\cdot,0))\psi\,\mathrm{d}x, \tag{4.2}$$

for any $\psi \in X^\alpha$. We set $w = E_\alpha(u_n)$, $u = u_n = w(\cdot,0)$ and $g(x,w(\cdot,0)) = f_h(x,u) - V(x)u - k(x)\phi_u u$. For each $L > 0$ we define $w_L := \min\{w, L\}$ and consider $\psi := w_L^{2\theta} w \in X^\alpha$, where $\theta > 0$ will be chosen later. By using ψ as test function in (4.2) we obtain

$$\begin{aligned}&\frac{m([u]_\alpha^2)}{\kappa_\alpha}\Big[\int_{\mathbb{R}^4_+} y^{1-2\alpha} w_L^{2\theta}|\nabla w|^2\,\mathrm{d}x\mathrm{d}y + \int_{\{w\le L\}} 2\theta y^{1-2\alpha} w_L^{2\theta}|\nabla w|^2\,\mathrm{d}x\mathrm{d}y\Big]\\ &= \int_{\mathbb{R}^3} f_h(x,u)u_L^{2\theta}u\,\mathrm{d}x - \int_{\mathbb{R}^3} V(x)u^2u_L^{2\theta}\,\mathrm{d}x - \int_{\mathbb{R}^3} k(x)\phi_u u^2 u_L^{2\theta}\,\mathrm{d}x.\end{aligned} \tag{4.3}$$

Taking into account (2.6), (4.3) and using Hölder inequality we deduce that

$$\begin{aligned}\frac{m_0}{\kappa_\alpha}\int_{\mathbb{R}^4_+} y^{1-2\alpha} w_L^{2\theta}|\nabla w|^2\,\mathrm{d}x\mathrm{d}y &\le \nu\int_{\mathbb{R}^3}\xi(x)u^\nu u_L^{2\theta}\,\mathrm{d}x\\ &\le \nu\|\xi\|_{L^\mu(\mathbb{R}^3)}\|u^\nu u_L^{2\theta}\|^{2\theta+\nu}_{L^{\frac{\mu}{\mu-1}}(\mathbb{R}^3)}.\end{aligned} \tag{4.4}$$

Let us denote $\overline{w}_L = w w_L^\theta$. Following [3, Lemma 4.1], one has

$$\|\overline{w}_L(\cdot,0)\|^2_{L^{2^*_\alpha}(\mathbb{R}^3)} \le 4S_\alpha(\theta+1)^2\int_{\mathbb{R}^4} y^{1-2\alpha} w_L^{2\theta}|\nabla w|^2\,\mathrm{d}x\mathrm{d}y. \tag{4.5}$$

Using (4.4) and (4.5) we deduce that

$$\|\overline{w}_L(\cdot,0)\|^2_{L^{2^*_\alpha}(\mathbb{R}^3)} \le C(\theta+1)^2\|\xi\|_{L^\mu(\mathbb{R}^3)}\|u^\nu u_L^{2\theta}\|^{2\theta+\nu}_{L^{\frac{\mu}{\mu-1}}(\mathbb{R}^3)}. \tag{4.6}$$

Now, by passing to the limit as $L\to+\infty$ in (4.6), Fatou's Lemma yields

$$\|u\|_{L^{(\theta+1)2^*_\alpha}(\mathbb{R}^3)} \le C^{\frac{1}{(\theta+1)}}(\theta+1)^{\frac{1}{(\theta+1)}}\|u\|^{\frac{2\theta+\nu}{2(\theta+1)}}_{L^{\alpha_*}(\mathbb{R}^3)}, \tag{4.7}$$

where $\alpha_* = \mu(2\theta+\nu)/(\mu-1)$. For each $n\in\mathbb{N}$, define $(\theta_{n-1}+1)2^*_\alpha = \mu(2\theta_n+\nu)/(\mu-1)$. Since $\mu>3/(2\alpha)$, it follows that θ_n is positive, increasing and unbounded. Thus, set

$$\zeta_n = \sum_{i=0}^{n-1}\frac{\ln(c_0(\beta_i+1))}{\beta_i+1} \quad\text{and}\quad \sigma_n = \prod_{i=0}^{n-1}\frac{2\beta_i+\nu}{2\beta_i+2}.$$

Notice that ζ_n and σ_n are convergent sequences (see also [18, Lemma 3.4]) with $\zeta_n\to\zeta>0$ and $\sigma_n\to\sigma\in(0,1)$. We can now iterate (4.7) to obtain

$$\|u\|_{L^{\mu(2\beta_n+\nu)/(\mu-1)}(\mathbb{R}^3)} \le \mathrm{e}^{\zeta_n}\|u\|^{\sigma_n}_{L^{\mu^*}(\mathbb{R}^3)}, \quad \text{for all } n\in\mathbb{N}. \tag{4.8}$$

Letting $n\to+\infty$ in (4.8) follows $\|u\|_{L^\infty(\mathbb{R}^3)} \le \mathrm{e}^{\zeta}\|u\|^{\sigma}_{L^{\mu^*}(\mathbb{R}^3)}$. Therefore, $u_n\to 0$ strongly in $L^\infty(\mathbb{R}^3)$ as $n\to+\infty$, which completes the proof. □

Remark 4.2. Note that for the local case $\alpha=1$, estimate (4.6) can be directly obtained by the continous Sobolev embedding $H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$.

Proof of Theorem 1.1. We now look back to the modified problem (2.4). In Section 3, we applied Theorem 3.3 to guarantee the existence of a sequence $(u_n)_{n\in\mathbb{N}}$ of critical points for the functional I_h. Hence, in view of Lemma 4.1, there exists $n_0\in\mathbb{N}$ such that (u_n,ϕ_{u_n}) is a solution for (1.5), for all $n\ge n_0$, from which the assertions of Theorem 1.1 follows. □

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