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SUPERLINEAR FRACTIONAL BOUNDARY VALUE PROBLEMS WITHOUT THE AMBROSETTI-RABINOWITZ CONDITION

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Abstract. In this article, by means of a direct variational approach and the theory of the fractional differential space, we prove the existence of a nontrivial solution for superlinear fractional boundary value problems without Ambrosetti and Rabinowitz condition.

1. Introduction

Fractional differential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, neurons, electrochemistry, control, porous media, electromagnetism, etc., (see [4, 10, 11, 15, 17, 20]). Recently, the study of various mathematical problems with the existence of solutions of various BVP of fractional differential equations has been received considerable attention, we refer the reader to [1, 2, 5, 6, 8, 9, 12, 19, 24, 25, 26] for an overview of and references on this subject.

In this article we are concerned with the existence of a nontrivial solution for the following Dirichlet problem of fractional order differential equation

$$\begin{gathered}\frac{d}{dt}\Big(\frac{1}{2}{}_0D_t^{-\beta}(u'(t))+\frac{1}{2}{}_0D_T^{-\beta}(u'(t))\Big)+\lambda\nabla F(t,u(t))=0,\quad \text{a.a. } t\in[0,T],\\ u(0)=u(T)=0,\end{gathered}\tag{1.1}$$

where ${}_0D_t^{-\beta}$ and ${}_0D_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0\le\beta<1$, respectively, $\lambda>0$ is a real number, $F:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ is a given function and $\nabla F(t,x)$ is the gradient of F at x.

Such a type of behaviour occurs, for example, when $\lambda=1$, in this case (1.1) becomes

$$\begin{gathered}\frac{d}{dt}\Big(\frac{1}{2}{}_0D_t^{-\beta}(u'(t))+\frac{1}{2}{}_0D_T^{-\beta}(u'(t))\Big)+\nabla F(t,u(t))=0,\quad \text{a.a. } t\in[0,T],\\ u(0)=u(T)=0.\end{gathered}\tag{1.2}$$

There have been many works about the existence of nontrivial solutions to (1.2) by using variational methods. Jiao and Zhou [3] obtained the existence of solutions

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for (1.2) by the Mountain Pass theorem under the Ambrosetti-Rabinowitz (AR) condition. Chen and Tang [13] studied the existence and multiplicity of solutions for the system (1.2) when the nonlinearity $F(t,\cdot)$ are superquadratic, asymptotically quadratic, and subquadratic, respectively. For more recent results, we can refer to [7, 14, 18] and the references therein.

It is well known, the (AR) condition is quite important not only to ensure that the Euler-lagrange functional associated to problem (1.2) has a mountain pass geometry, but also to guarantee that Palais-Smale sequence of the Euler-Lagrange functional is bounded. However, this condition is very restrictive and eliminates many interesting and important nonlinearities.

Motivated by the works described above, we try to get the existence of a nontrivial solution for problem (1.1) without (AR) condition. To state our main result, we assume that $F(t,x)$ satisfies the following general conditions:

(A1) $F(t,0)=0$, $\lim_{|x|\to 0}\frac{F(t,x)}{|x|^2}=0$ uniformly a.e. $t\in[0,T]$.

(A2) There are positive constant positive constants a, b and $p>1$ such that

$$|\nabla F(t,x)|\le a+b|x|^p,\ \text{for a.e. } t\in[0,T],\ \text{all } x\in\mathbb{R}^N.$$

(A3) $\lim_{|x|\to\infty}\frac{F(t,x)}{|x|^2}=+\infty$ uniformly a.e. $t\in[0,T]$.

(A4) There exists a constant $C_*>0$ such that

$$H(t,y)\le H(t,x)+C_*,$$

for any $t\in[0,T]$, $0<|y|<|x|$ or $0<|x|<|y|$, where $H(t,x)=(x,\nabla F(t,x))-2F(t,x)$.

This article is organized as follows. In Sect.2 we introduce the fractional differential space setting that we adopt throughout the paper. In Sect.3 we give the main result and its proof.

2. Preliminary results

In this section, we introduce some basic definitions and properties of the fractional calculus which are used further in this paper. For the proofs, which are omitted, we refer the reader to [3, 7, 13, 16, 21, 22] or other texts on basic fractional calculus.

Definition 2.1 ([16])**.** Let $g(t)$ be a function defined on $[a,b]$ and $\mu>0$. The left and right Riemann-Liouville fractional integrals of order μ for function $g(t)$ denoted by ${}_aD_t^{-\mu}g(t)$ and ${}_tD_b^{-\mu}g(t)$, respectively, are defined by

$$\begin{aligned}{}_aD_t^{-\mu}g(t)&=\frac{1}{\Gamma(\mu)}\int_a^t(t-s)^{\mu-1}g(s)ds,\quad t\in[a,b],\\ {}_tD_b^{-\mu}g(t)&=\frac{1}{\Gamma(\mu)}\int_t^b(t-s)^{\mu-1}g(s)ds,\quad t\in[a,b],\end{aligned}$$

where $\Gamma(\mu)=\int_0^\infty t^{\mu-1}e^{-t}dt$.

Definition 2.2 ([16])**.** Let $g(t)$ be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order μ for function $g(t)$ denoted by ${}_aD_t^{\mu}g(t)$ and ${}_tD_b^{\mu}g(t)$, respectively, are defined by

$${}_aD_t^{\mu}g(t)=\frac{d^n}{dt^n}\,{}_aD_t^{\mu-n}g(t)=\frac{1}{\Gamma(n-\mu)}\frac{d^n}{dt^n}\Big(\int_a^t(t-s)^{n-\mu-1}g(s)ds\Big),$$

$$ {}_tD_b^{\mu}g(t) = (-1)^n \frac{d^n}{dt^n}\, {}_tD_b^{\mu-n}g(t) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dt^n} \Big(\int_t^b (t-s)^{n-\mu-1} g(s)ds \Big), $$

where $t \in [a,b]$, $n-1 \le \mu < n$, and $n \in \mathbb{N}$.

The left and right Caputo fractional derivatives are defined via the Riemann-Liouville fractional derivatives. In particular, they are defined for the function belonging to the space of absolutely continuous functions, which we denote by $AC([a,b],\mathbb{R}^N)$. $AC^k([a,b],\mathbb{R}^N)$ $(k=1,2,\dots)$ is the space of functions g such that $g \in C^k([a,b],\mathbb{R}^N)$. In particular, $AC([a,b],\mathbb{R}^N) = AC^1([a,b],\mathbb{R}^N)$.

Definition 2.3 ([16])**.** Let $\mu \ge 0$ and $n \in \mathbb{N}$. If $\mu \in [n-1,n)$ and $g(t) \in AC^n([a,b],\mathbb{R}^N)$, then the left and right Caputo fractional derivatives of order μ for function $g(t)$ denoted by ${}_a^cD_t^{\mu}g(t)$ and ${}_t^cD_b^{\mu}g(t)$, respectively, exist almost everywhere on $[a,b]$. ${}_a^cD_t^{\mu}g(t)$ and ${}_t^cD_b^{\mu}g(t)$ are represented by

$$ {}_a^cD_t^{\mu}g(t) = {}_a D_t^{\mu-n}g^{(n)}(t) = \frac{1}{\Gamma(n-\mu)} \Big(\int_a^t (t-s)^{n-\mu-1} g^{(n)}(s)ds \Big), $$

$$ {}_t^cD_b^{\mu}g(t) = (-1)^n_t D_b^{\mu-n}g^{(n)}(t) = \frac{1}{\Gamma(n-\mu)} \Big(\int_t^b (t-s)^{n-\mu-1} g^{(n)}(s)ds \Big), $$

respectively, where $t \in [a,b]$.

Definition 2.4 ([13])**.** Define $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^{\infty}([0,T],\mathbb{R}^N)$ with respect to the norm

$$ \|u\|_{\alpha,p} = \Big(\int_0^T |u(t)|^p dt + \int_0^T |{}_0^cD_t^{\alpha}u(t)|^p dt \Big)^{1/p}, \quad \forall u \in E_0^{\alpha,p}, $$

where $C_0^{\infty}([0,T],\mathbb{R}^N)$ denotes the set of all functions $u \in C^{\infty}([0,T],\mathbb{R}^N)$ with $u(0)=u(T)=0$. It is obvious that the fractional derivative space $E_0^{\alpha,p}$ is the space of functions $u \in L^p([0,T],\mathbb{R}^N)$ having an α-order Caputo fractional derivative ${}_0^cD_t^{\alpha}u \in L^p([0,T],\mathbb{R}^N)$ and $u(0)=u(T)=0$.

Proposition 2.5 ([13])**.** *Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable space.*

Throughout this paper, we denote the norm of u in $E_0^{\alpha,p}([0,T])$ and $L^p([0,T])$, $1 < p \le \infty$, respectively, by

$$ \|u\|_{\alpha,p} = \Big(\int_0^T |{}_0^cD_t^{\alpha}u|^p dt \Big)^{1/p}, \quad \|u\|_p = \Big(\int_0^T |u|^p dt \Big)^{1/p}, \ \|u\|_{\infty} = \max_{t \in [0,T]} |u(t)|. $$

Proposition 2.6 ([13])**.** *Let $0 < \alpha \le 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, one has*

$$ \|u\|_p \le \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|{}_0^cD_t^{\alpha}u\|_p. \tag{2.1} $$

Moreover, if $\alpha > 1/p$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ \|u\|_{\infty} \le \frac{T^{\frac{\alpha-1}{p}}}{\Gamma(\alpha)\Big((\alpha+1)q+1\Big)^{\frac{1}{q}}} \|{}_0^cD_t^{\alpha}u\|_p. \tag{2.2} $$

According to (2.1), one can consider $E_0^{\alpha,p}$ with respect to the norm

$$ \|u\|_{\alpha,p} = \|{}_0^cD_t^{\alpha}u\|_p = \Big(\int_0^T |{}_0^cD_t^{\alpha}u|^p dt \Big)^{1/p}. \tag{2.3} $$

Proposition 2.7 ([13]). *Define $0 < \alpha \le 1$ and $1 < p < \infty$. Assume that $\alpha > \frac{1}{p}$, and the sequence u_k converges weakly to $u \in E_0^{\alpha,p}$, that is, $u_k \rightharpoonup u$. Then $u_k \to u$ in $C([0,T],\mathbb{R}^N)$, that is, $\|u_k - u\|_\infty \to 0$, as $k \to \infty$.*

By Definition 2.3, for any $u \in AC([0,T],\mathbb{R}^N)$, problem (1.1) is equivalent to the problem

$$\begin{gathered}\frac{d}{dt}\Big(\frac{1}{2}{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \frac{1}{2}{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\Big) + \lambda\nabla F(t,u(t)) = 0, \\ \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0,\end{gathered} \tag{2.4}$$

where $\alpha = 1 - \frac{\beta}{2} \in (\frac{1}{2}, 1]$.

In the following, we will treat problem (2.4) in the Hilbert space $E^\alpha = E_0^{\alpha,2}$ and corresponding norm $\|u\|_\alpha = \|u\|_{\alpha,2}$. It follows from that the functional $\phi : E^\alpha \to \mathbb{R}$ given by

$$\phi_\lambda(u) = -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))dt - \lambda\int_0^T F(t,u(t))dt, \quad u \in E^\alpha, \tag{2.5}$$

is continuously differentiable on E^α. Moreover, for $u, v \in E^\alpha$, we have

$$\begin{aligned}\langle \phi_\lambda'(u), v\rangle = &-\frac{1}{2}\int_0^T [({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha v(t)) + ({}_t^cD_T^\alpha u(t), {}_0^cD_t^\alpha v(t))]dt \\ &- \lambda\int_0^T (\nabla F(t,u(t)), v(t))dt.\end{aligned} \tag{2.6}$$

Proposition 2.8 ([13]). *A function $u \in AC([0,T],\mathbb{R}^N)$ is a solution of (2.4) if*

(i) *$D^\alpha(u(t))$ is derivative for almost every $t \in [0,T]$,*
(ii) *u satisfies (2.4), where $D^\alpha(u(t)) = \frac{1}{2}{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \frac{1}{2}{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))$.*

Proposition 2.9 ([13]). *If $\frac{1}{2} < \alpha \le 1$ then for any $u \in E^\alpha$, one has*

$$|\cos(\pi\alpha)|\|u\|_\alpha^2 \le -\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))dt \le \frac{1}{|\cos(\pi\alpha)|}\|u\|_\alpha^2. \tag{2.7}$$

Proposition 2.10 ([13]). *Let $\frac{1}{2} < \alpha \le 1$ and ϕ_λ be defined by (2.5). If $u \in E^\alpha$ is a solution of $\phi_\lambda'(u) = 0$, then u is a solution of problem (2.4) which satisfies the problem (1.1).*

3. Main result and its proof

Lemma 3.1. *Suppose that (A1)–(A4) holds. Then we have the following assertions:*

(a) *ϕ_λ is unbounded from below on E^α;*
(b) *$u = 0$ is a strict local minimum for the functional ϕ_λ.*

Proof. By (A3), for any $M > 0$, there exists $K_0 > 0$, such that

$$F(t,x) \ge M|x|^2, \quad \text{for all } |x| \ge K_0, \text{ a.e. } t \in [0,T]. \tag{3.1}$$

On the other hand, by the mean value theorem and (A2), we obtain

$$\begin{aligned} |F(t,x)| =& |(\nabla F(t,\lambda_0 x),x)| \\ \le& |\nabla F(t,\lambda_0 x)|\cdot|x| \\ \le& (a+b\lambda_0^p|x|^p)\cdot|x| \\ \le& (a+b|x|^p)\cdot|x| \\ \le& aK_0+bK_0^{p+1}=:C_M, \end{aligned} \tag{3.2}$$

for for some $\lambda_0\in(0,1), |x|\le K_0$ and a.e. $t\in[0,T]$.

Hence, for any $M>0$, there exists $C_M>0$, such that

$$F(t,x)\ge M_0|x|^2-C_{M_0},\quad \text{a.e. } t\in[0,T],\ \text{all } x\in\mathbb{R}^N, \tag{3.3}$$

where $M_0=M+\frac{C_M}{K_0^2}$.

Choosing $u_0=(0,\dots,0,\frac{T}{\pi}\sin(\frac{\pi t}{T}))\in E^\alpha$, then

$$\|u_0\|_2^2=\int_0^T|u_0|^2dt=\frac{T^3}{2\pi^2}\quad\text{and}\quad \|u_0\|_\alpha^2\le\frac{T^{3-2\alpha}}{\Gamma^2(2-\alpha)(3-2\alpha)}. \tag{3.4}$$

For $\eta>0$, and noting that (3.3) and (3.4), we have

$$\begin{aligned} \phi_\lambda(\eta u_0) =& -\frac{1}{2}\int_0^T({}_0^cD_t^\alpha\eta u_0(t),{}_t^cD_T^\alpha\eta u_0(t))dt-\lambda\int_0^T F(t,\eta u_0(t))dt \\ \le& \frac{\eta^2}{2|\cos(\pi\alpha)|}\|u_0\|_\alpha^2-\lambda\int_0^T(M_0\eta^2|u_0|^2-C_{M_0})dt \\ =& \frac{\eta^2}{2|\cos(\pi\alpha)|}\|u_0\|_\alpha^2-\lambda M_0\eta^2\int_0^T|u_0|^2dt+\lambda C_{M_0}T \\ \le& \frac{\eta^2}{2|\cos(\pi\alpha)|}\cdot\frac{T^{3-2\alpha}}{\Gamma^2(2-\alpha)(3-2\alpha)}-\lambda M_0\eta^2\cdot\frac{T^3}{2\pi^2}+\lambda C_{M_0}T \\ =& \Big(\frac{1}{2|\cos(\pi\alpha)|}\cdot\frac{T^{3-2\alpha}}{\Gamma^2(2-\alpha)(3-2\alpha)}-\lambda M_0\frac{T^3}{2\pi^2}\Big)\eta^2+\lambda C_{M_0}T. \end{aligned} \tag{3.5}$$

If M_0 is large enough so that

$$\frac{1}{2|\cos(\pi\alpha)|}\frac{T^{3-2\alpha}}{\Gamma^2(2-\alpha)(3-2\alpha)}-\lambda M_0\frac{T^3}{2\pi^2}<0, \tag{3.6}$$

then

$$\phi_\lambda(\eta u_0)\to-\infty,\quad \text{as } \eta\to+\infty. \tag{3.7}$$

This proves (a).

By (A1), for any $\varepsilon>0$, there exists $\delta(\varepsilon)>0$, such that

$$|F(t,x)|<|x|^2\varepsilon,\quad |x|<\delta. \tag{3.8}$$

Analogously, by the mean value theorem and (A2), we have

$$\begin{aligned} |F(t,x)| \le& (a+b|x|^p)|x| \\ =& |x|^{p+1}(a\cdot\frac{1}{|x|^p}+b) \\ \le& |x|^{p+1}(\frac{a}{\delta^p}+b) \\ =& C_\varepsilon|x|^{p+1}, \end{aligned} \tag{3.9}$$

where $|x| \geq \delta, t \in [0,T], 1 < p < \infty$. Hence, for almost all $t \in [0,T]$ and all $x \in \mathbb{R}^N$, we have

$$F(t,x) \leq |x|^2\varepsilon + C_\varepsilon |x|^{p+1}. \tag{3.10}$$

Then

$$\begin{aligned} \phi_\lambda(u) &= -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))dt - \lambda \int_0^T F(t,u(t))dt \\ &\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \lambda \int_0^T (|u|^2\varepsilon + C_\varepsilon |u|^{p+1})dt \\ &= \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \lambda\varepsilon\|u\|_2^2 - \lambda C_\varepsilon \|u\|_{p+1}^{p+1}. \end{aligned} \tag{3.11}$$

By Proposition 2.6, we have

$$\|u\|_2 \leq \frac{T^\alpha}{\Gamma(\alpha+1)}\|{}_0^cD_t^\alpha u\|_2 = \frac{T^\alpha}{\Gamma(\alpha+1)}\|u\|_\alpha. \tag{3.12}$$

Hence,

$$\phi_\lambda(u) \geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \lambda\varepsilon(\frac{T^\alpha}{\Gamma(\alpha+1)})^2\|u\|_\alpha^2 - C_\varepsilon\|u\|_{p+1}^{p+1}. \tag{3.13}$$

Since embedding $E^\alpha \hookrightarrow C[0,T]$ is continous, then there exists a constant $c > 0$ such that

$$\|u\|_{p+1} \leq c\|u\|_\alpha, \quad \forall u \in E^\alpha,$$

which implies

$$\begin{aligned} \phi_\lambda(u) &\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \frac{\lambda\varepsilon T^{2\alpha}}{\Gamma^2(\alpha+1)}\|u\|_\alpha^2 - c_0\|u\|_\alpha^{p+1} \\ &= \Big(\frac{|\cos(\pi\alpha)|}{2} - \frac{\lambda\varepsilon T^{2\alpha}}{\Gamma^2(\alpha+1)}\Big)\|u\|_\alpha^2 - c_0\|u\|_\alpha^{p+1}. \end{aligned} \tag{3.14}$$

For a given λ, choose that $\varepsilon = \varepsilon(\lambda)$ satisfies $\varepsilon < |\cos(\pi\alpha)|\Gamma^2(\alpha+1)/(2\lambda T^{2\alpha})$, then we have

$$\phi_\lambda(u) > 0, \quad \|u\|_\alpha < \rho \tag{3.15}$$

for some $\rho \in (0,1)$. So $u = 0$ is a strict local minimum for ϕ_λ. □

Lemma 3.2. *Assume* (A1)–(A3) *hold and* $0 < \lambda_0 < \mu_0, \lambda_0 \leq \lambda \leq \mu_0$, $c_\lambda = \inf_{\gamma\in P}\max_{z\in[0,1]}\phi_\lambda(\gamma(z))$, *then* c_λ/λ *is monotone decreasing and is left semi-continuous.*

Proof. By choosing $\varepsilon > 0$, such that

$$\Big(\frac{|\cos(\pi\alpha)|}{2} - \frac{\lambda\varepsilon T^{2\alpha}}{\Gamma^2(\alpha+1)}\Big) \geq \frac{|\cos(\pi\alpha)|}{4}.$$

By (3.14), we have

$$\phi_\lambda(u) \geq \frac{|\cos(\pi\alpha)|}{4}\|u\|_\alpha^2 - c_0\|u\|_\alpha^{p+1}, \quad \forall u \in E^\alpha,\ 0 < \lambda_0 < \mu_0,\ c_0 > 0. \tag{3.16}$$

That is, there exist $\rho > 0$ and $R > 0$, such that

$$\phi_\lambda(u) \geq R, \quad \|u\|_\alpha = \rho, \quad \forall \lambda \leq \mu_0. \tag{3.17}$$

By choosing $e \in E^\alpha$ such that $\phi_{\lambda_0}(e) < 0$, we infer that

$$\frac{\phi_\lambda(e)}{\lambda} < \frac{\phi_{\lambda_0}(e)}{\lambda_0} < 0, \quad \forall \lambda \in [\lambda_0, \mu_0].$$

The same can be done to obtain

$$\frac{\phi_\lambda(u)}{\lambda} \le \frac{\phi_\mu(u)}{\mu}, \quad \forall u \in E^\alpha,\ \mu < \lambda. \tag{3.18}$$

Define

$$P = \big\{\gamma : [0,1] \to E^\alpha : \gamma \text{ is continuous and } \gamma(0) = 0,\ \gamma(1) = e\big\}.$$

According to the definition of c_λ, we have $c_{\mu_0} > 0$. Thus map $c : [\lambda_0, \mu_0] \to \mathbb{R}_+$ is defined as $c(\lambda) = c_\lambda$. In fact, the formula (3.18) contains $\frac{c_\lambda}{\lambda}$ is monotone decreasing, the formula (3.17) contains $c_\lambda \ge R > 0$ and c_λ is bounded from below.

Now we prove left semi-continuity of c_λ/λ. Fix $\mu \in [\lambda_0, \mu_0]$ and $\varepsilon > 0$. Then exists $\gamma \in P$, such that

$$c_\mu \le \max_{z\in[0,1]} \phi_\mu(\gamma(z)) \le c_\mu + \frac{\varepsilon\mu}{8}. \tag{3.19}$$

Let $r_0 = \max_{z\in[0,1]} \big|\int_0^T F(t, \gamma(z))dt\big|$. Then, for $\mu < 2\lambda$ and $\frac{1}{\lambda} < \frac{1}{\mu} + \frac{\varepsilon}{2c_\mu}$, we can obtain

$$\begin{aligned} \phi_\lambda(\gamma(z)) &= \phi_\mu(\gamma(z)) + (\phi_\lambda(\gamma(z)) - \phi_\mu(\gamma(z))) \\ &= \phi_\mu(\gamma(z)) + (\mu - \lambda) \int_0^T F(t, \gamma(z))dt \\ &\le c_\mu + \frac{\varepsilon\mu}{8} + r_0|\mu - \lambda|. \end{aligned} \tag{3.20}$$

If $|\mu - \lambda| < \frac{\varepsilon\mu}{8r_0}$, then $c_\lambda \le c_\mu + \frac{\varepsilon\mu}{4}$. Hence, if $\lambda < \mu$, then

$$\begin{aligned} \frac{c_\mu}{\mu} - \varepsilon < \frac{c_\mu}{\mu} < \frac{c_\lambda}{\lambda} &\le \frac{c_\mu + \frac{\varepsilon\mu}{4}}{\lambda} = \frac{c_\mu}{\lambda} + \frac{\varepsilon\mu}{4\lambda} \\ &\le \frac{c_\mu}{\lambda} + \frac{\varepsilon\mu}{4}\frac{2}{\mu} \le c_\mu(\frac{1}{\mu} + \frac{\varepsilon}{2c_\mu}) + \frac{\varepsilon}{2} = \frac{c_\mu}{\mu} + \varepsilon. \end{aligned} \tag{3.21}$$

Hence, c_λ/λ and c_λ are left semi-continuous. □

Remark 3.3. We recall that the map $b : [\lambda_0, \mu_0] \to \mathbb{R}_+$, given by $b(\lambda) = \frac{c_\lambda}{\lambda}$, is monotone decreasing. Thus, b_λ and c_λ are differentiable at almost all values $\lambda \in (\lambda_0, \mu_0)$.

Lemma 3.4. *There exists $C > 0$, such that*

$$\|\phi'_\mu(u) - \phi'_\lambda(u)\|_{(E^\alpha)^*} \le C(1 + \|u\|_\alpha^p)|\mu - \lambda|, \quad \forall \lambda,\ \mu > 0. \tag{3.22}$$

Proof. By (A2), we have

$$|\nabla F(t, u)| \le a + b|u|^p. \tag{3.23}$$

For all $v \in E^\alpha$ with $\|v\|_\alpha \le 1$, we have

$$\begin{aligned} |\langle\phi'_\mu(u) - \phi'_\lambda(u), v\rangle| &= |\lambda - \mu|| \int_0^T (\nabla F(t, u(t)), v)dt| \\ &\le |\lambda - \mu|| \int_0^T |\nabla F(t, u(t))||v|dt \\ &\le |\lambda - \mu|| \int_0^T |(a + b|u|^p)|v|dtr \\ &\le |\lambda - \mu|| \int_0^T |(a + b|u|^p) \max_{t\in[o,T]} |v|dt. \end{aligned} \tag{3.24}$$

Since the embedding $E^\alpha \hookrightarrow C[0,T]$ is continuous, there exists a constant $c_1 > 0$ such that

$$\|v\|_\infty \le c_1\|v\|_\alpha,\ \|v\|_p \le c_1\|v\|_\alpha,\ \ \forall\, v \in E^\alpha,$$

which implies that

$$\begin{aligned} |\langle \phi'_\mu(u) - \phi'_\lambda(u), v\rangle| &\le |\lambda-\mu|\|v\|_\infty(aT + b\|u\|_p^p)\\ &\le |\lambda-\mu|c_1\|v\|_\alpha(aT + bc_1\|u\|_\alpha^p)\\ &\le |\lambda-\mu|c_1\|v\|_\alpha(aT + bc_1\|u\|_\alpha^p). \end{aligned} \tag{3.25}$$

So that there exists $C > 0$, such that

$$\|\phi'_\mu(u) - \phi'_\lambda(u)\|_{(E^\alpha)^*} \le C(1 + \|u\|_\alpha^p)|\mu-\lambda|, \quad \forall\lambda,\ \mu > 0.$$

□

Lemma 3.5. *Assume that map $c : [\lambda_0, \mu_0] \to \mathbb{R}_+$, satisfies $c(\lambda) = c_\lambda$ and $c(\lambda)$ is differentiable at point μ, then there exists a sequence $\{u_n\} \in E^\alpha$, such that*

$$\phi_\mu(u_n) \to c_\mu, \quad \phi'_\mu(u_n) \to 0, \quad -\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))dt \le C_2,$$

as $n \to \infty$, where $C_2 = 2c_\mu + 2\mu(2 - c'(\mu)) + 1$.

Proof. Assume, by contradiction, that the lemma is false. Then there exists $\delta > 0$, such that

$$\|\phi'_\mu(u)\| \ge 2\delta,\ \ \forall\, u \in \mathbb{N}_\delta,$$

where $N_\delta = \{u \in E^\alpha : -\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))dt \le C_2, |\phi_\mu(u) - c_\mu| < \delta\}$.

There exists constant C_3, such that

$$\begin{aligned} \Big|\int_0^T F(t,u)dt\Big| &= \frac{1}{2\mu}|2\phi_\mu(u) - \int_0^T -({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))dt|\\ &\le C_3,\ \ \forall\, u \in \mathbb{N}_\delta. \end{aligned} \tag{3.26}$$

Let $V : N_\delta \to E^\alpha$ be the pseudo-gradient vector field for ϕ_μ in N_δ, that is, V is locally Lipschitz, $\|V\| \le 1$ and

$$\phi'_\mu(u)\cdot(V(u)) \le -\delta, \quad \forall u \in \mathbb{N}_\delta \tag{3.27}$$

(see [23]). Now, fix a sequence $\{\lambda_n\} \subseteq (\lambda_0, \mu_0)$ such that $\mu \le \lambda_{n+1} < \lambda_n$, $\lambda_n \to \mu$, $|\lambda_n - \mu| \le \frac{\delta}{4}$, $|c_\mu - c_{\lambda_n}| \le \frac{\delta}{4}$, for each n, let $\gamma_n \in P$ be such that

$$\max_{z\in[0,1]} \phi_\mu(\gamma_n(z)) \le c_\mu + (\lambda_n - \mu). \tag{3.28}$$

Consider the set

$$A_n = \{z \in [0,1] : \phi_{\lambda_n}(\gamma_n(z)) > c_{\lambda_n} - (\lambda_n - \mu)\}.$$

By definition of c_{λ_n}, A_n is nonempty. If $v \in \gamma_n(A_n)$, we have

$$\begin{aligned} \int_0^T F(t,v)dt &= \frac{\phi_\mu(v) - \phi_{\lambda_n}(v)}{\lambda_n - \mu}\\ &\le \frac{c_\mu + (\lambda_n-\mu) - c_{\lambda_n} + (\lambda_n - \mu)}{\lambda_n - \mu}\\ &= \frac{c_\mu - c_{\lambda_n}}{\lambda_n - \mu} + 2. \end{aligned} \tag{3.29}$$

Since $c(\mu)$ is differentiable, we have

$$\begin{aligned} c_{\lambda_n} =& c_\mu + c'(\mu)(\lambda_n-\mu) + o((\lambda_n-\mu)^2) \\ =& c_\mu + c'(\mu)(\lambda_n-\mu) + o_n(1)(\lambda_n-\mu); \end{aligned} \tag{3.30}$$

that is, $c_\mu - c_{\lambda_n} = [c'_\mu + o_n(1)](\mu-\lambda_n)$. So $\int_0^T F(t,v)dt = -c'_\mu + 2 + o_n(1)$.

Since $\int_0^T -({}_0^cD_t^\alpha v(t), {}_t^c D_T^\alpha v(t))dt = 2\phi_\mu(v) + 2\mu\int_0^T F(t,v(t))dt$, we have

$$\begin{aligned} &\int_0^T -({}_0^cD_t^\alpha v(t), {}_t^c D_T^\alpha v(t))dt \\ &\le 2c_\mu + 2(\lambda_n-\mu) + 2\mu(-c'(\mu)+2+o_n(1)) \le C_2, \end{aligned} \tag{3.31}$$

for n large.

It is easy to see that inequality (3.26) is satisfied for $v \in \gamma_n(A_n)$. Thus $\gamma_n(A_n) \subset N_\delta$, since

$$\begin{gathered} c_{\lambda_n} - (\lambda_n-\mu) \le \phi_{\lambda_n}(v), \quad \phi_\mu(v) \le c_\mu + (\lambda_n-\mu), \\ |\phi_{\lambda_n}(v) - \phi_\mu(v)| = |\lambda_n-\mu|\,|\int_0^T F(t,v)dt| \le c_3|\lambda_n-\mu|. \end{gathered} \tag{3.32}$$

So $v \in \mathbb{N}_\delta$, that is $\gamma_n(A_n) \subset N_\delta$.

Using (3.27) we have $\phi'_{\lambda_n}\cdot(V(u)) < -\frac{\delta}{2}$, for all $u \in \mathbb{N}_\delta$. Now consider a Lipschitz continuous cut-off function η such that $0 \le \eta \le 1$, $\eta(u) = 0$ in $u \notin \mathbb{N}_\delta$, and $\eta(u) = 1$ for $u \in \mathbb{N}_{\frac{\delta}{2}}$.

Let ϕ be the flow generated by ηV, that is,

$$\begin{gathered} \frac{\partial\phi}{\partial r}(u,r) = \eta(\phi(u,r))V(\phi(u,r)), \quad r \ge 0, \\ \phi(u,0) = u. \end{gathered} \tag{3.33}$$

Integrating the both sides of the equation, we have

$$\int_0^r \frac{\partial\phi(u,t)}{\partial t}dt = \int_0^r \eta(\phi(u,t))V(\phi(u,t))dt,$$

That is $\phi(u,t) = u + \int_0^r \eta(\phi(u,t))V(\phi(u,t))dt$ is the solution of (3.33). Applying the ODE uniqueness we have:

If $u \notin \mathbb{N}_\delta$, then $\phi(u,r) = u$ for all $r \ge 0$.

If $u \in \mathbb{N}_\delta$, then $\phi(u,r) \in \mathbb{N}_\delta$ for all $r \ge 0$,

If $u \in E^\alpha$, then $\frac{\partial\phi}{\partial r}(u,r) = V(\phi(u,r))$ and

$$\phi_{\lambda_n}(\phi(u,r))\frac{\partial\phi}{\partial r}(u,r) < -\frac{\delta}{2} \le 0, \ \forall r \ge 0,$$

If $\phi(u,r) \in \mathbb{N}_{\frac{\delta}{2}}$, for all $r \in [0,r_0]$, then $\phi_{\lambda_n}(\phi(u,r)) \le \phi_{\lambda_n}(u) - \frac{\delta r_0}{2}$.

It is easy to see that if $u \in \mathbb{N}_{\frac{\delta}{2}}$, then $\phi_{\lambda_n}(\phi(u,1)) \le \phi_{\lambda_n}(u) - \frac{\delta}{2}$.

Since $e \notin \mathbb{N}_\delta$, we have $\phi(e,r) = e$ and $\phi(0,r) = 0$, for all $r \ge 0$, and then $\phi(\gamma,r) \in P$, for all $r \in \mathbb{R}$ and $\gamma \in P$. This implies that $h_n(z) = \phi(\gamma_n(z),1)$ is continuous path in P such that

$$\phi_{\lambda_n}(h_n(z)) \le \phi_{\lambda_n}(\gamma_n(z)),$$

and then for its maximum point $s_n \in [0,1]$, we have $s_n \in A_n$ and

$$c_\mu - o_n(1) = c_{\lambda_n} \le \max_{z\in[0,1]} \phi_{\lambda_n}(h_n(z)) = \phi_{\lambda_n}(h_n(s_n)) \le \phi_{\lambda_n}(\gamma_n(s_n)) - \frac{\delta}{2}.$$

That is,

$$\phi_{\lambda_n}(\gamma_n(s_n)) \geq c_\mu - o_n(1) + \frac{\delta}{2}. \tag{3.34}$$

On the other hand, by (3.28) and (3.32), we obtain

$$\phi_{\lambda_n}(\gamma_n(s_n)) \leq \phi_\mu(\gamma_n(s_n)) + C_3|\lambda_n - \mu| \leq c_\mu + (1 + C_3)|\lambda_n - \mu|. \tag{3.35}$$

According to (3.34) and (3.35), we have

$$c_\mu + \frac{\delta}{2} \leq \lim_{n\to\infty} \phi_{\lambda_n}(\gamma_n(s_n)) \leq c_\mu,$$

which is a contradiction. So the original conclusion is true. □

From Lemma 3.5 we can obtain the following statement.

Lemma 3.6. *For almost all $\lambda > 0$, c_λ is a critical value for ϕ_λ.*

Theorem 3.7. *Suppose that* (A1)–(A4) *holds, then for any $\lambda > 0$, problem* (1.1) *has a nontrivial solution.*

Proof. For c_λ is left semi-continuity, applying Lemma 3.6, for any $\mu > 0$, there exists a sequences $\{u_n\} \subseteq E^\alpha$ and $\lambda_n \subseteq \mathbb{R}$, such that

$$\begin{gathered} \lambda_n \to \mu, \quad c_{\lambda_n} \to c_\mu, \quad n \to \infty. \\ \phi_{\lambda_n}(u_n) = c_{\lambda_n}, \quad \phi'_{\lambda_n}(u_n) = 0. \end{gathered} \tag{3.36}$$

We only need to show that $\{u_n\}$ is bounded in E^α. If $\{u_n\}$ is unbounded, we may assume, without loss of generality, that $\|u_n\|_\alpha \to \infty$ as $n \to \infty$. Let $\omega_n = \frac{u_n}{\|u_n\|_\alpha}$, then $\omega_n \in E^\alpha$ with $\|\omega_n\|_\alpha = 1$. Then there are $\omega \in E^\alpha$ and $h \in L^{p+1}([0,T])$ such that $\omega_n \rightharpoonup \omega$ in E^α, $\omega_n \to \omega$ in $C([0,T],\mathbb{R}_+)$ and $L^{p+1}([0,T])$, $\omega_n(t) \to \omega(t)$, a.e. $t \in [0,T]$, $n \to \infty$, $|\omega_n(t)| \leq h(t)$, a.e. $t \in [0,T]$, for all $n \in \mathbb{N}$.

Let $\Omega_0 = \{t \in [0,T] : \omega(t) \neq 0\}$. If $t \in \Omega_0$, then by (A3),

$$\lim_{n\to+\infty} \frac{F(t,u_n(t))}{u_n^2(t)} \omega_n^2(t) = +\infty \tag{3.37}$$

and

$$\begin{aligned} &\lim_{n\to+\infty} \frac{\int_0^T F(t,u_n(t))dt}{\int_0^T -({}_0^cD_t^\alpha u_n(t), {}_t^cD_T^\alpha u_n(t))dt} \\ &= \lim_{n\to+\infty} \Big(\frac{1}{2\lambda_n} - \frac{\phi_{\lambda_n(u_n)}}{\lambda_n \int_0^T -({}_0^cD_t^\alpha u_n(t), {}_t^cD_T^\alpha u_n(t))dt}\Big) = \frac{1}{2\mu}. \end{aligned} \tag{3.38}$$

By Proposition 2.9 and Fatou's lemma

$$\begin{aligned} +\infty &= \int_0^T \lim_{n\to\infty} |\cos(\pi\alpha)| \frac{F(t,u_n(t))}{u_n^2(t)} \cdot \omega_n^2(t)dt \\ &\leq \lim_{n\to\infty} \int_0^T |\cos(\pi\alpha)| \frac{F(t,u_n(t))}{u_n^2(t)} \cdot \omega_n^2(t)dt \\ &\leq \lim_{n\to\infty} \int_0^T \frac{F(t,u_n(t))}{\frac{1}{|\cos(\pi\alpha)|}\|u_n(t)\|_\alpha^2}dt \\ &\leq \lim_{n\to+\infty} \frac{\int_0^T F(t,u_n(t))dt}{\int_0^T -({}_0^cD_t^\alpha u_n(t), {}_t^cD_T^\alpha u_n(t))dt} = \frac{1}{2\mu}. \end{aligned} \tag{3.39}$$

This is a contradiction. This shows that Ω_0 has zero measure. Hence $\omega = 0$ a.e. $t \in [0, T]$.

Let $z_n \in [0, 1]$, such that $\phi_{\lambda_n}(z_n u_n) = \max_{z\in[0,1]} \phi_{\lambda_n}(zu_n)$. By (2.5) and (2.6), we have

$$2\phi_{\lambda_n}(z_n u_n) = -\int_0^T ({}_0^cD_t^\alpha z_n u_n, {}_t^cD_T^\alpha z_n u_n)dt - 2\lambda_n \int_0^T F(t, z_n u_n)dt \tag{3.40}$$

and

$$\begin{aligned} \phi'_{\lambda_n}(z_n u_n)(z_n u_n) = &-\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha z_n u_n(t), {}_t^cD_T^\alpha z_n u_n(t))dt \\ &-\frac{1}{2}\int_0^T ({}_t^cD_T^\alpha z_n u_n(t), {}_0^cD_t^\alpha z_n u_n(t))dt \\ &-\lambda_n \int_0^T (\nabla F(t, z_n u_n(t)), z_n u_n(t))dt. \end{aligned} \tag{3.41}$$

From $\phi'_{\lambda_n}(z_n u_n)(z_n u_n) = 0$, we have

$$\begin{aligned} 2\phi_{\lambda_n}(zu_n) \le& 2\phi_{\lambda_n}(z_n u_n) - \phi'_{\lambda_n}(z_n u_n)(z_n u_n) \\ =&\lambda_n \int_0^T (\nabla F(t, z_n u_n(t)), z_n u_n(t)) - 2F(t, z_n u_n))dt. \end{aligned} \tag{3.42}$$

By assumption (A4), it follows that

$$H(t, z_n u_n) = (z_n u_n, \nabla F(t, z_n u_n)) - 2F(t, z_n u_n), \tag{3.43}$$

and $|z_n u_n| \le |u_n|$. Therefore,

$$H(t, z_n u_n) \le (u_n, \nabla F(t, u_n)) - 2F(t, u_n) + C_*. \tag{3.44}$$

By (3.42) and (3.44), we obtain

$$2\phi_{\lambda_n}(zu_n) \le \lambda_n \int_0^T [(u_n, \nabla F(t, u_n)) - 2F(t, u_n) + C_*]dt. \tag{3.45}$$

Since $\phi_{\lambda_n}(u_n) = c_{\lambda_n}$, $\phi'_{\lambda_n}(u_n) = 0$, it follows that

$$\begin{gathered} 2c_{\lambda_n} = -\int_0^T ({}_0^cD_t^\alpha u_n(t), {}_t^cD_T^\alpha u_n(t))dt - 2\lambda_n \int_0^T F(t, u_n(t))dt, \\ \phi'_{\lambda_n}(u_n)(u_n) = -\int_0^T ({}_0^cD_t^\alpha u_n(t), {}_t^cD_T^\alpha u_n(t))dt - \lambda_n \int_0^T (\nabla F(t, u_n(t)), u_n(t))dt. \end{gathered}$$

So $2\phi_{\lambda_n}(zu_n) \le \lambda_n C_* T + 2c_{\lambda_n}$, for all $z \in [0, 1]$.

On the other hand, for all $r_0 > 0$,

$$\begin{aligned} 2\phi_{\lambda_n}(r_0\omega_n) = &-\int_0^T ({}_0^cD_t^\alpha r_0\omega_n, {}_t^cD_T^\alpha r_0\omega_n)dt - 2\lambda_n \int_0^T F(t, r_0\omega_n)dt, \\ =&|\cos(\pi\alpha)|r_0^2 - O_n(1), \end{aligned} \tag{3.46}$$

which contradicts $2\phi_{\lambda_n}(zu_n) \le \lambda_n C_* T + 2c_{\lambda_n}$. This contradiction shows $\{u_n\}$ is unbounded in E^α. The proof is complete. □

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