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EXISTENCE OF SOLUTIONS FOR A QUASI-LINEAR PHASE SEPARATION OF MULTI-COMPONENT SYSTEM

DONGPEI ZHANG, RUIKUAN LIU

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Abstract. This article formulates a new model of the phase separation of multi-component system, which is a fourth-order quasi-linear evolution partial differential equation. By using the acute angle principle, we obtain a weak solution of the corresponding steady-state equations. In addition, we show that the quasi-linear dynamic equations have at least one global weak solution, based on the T-weakly continuous operators theory.

1. Introduction and statement of main results

Phase separation of multi-component, which consists of N ($N \geq 2$) different kinds of components, is a fundamental physical phenomenon. When the temperature of the system $T > T_c$ (T_c is the critical temperature), the concentration of N different kinds of components is homogeneous distribution. However, the temperature $T < T_c$, the multi-component system may lead to phase separation, i.e., the concentration which is homogeneous distribution undergoes changes leading to heterogeneous spatial distribution. In the case that $N = 2$, it is the binary mixture system described by the well-known Cahn-Hilliard equations [7]. There have been many mathematical studies on the dynamics of the Cahn-Hilliard equations, see [1, 2, 8, 10, 12, 13, 16, 17, 19, 20, 21, 25, 26, 27, 28] and the references therein.

Note that the existence, uniqueness, regularity and numerical approximate solution of the version of stochastic Cahn-Hilliard equation have attracted much attentions [9, 15, 30]. As we known, there are few mathematical researches for the phase separation of multi-component systems. For the phase separation of a multi-component alloy by the finite element method, we refer the readers to [3, 4, 5, 6]. For the phase separation of multi-component mixture with interfacial free energy, Elliott and Luckhaus[11] studied a nonlinear multi-component diffusion equation incorporating uphill diffusion and capillarity effects. Moreover, Elliott and Garcke[12] derived a model of fourth-order degenerate parabolic partial differential equations for the phase separation in multi-component systems by considering the possibility of a concentration dependence of the mobility matrix. It is worth pointing out that

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they also showed some properties of the model and proved a global existence result for the degenerate system.

Based on the equilibrium phase transition dynamics theory established by Ma and Wang [22, 23], we derive a fourth-order quasi-linear dynamic model for phase separation of multi-component system with Ginzburg-Landau free energy. The fourth-order quasi-linear dynamic equations can be expressed as follows

$$\frac{\partial u_k}{\partial t} = D_i[a_{ij}^{kl}(x, \mathbf{u}, \nabla\mathbf{u}, D^2\mathbf{u})D_j\Delta u_l] - f^k(x, \mathbf{u}, \nabla\mathbf{u}, \Delta\mathbf{u}), \tag{1.1}$$

with the initial-boundary value conditions

$$\mathbf{u}(x, 0) = \varphi(x), \tag{1.2}$$

$$\mathbf{u}|_{\partial\Omega} = 0, \ \Delta\mathbf{u}|_{\partial\Omega} = 0, \tag{1.3}$$

and the physical condition

$$\int_\Omega \mathbf{u} dx = 0, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $\mathbf{u} = (u_1, u_2, \dots, u_m)$ $(m \geq 2)$ is the unknown function, $1 \leq k, l \leq m$, $1 \leq i, j \leq n$. The boundary conditions (1.3) show that there is no component on the boundary. And the physical condition (1.4) indicates that the system satisfies the certain physical conservation laws.

When $\mathbf{u}$ is in equilibrium state, i.e., $\frac{\partial\mathbf{u}}{\partial t} = 0$, the corresponding stationary equations of (1.1)–(1.4) can be expressed as

$$\begin{gathered} D_i[a_{ij}^{kl}(x, \mathbf{u}, \nabla\mathbf{u}, D^2\mathbf{u})D_j\Delta u_l] - f^k(x, \mathbf{u}, \nabla\mathbf{u}, \Delta\mathbf{u}) = 0, \\ \mathbf{u}|_{\partial\Omega} = 0, \quad \Delta\mathbf{u}|_{\partial\Omega} = 0, \\ \int_\Omega \mathbf{u} dx = 0, \end{gathered} \tag{1.5}$$

where $x \in \Omega \subset \mathbb{R}^n$, $\mathbf{u} = (u_1, \dots, u_m)$, $1 \leq k$, $l \leq m$, $1 \leq i, j \leq n$.

The main aim of this article is to study the existence of global weak solution for the dynamic system (1.1)–(1.4) and the existence of weak solution for the corresponding stationary equations (1.5). The main techniques are the T-weakly continuous operators theory for the evolution partial differential equations established by Ma et al [22, 23, 24] and the acute angle principle for weakly continuous operators proposed by Ma et al [18, 23, 24], respectively.

First, we define the following two spaces, which are crucial to our theorems and the proofs.

$$H_2 = \Big\{\mathbf{u} \in H^2(\Omega, \mathbb{R}^m) : \int_\Omega \mathbf{u} dx = 0, \mathbf{u}|_{\partial\Omega} = 0\Big\},$$

$$X_2 = \Big\{\mathbf{u} \in W^{3,2}(\Omega, \mathbb{R}^m) \cap W^{2,p_2}(\Omega, R^m) : \int_\Omega \mathbf{u} dx = 0, \mathbf{u}|_{\partial\Omega} = 0, \Delta\mathbf{u}|_{\partial\Omega} = 0\Big\},$$

where $p_2 > 2$.

We make the following assumptions:

(A1) $a_{ij}^{kl}(x, z, \xi, \eta)$ and $f^k(x, z, \xi, \eta)$, $1 \leq k, l \leq m$, $1 \leq i, j \leq n$, satisfy the Carathéodory conditions.

(A2) There exists a $\lambda > 0$, such that

$$a_{ij}^{kl}\zeta_i^k\zeta_j^l \geq \lambda|\zeta|^2, \quad \text{for any } \zeta \in \mathbb{R}^{nm}\backslash\{0\}.$$

(A3) $f^k(x,z,\xi,\eta)(1\le k\le m)$ satisfy the structural conditions

$$D_\eta f^k(x,z,\xi,\eta)\ge\delta>0,$$
$$f^k(x,z,\xi,\eta)\eta_k\ge C_1|\eta|^{p_2}-C_2,$$

where $\delta>0$, C_1, $C_2\ge 0$ are constants, $p_2>2$.

(A4) $a_{ij}^{kl}(x,z,\xi,\eta)$ and $f^k(x,z,\xi,\eta)$ satisfy the increasing conditions

$$|a_{ij}^{kl}(x,z,\xi,\eta)|\le\begin{cases}C(|\eta|^{\frac{q_3}{2}}+|\xi|^{\frac{q_2}{2}}+|z|^{\frac{q_1}{2}}+1), & n>\max\{6,2p_2\},\\ \mu_3(|z|)(|\eta|^{\frac{q_3}{2}}+|\xi|^{\frac{q_2}{2}}+1), & \max\{4,p_2\}<n<\max\{6,2p_2\},\\ \mu_4(|\xi|,|z|)(|\eta|^{\frac{q_3}{2}}+1), & p_2<n<\max\{4,p_2\}.\end{cases}$$

$$|f^k(x,z,\xi,\eta)|\le\begin{cases}C(|\eta|^{\frac{q_3}{p_2}}+|\xi|^{\frac{q_2}{p_2}}+|z|^{\frac{q_1}{p_2}}+1), & n>\max\{6,2p_2\},\\ \mu_1(|z|)(|\eta|^{\frac{q_3}{p_2}}+|\xi|^{\frac{q_2}{p_2}}+1), & \max\{4,p_2\}<n<\max\{6,2p_2\},\\ \mu_2(|\xi|,|z|)(|\eta|^{\frac{q_3}{p_2}}+1), & p_2<n<\max\{4,p_2\}.\end{cases}$$

where $C>0$ is a constant, $\mu_i(i=1,2,3,4)$ are monotonically increasing and continuous functions. $q_1<\max\{\frac{2n}{n-6},\frac{np_2}{n-2p_2}\}$, $q_2<\max\{\frac{np_2}{n-p_2},\frac{2n}{n-4}\}$, $q_3<\max\{p_2,\frac{2n}{n-2}\}$.

For the stationary equations (1.5), we have the following existence result.

Theorem 1.1. *Assume that* (A1)–(A4) *hold, then* (1.5) *have at least one weak solution* $\mathbf{u}\in X_2$.

For the evolution equations (1.1)–(1.4), the structural condition (A3) can be replaced by the following condition:

(A3') $f^k(x,z,\xi,\eta)(1\le k\le m)$ satisfy the structural condition

$$f^k(x,z,\xi,\eta)\eta_k\ge C_1|\eta|^{p_2}-C_2(|\eta|^2+|\xi|^2+|z|^2)-g_1(x),$$

where $C_1,\ C_2\ge 0$ are constants, $p_2>2$, $g_1(x)\in L^1(\Omega)$.

Now, we give the existence of global weak solution for system (1.1)–(1.4).

Theorem 1.2. *Let* $\varphi\in H_2$, *and* (A1), (A2), (A3') (A4) *hold. Then* (1.1)–(1.4) *have at least one global weak solution*

$$\mathbf{u}\in L^p_{\rm loc}((0,\infty),X_2)\cap L^\infty_{\rm loc}((0,\infty),H_2).$$

Remark 1.3. Here we need to introduce the space mentioned in Theorem 1.2. For a Banach space X, we let

$$L^p((0,T),X)=\Big\{u:(0,T)\to X:\Big(\int_0^T\|u\|^p dt\Big)^{1/p}<\infty\Big\},$$

where $p=(p_1,p_2,\dots,p_m)$, $p_i\ge 1$ $(1\le i\le m)$, $\|u\|^p=\sum_{i=1}^m|u|_i^{p_i}$, $|\cdot|_i$ is the semi-norm in X and $\|\cdot\|_X=\sum_{i=1}^m|\cdot|_i$.

Then we can define

$$L^p_{\rm loc}((0,\infty),X)=\{u(t)\in X:u\in L^p((0,T),X),\ \text{for any } T>0\}.$$

Remark 1.4. According to the definition of the space $L^p((0,T),X)$, it is easy to see that $p=(2,p_2)$ in Theorem 1.2.

The rest of this paper is organized as follows. The preliminaries, the acute angle principle for weakly continuous operators and the T-weakly continuous operators theory for parabolic equations are given in Section 2. In Section 3, we first introduce some basic physical quantities and then derive the fourth-order quasi-linear dynamic equations of phase separation of multi-component system. Section 4 is devoted to proving the main results.

2. Preliminaries

In this section, we introduce the acute principle for the weakly continuous operators and the T-weakly continuous operators theory for the evolution equations respectively.

2.1. **Acute angle principle for weakly continuous operators.**

Weakly continuous operators theory is a useful tool to solve the existence of elliptic equations [14]. Here, we mainly introduce the definition and the acute angle principle for weakly continuous operators proposed by Ma in [23, 24].

Let X be a linear space and $X_1,\ X_2$ be the completion of X with the norm $\|\cdot\|_1,\ \|\cdot\|_2$, respectively. Let X_1 be a separable Banach space and X_2 be a reflexive Banach space. X_1^* is the dual space of X_1 and $X \subset X_2$. There is a linear operator L satisfying

$$L : X \to X_1 \text{ is a one-to-one and dense linear operator.}$$

Definition 2.1. A mapping $G : X_2 \to X_1^*$ is called weakly continuous. If for any $\{u_n\} \subset X_2,\ u_n \rightharpoonup u_0$ in X_2, we have

$$\lim_{n\to\infty} \langle G(u_n), v\rangle = \langle G(u_0), v\rangle, \quad \text{for any } v \in X_1.$$

The following lemma for weakly continuous operator is crucial to our proof.

Lemma 2.2 (Acute angle principle). *Suppose that $G : X_2 \to X_1^*$ is weakly continuous. Let $U \subset X_2$ be a bounded open set and $0 \in U$. If*

$$\langle G(u), Lu\rangle \geq 0, \quad \text{for any } u \in \partial U \cap X,$$

then the equation $G(u) = 0$ has a solution in X_2.

2.2. **T-weakly continuous operators theory for parabolic equations.**

The T-weakly continuous operators theory was established by Ma [23], which can effectively solve the global weak solutions for many nonlinear problems [22, 23, 24, 29].

Assume that the nonlinear evolution equations can be expressed as the abstract form

$$\begin{aligned} \frac{du}{dt} &= \widetilde{G}u,\ 0 < t < \infty, \\ u(0) &= \varphi, \end{aligned} \tag{2.1}$$

where $\varphi \in H$, H is a Hilbert space. $u : [0, \infty) \to H$ is the unknown function.

Let Y_1 and Y_2 be Banach spaces, $Y_1, Y_2 \subset H$ and Y_1^* be the dual space of Y_1.

Basic definitions and lemmas. First, we introduce the definition of global weak solution for the equations (2.1).

Definition 2.3. Let $\varphi \in H$. $u \in L^p_{\rm loc}((0,\infty), Y_2) \cap L^\infty_{\rm loc}((0,\infty), H)$ is called a global weak solution of (2.1), if u satisfies the following equality:

$$\langle u(t), v\rangle_H = \int_0^t \langle \widetilde{G}u, v\rangle d\tau + \langle \varphi, v\rangle_H.$$

for any $v \in Y_1 \subset H$.

Next we give the definitions of uniformly weak convergence and T-weak continuity.

Definition 2.4. Let $\{u_n\} \subset L^p((0,T), Y_2)$, $u_0 \in L^p((0,T), Y_2)$. We say that $u_n \rightharpoonup u_0$ in $L^p((0,T), Y_2)$ is uniformly weakly convergent, if $\{u_n\} \subset L^\infty((0,T), H)$ is bounded and satisfies

$$u_n \rightharpoonup u_0 \text{ in } L^p((0,T), Y_2),$$
$$\lim_{n\to\infty} \int_0^T |\langle u_n - u_0, v\rangle_H|^2 dt = 0, \quad \text{for any } v \in H.$$

Definition 2.5. A mapping $\widetilde{G} : Y_2 \times (0,\infty) \to Y_1^*$ is called T-weakly continuous. If for any $p = (p_1, p_2, \dots, p_m)$, $0 < T < \infty$ and $u_n \rightharpoonup u_0$ is uniformly weakly convergent in $L^p((0,T), Y_2)$, we have

$$\lim_{n\to\infty} \int_0^T \langle \widetilde{G}u_n, v\rangle dt = \int_0^T \langle \widetilde{G}u_0, v\rangle dt, \quad \text{for any } v \in Y_1.$$

The following two elementary lemmas will be used later. Their proofs can be found in [23] .

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set, $\{u_n\} \subset L^p((0,T), W^{s,p}(\Omega))(s \geq 1, p \geq 2)$ be a bounded sequence and $\{u_n\}$ is uniformly weakly convergent to $u_0 \in L^p((0,T), W^{s,p}(\Omega))$. Then for any $|\alpha| \leq s-1$, we have*

$$D^\alpha u_n \to D^\alpha u_0 \quad \text{in } L^2((0,T) \times \Omega).$$

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^n$ be an open set, the function $f : \Omega \times \mathbb{R}^N \to \mathbb{R}^1$ satisfy the Carathéodory conditions and*

$$|f(x,\xi)| \leq C \sum_{i=1}^N |\xi_i|^{p_i/p} + b(x),$$

where $C > 0$ is a constant and $p_i, p > 1$, $b(x) \in L^p(\Omega)$.

If $\{u_{i_k}\} \subset L^{p_i}(\Omega)$ $(1 \leq i \leq N)$ is bounded and $\{u_{i_k}\}$ converges to $\{u_i\}$ by measure in Ω_0 for any bounded subregion $\Omega_0 \subset \Omega$, then for any $v \in L^{p'}(\Omega)$, we have

$$\lim_{k\to\infty} \int_\Omega f(x, u_{1_k}, \dots, u_{N_k}) v\, dx = \int_\Omega f(x, u_1, \dots, u_N) v\, dx,$$

where p' satisfies $\frac{1}{p'} + \frac{1}{p} = 1$.

Existence of a global weak solution for nonlinear parabolic equations. First, we introduce the following function spaces

$$Y \subset Y_2 \subset Y_1 \subset H,$$
$$Y_2 \subset H_2 \subset H_1 \subset H,$$

where Y is a linear space, Y_1, Y_2 are Banach spaces, H, H_1 and H_2 are Hilbert spaces. We remark that all inclusion relations are dense embedding.

Moreover, suppose that there exists an operator $\overline{L}$ satisfying the following conditions

$$\begin{gathered} \overline{L}: Y \to Y_1 \text{ is a one-to-one and dense linear operator,} \\ \langle \overline{L}u, v\rangle_H = \langle u, v\rangle_{H_2}, \quad \text{for any } u, v \in Y. \end{gathered} \tag{2.2}$$

In addition, there exists a sequence $\{e_k\}_{k=1}^{\infty} \subset Y$ such that

$$\overline{L}e_k = \rho_k e_k, \quad k = 1, 2, \dots, \tag{2.3}$$

where $\rho_k \neq 0$, $\{e_k\}_{k=1}^{\infty}$ is the common orthogonal basis of H.

Here we also assume that $\widetilde{G}: Y_2 \times (0, \infty) \to Y_1^*$ satisfies the following inequality,

$$\langle \widetilde{G}u, \overline{L}u\rangle \leq -C_1\|u\|_{Y_2}^p + C_2\|u\|_{H_2}^2 + f(t), \tag{2.4}$$

where $p = (p_1, p_2, \dots, p_m)$, $p_i > 1$ $(1 \leq i \leq m)$, $\|u\|_{Y_2}^p = \sum_{i=1}^m |u|_i^{p_i}$, $|\cdot|_i$ is the semi-norm in Y_2, $\|u\|_{Y_2} = \sum_{i=1}^m |u|_i$, $C_1, C_2 > 0$ are constants, $f \in L^1_{\rm loc}(0, \infty)$.

Then we give the following existence result of global weak solutions for the nonlinear parabolic equations (2.1).

Lemma 2.8. *Assume that* (2.2)–(2.4) *hold. If* $\widetilde{G}: Y_2 \times (0, \infty) \to Y_1^*$ *is* T*-weakly continuous, then problem* (2.1) *has a global weak solution*

$$u \in L^p_{\rm loc}((0, \infty), Y_2) \cap L^\infty_{\rm loc}((0, \infty), H_2)$$

for any $\varphi \in H_2$.

3. Dynamic equations of phase separation of multi-component system

In this section, we devote to deriving the new dynamic model (1.1)–(1.4) of phase separation of multi-component system by using the equilibrium phase transition dynamics theory founded by Ma and Wang[22].

3.1. Basic physical quantities.

Let Σ be a multi-component system mixed by $m+1$ different kinds of components $A_1, \dots, A_{m+1}$ $(m \geq 2)$. u_k $(1 \leq k \leq m+1)$ is the molar density of A_k, i.e.,

$$u_k(x) = \text{the molar number of } A_k \text{ in unit volume at } x \in \Omega.$$

Note that $u_1, \dots, u_{m+1}$ satisfy the relation

$$u_1 + u_2 + \cdots + u_{m+1} = \text{constant}.$$

It is worth noticing that the order parameter $\mathbf{u}$ contains only m independent variables, i.e., $\mathbf{u} = (u_1, u_2, \dots, u_m)$. In fact $\mathbf{u} = (u_1, u_2, \dots, u_m)$ is the unknown function.

Based on the physical experiments, this system is also related to the temperature T and the container volume $|\Omega|$. Hence, we regard T and $|\Omega|$ as the control parameters. More generally, the control parameter can be expressed as

$$\kappa = (T, |\Omega|, \omega_1, \dots, \omega_m),$$

where ω_k is the proportion of A_k in the multi-component system.

3.2. **A new dynamic model.** In this subsection, we are focused on obtaining the dynamic equations (1.1) for the order parameter $\mathbf{u}$.

According to the Ginzburg-Landau mean field theory, the free energy of a $m+1$-components system(see[22]) can be expressed as

$$H(\mathbf{u},\kappa) = \int_\Omega \Big[\frac{1}{2}\sum_{k=1}^m \mu_k|\nabla u_k|^2 + g(\mathbf{u},\kappa)\Big]dx, \tag{3.1}$$

where $\mu_k = \mu_k(\kappa) \geq 0$ is the physical parameter. $g(\mathbf{u},\kappa)$ is a polynomial on $\mathbf{u}$, which can be given by

$$g(\mathbf{u},\kappa) = \sum_{1\leq|\gamma|\leq 2r} a_\gamma u_1^{\gamma_1} u_2^{\gamma_2} \dots u_m^{\gamma_m}, \quad \gamma = (\gamma_1,\gamma_2,\dots\gamma_m). \tag{3.2}$$

Based on the equilibrium phase transition dynamics theory (see[22]), the following dynamic equations can be deduced from (3.1)–(3.2):

$$\begin{aligned} \frac{\partial u_k}{\partial t} &= -\beta_k \nabla\cdot\Big[\sum_{l=1}^m L_{kl}\nabla(\mu_l\Delta u_l - g_l(\mathbf{u},\kappa))\Big] \\ &\quad + \nabla\cdot\Big(\sum_{l=1}^m L_{kl}\nabla\phi_l(\mathbf{u},\kappa)\Big), \end{aligned} \tag{3.3}$$

where $\beta_k > 0$, $L_{kl} = L_{kl}(\mathbf{u}, D\mathbf{u})$ $(1 \leq k,l \leq m)$ is positive and symmetric, and $g_l(\mathbf{u},\kappa) = \frac{\partial}{\partial u_l} g(\mathbf{u},\kappa)$. ϕ_l is independent of u_l and satisfies

$$\int_\Omega \sum_{k,l=1}^m L_{kl}\nabla(\mu_k\Delta u_k - g_k)\cdot\nabla\phi_l dx = 0, \tag{3.4}$$

where $g_k(\mathbf{u},\kappa) = \frac{\partial}{\partial u_k} g(\mathbf{u},\kappa)$.

In this paper, we consider the more general case that the equations (3.3) are quasi-linear. Meanwhile, we take $\phi_l(\mathbf{u},\kappa) = 0$ in (3.3) and (3.4), which has no material impact to the main characteristics of this physical system. Furthermore, we supplement with the initial-boundary conditions (1.2)–(1.3) and the physical conservation laws condition (1.4). Therefore, we obtain the modified dynamic model (1.1)–(1.4), which is a fourth-order quasi-linear evolution partial differential equations.

4. Proofs of main results

4.1. **Proof of Theorem 1.1.** Now we will apply Lemma 2.2 and Lemma 2.7 to prove the existence of a weak solution for the steady state equations (1.5). We will prove Theorem 1.1 in three steps.

Step 1. Define the operator G. Let

$$X = \Big\{\mathbf{u} \in C^\infty(\Omega,\mathbb{R}^m) : \int_\Omega \mathbf{u}dx = 0, \mathbf{u}|_{\partial\Omega} = 0, \Delta\mathbf{u}|_{\partial\Omega} = 0\Big\},$$
$$X_1 = \{\mathbf{u} \in C^\infty(\Omega,\mathbb{R}^m) : \mathbf{u}|_{\partial\Omega} = 0\},$$
$$X_2 = \Big\{\mathbf{u} \in W^{3,2}(\Omega,\mathbb{R}^m) \cap W^{2,p_2}(\Omega,\mathbb{R}^m) : \int_\Omega \mathbf{u}dx = 0, \mathbf{u}|_{\partial\Omega} = 0, \Delta\mathbf{u}|_{\partial\Omega} = 0\Big\}.$$

According to the general definition of weak solution, we define the operator $G : X_2 \to X_1^*$ by the inner product from

$$\langle G\mathbf{u}, \mathbf{v}\rangle = \int_\Omega [a_{ij}^{kl}(x, \mathbf{u}, \nabla\mathbf{u}, D^2\mathbf{u}) D_j\Delta u_l D_i v_k + f^k(x, \mathbf{u}, \nabla\mathbf{u}, \Delta\mathbf{u}) v_k] dx,$$

where $\mathbf{v} = (v_1, v_2, \dots, v_m) \in X_1$, X_1^* is the dual space of X_1. From (A4), it is easy to show that the operator G is a bounded operator.

Step 2. Check the conditions for the acute angle principle. Let $L = \Delta : X \to X_1$. The conditions (A2) and (A3) imply that

$$\begin{aligned}\langle G\mathbf{u}, \Delta\mathbf{u}\rangle &= \int_\Omega [a_{ij}^{kl}(x, \mathbf{u}, \nabla\mathbf{u}, D^2\mathbf{u}) D_j\Delta u_l D_i \Delta u_k + f^k(x, \mathbf{u}, \nabla\mathbf{u}, \Delta\mathbf{u}) \Delta u_k] dx \\ &\geq \lambda \int_\Omega |\nabla(\Delta\mathbf{u})|^2 dx + C_1 \int_\Omega |\Delta\mathbf{u}|^{p_2} dx - C_2.\end{aligned} \tag{4.1}$$

By (4.1), it is clear that

$$\langle G\mathbf{u}, \Delta\mathbf{u}\rangle \geq 0, \quad \text{for any } \mathbf{u} \in X_2 \text{ and } \|\mathbf{u}\|_{X_2} \text{ is large enough,}$$

which implies that the operator $G : X_2 \to X_1^*$ satisfies the condition of Lemma 2.2.

Step 3. Verify the weak continuity of the operator G. Let $\{\mathbf{u}_n\} \subset X_2$, $\mathbf{u}_n \rightharpoonup \mathbf{u}_0$ in X_2. Based on the Definition 2.1, we only need to prove that the following limit holds

$$\begin{aligned}&\lim_{n\to\infty} \int_\Omega [a_{ij}^{kl}(x, \mathbf{u}_n, \nabla\mathbf{u}_n, D^2\mathbf{u}_n) D_j\Delta u_{nl} D_i v_k + f^k(x, \mathbf{u}_n, \nabla\mathbf{u}_n, \Delta\mathbf{u}_n) v_k] dx \\ &= \int_\Omega [a_{ij}^{kl}(x, \mathbf{u}_0, \nabla\mathbf{u}_0, D^2\mathbf{u}_0) D_j\Delta u_{0l} D_i v_k + f^k(x, \mathbf{u}_0, \nabla\mathbf{u}_0, \Delta\mathbf{u}_0) v_k] dx.\end{aligned} \tag{4.2}$$

for any $\mathbf{v} \in X_1$.

We should divide (4.2) into the following two parts.

$$\lim_{n\to\infty} \int_\Omega f^k(x, \mathbf{u}_n, \nabla\mathbf{u}_n, \Delta\mathbf{u}_n) v_k dx = \int_\Omega f^k(x, \mathbf{u}_0, \nabla\mathbf{u}_0, \Delta\mathbf{u}_0) v_k dx, \tag{4.3}$$

$$\begin{aligned}&\lim_{n\to\infty} \int_\Omega a_{ij}^{kl}(x, \mathbf{u}_n, \nabla\mathbf{u}_n, D^2\mathbf{u}_n) D_j\Delta u_{nl} D_i v_k dx \\ &= \int_\Omega a_{ij}^{kl}(x, \mathbf{u}_0, \nabla\mathbf{u}_0, D^2\mathbf{u}_0) D_j\Delta u_{0l} D_i v_k dx.\end{aligned} \tag{4.4}$$

By the compact embedding theorem, it is easy to check the following relations

$$(\mathbf{u}_n, D\mathbf{u}_n, D^2\mathbf{u}_n) \to (\mathbf{u}_0, D\mathbf{u}_0, D^2\mathbf{u}_0) \quad \text{in } \begin{cases} L^{q_1} \times L^{q_2} \times L^{q_3}, \\ C^0 \times L^{q_2} \times L^{q_3}, \\ C^0 \times C^0 \times L^{q_3}, \end{cases} \tag{4.5}$$

where $q_1 < \max\{\frac{2n}{n-6}, \frac{np_2}{n-2p_2}\}$, $q_2 < \max\{\frac{np_2}{n-p_2}, \frac{2n}{n-4}\}$, $q_3 < \max\{p_2, \frac{2n}{n-2}\}$. Combining (A4), (4.5) and Lemma 2.7, it is easy to see that (4.3) is valid.

Notice that (4.4) is equivalent to

$$\begin{aligned}&\lim_{n\to\infty} \int_\Omega [a_{ij}^{kl}(x, \mathbf{u}_n, \nabla\mathbf{u}_n, D^2\mathbf{u}_n) D_j\Delta u_{nl} \\ &- a_{ij}^{kl}(x, \mathbf{u}_0, \nabla\mathbf{u}_0, D^2\mathbf{u}_0) D_j\Delta u_{0l}] D_i v_k dx = 0.\end{aligned} \tag{4.6}$$

Moreover, the left part of (4.6) can be rewritten as

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega [a_{ij}^{kl}(x,\mathbf{u}_n,\nabla\mathbf{u}_n,D^2\mathbf{u}_n)D_j\Delta u_{nl}\\
&- a_{ij}^{kl}(x,\mathbf{u}_0,\nabla\mathbf{u}_0,D^2\mathbf{u}_0)D_j\Delta u_{0l}]D_iv_kdx\\
&= \lim_{n\to\infty}\Big\{\int_\Omega [a_{ij}^{kl}(x,\mathbf{u}_n,\nabla\mathbf{u}_n,D^2\mathbf{u}_n) - a_{ij}^{kl}(x,\mathbf{u}_0,\nabla\mathbf{u}_0,D^2\mathbf{u}_0)]D_j\Delta u_{nl}D_iv_kdx\\
&\quad + \int_\Omega a_{ij}^{kl}(x,\mathbf{u}_0,\nabla\mathbf{u}_0,D^2\mathbf{u}_0)[D_j\Delta u_{nl} - D_j\Delta u_{0l}]D_iv_kdx\Big\}.
\end{aligned} \tag{4.7}$$

Analogously, under the assumption (A4), we get following equality basing on (4.5) and Lemma 2.7,

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega [a_{ij}^{kl}(x,\mathbf{u}_n,\nabla\mathbf{u}_n,D^2\mathbf{u}_n)\\
&- a_{ij}^{kl}(x,\mathbf{u}_0,\nabla\mathbf{u}_0,D^2\mathbf{u}_0)]D_j\Delta u_{nl}D_iv_kdx = 0.
\end{aligned} \tag{4.8}$$

For the second term on the right hand of (4.7), it is not difficult to derive the following result from $\mathbf{u}_n \rightharpoonup \mathbf{u}_0$ in X_2,

$$\lim_{n\to\infty}\int_\Omega a_{ij}^{kl}(x,\mathbf{u}_0,\nabla\mathbf{u}_0,D^2\mathbf{u}_0)[D_j\Delta u_{nl} - D_j\Delta u_{0l}]D_iv_kdx = 0. \tag{4.9}$$

Obviously, (4.8) and (4.9) infer that (4.4) holds true. Then the weak continuity of the operator $G : X_2 \to X_1^*$ is obtained.

Therefore, we can immediately get that problem (1.5) has a weak solution by using Lemma 2.2.

4.2. **Proof of Theorem 1.2.** We now apply Lemma 2.8 to prove the system (1.1)–(1.4) has a global weak solution. The proof is divided into three steps.

Step 1. Define the operator $\widetilde{G}$. Let

$$\begin{gathered}
X = \Big\{\mathbf{u}\in C^\infty(\Omega,\mathbb{R}^m) : \int_\Omega \mathbf{u}dx = 0, \mathbf{u}|_{\partial\Omega} = 0, \Delta\mathbf{u}|_{\partial\Omega} = 0\Big\},\\
X_1 = \{\mathbf{u}\in C^\infty(\Omega,\mathbb{R}^m) : \mathbf{u}|_{\partial\Omega} = 0\},\\
X_2 = \Big\{\mathbf{u}\in W^{3,2}(\Omega,\mathbb{R}^m)\cap W^{2,p_2}(\Omega,R^m) : \int_\Omega \mathbf{u}dx = 0, \mathbf{u}|_{\partial\Omega} = 0, \Delta\mathbf{u}|_{\partial\Omega} = 0\Big\},\\
H = \Big\{\mathbf{u}\in L^2(\Omega,\mathbb{R}^m) : \int_\Omega \mathbf{u}dx = 0\Big\},\\
H_1 = \Big\{\mathbf{u}\in H^1(\Omega,\mathbb{R}^m) : \int_\Omega \mathbf{u}dx = 0, \mathbf{u}|_{\partial\Omega} = 0\Big\},\\
H_2 = \Big\{\mathbf{u}\in H^2(\Omega,\mathbb{R}^m) : \int_\Omega \mathbf{u}dx = 0, \mathbf{u}|_{\partial\Omega} = 0\Big\}.
\end{gathered}$$

According to the Definition 2.3, we define the operator $\widetilde{G} : X_2 \times (0,\infty) \to X_1^*$ by the inner product form

$$\langle \widetilde{G}\mathbf{u},\mathbf{v}\rangle = \int_\Omega [-a_{ij}^{kl}(x,\mathbf{u},\nabla\mathbf{u},D^2\mathbf{u})D_j\Delta u_lD_iv_k - f^k(x,\mathbf{u},\nabla\mathbf{u},\Delta\mathbf{u})v_k]dx,$$

where $\mathbf{v}\in X_1$. By assumption (A4), it is easy to check that the $\widetilde{G}$ is a bounded operator.

Step 2. Check conditions (2.2)–(2.4). Let $\overline{L} = \Delta : X \to X_1$. It is obvious that (2.2) and (2.3) are valid. It follows from assumptions (A2) and (A3') that

$$\begin{aligned}
&\langle \widetilde{G}\mathbf{u}, \Delta\mathbf{u}\rangle \\
&= \int_\Omega [-a_{ij}^{kl}(x,\mathbf{u},\nabla\mathbf{u},D^2\mathbf{u})D_j\Delta u_l D_i\Delta u_k - f^k(x,\mathbf{u},\nabla\mathbf{u},\Delta\mathbf{u})\Delta u_k]dx \\
&\leq -\lambda\int_\Omega |\nabla(\Delta\mathbf{u})|^2 dx - C_1\int_\Omega |\Delta\mathbf{u}|^{p_2}dx \\
&\quad + C_2\int_\Omega (|\Delta\mathbf{u}|^2 + |\nabla\mathbf{u}|^2 + |\mathbf{u}|^2)dx + \int_\Omega g_1(x)dx,
\end{aligned} \tag{4.10}$$

which implies that (2.4) holds true.

Step 3. Verify the condition for the T-weak continuity of the operator $\widetilde{G}$. Let $\{\mathbf{u}_n\} \subset L^p((0,T),X_2) \cap L^\infty((0,T),H_2)$, $\mathbf{u}_n \rightharpoonup \mathbf{u}_0$ in $L^p((0,T),X_2)$ be uniformly weakly convergent. By definition 2.5, we only need to show the following limit holds,

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^t\int_\Omega [-a_{ij}^{kl}(x,\mathbf{u}_n,\nabla\mathbf{u}_n,D^2\mathbf{u}_n)D_j\Delta u_{nl}D_i v_k \\
&- f^k(x,\mathbf{u}_n,\nabla\mathbf{u}_n,\Delta\mathbf{u}_n)v_k]\,dx\,d\tau \\
&= \int_0^t\int_\Omega [-a_{ij}^{kl}(x,\mathbf{u}_0,\nabla\mathbf{u}_0,D^2\mathbf{u}_0)D_j\Delta u_{0l}D_i v_k \\
&\quad - f^k(x,\mathbf{u}_0,\nabla\mathbf{u}_0,\Delta\mathbf{u}_0)v_k]\,dx\,d\tau.
\end{aligned} \tag{4.11}$$

Obviously, (4.11) can be divided into the following two parts.

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^t\int_\Omega f^k(x,\mathbf{u}_n,\nabla\mathbf{u}_n,\Delta\mathbf{u}_n)v_k\,dx\,d\tau \\
&= \int_0^t\int_\Omega f^k(x,\mathbf{u}_0,\nabla\mathbf{u}_0,\Delta\mathbf{u}_0)v_k\,dx\,d\tau.
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^t\int_\Omega a_{ij}^{kl}(x,\mathbf{u}_n,\nabla\mathbf{u}_n,D^2\mathbf{u}_n)D_j\Delta u_{nl}D_i v_k\,dx\,d\tau \\
&= \int_0^t\int_\Omega a_{ij}^{kl}(x,\mathbf{u}_0,\nabla\mathbf{u}_0,D^2\mathbf{u}_0)D_j\Delta u_{0l}D_i v_k\,dx\,d\tau.
\end{aligned} \tag{4.13}$$

Owing to $\{\mathbf{u}_n\} \subset L^p((0,T),X_2) \cap L^\infty((0,T),H_2)$, $\mathbf{u}_n \rightharpoonup \mathbf{u}_0$ in $L^p((0,T),X_2)$ is uniformly weakly convergent, we can derive the following convergence properties by using the Lemma 2.6,

$$\begin{gathered}
\mathbf{u}_n \to \mathbf{u}_0 \text{ in } L^2((0,T)\times\Omega), \\
D\mathbf{u}_n \to D\mathbf{u}_0 \text{ in } L^2((0,T)\times\Omega), \\
D^2\mathbf{u}_n \to D^2\mathbf{u}_0 \text{ in } L^2((0,T)\times\Omega),
\end{gathered} \tag{4.14}$$

which infer that $\{\mathbf{u}_n\}$, $\{D\mathbf{u}_n\}$ and $\{D^2\mathbf{u}_n\}$ converge to $\mathbf{u}_0$, $D\mathbf{u}_0$ and $D^2\mathbf{u}_0$ by measure in $\Omega \times (0,T)$, respectively. Then, together the assumption (A4) with Lemma 2.7, we see that (4.12) holds.

Note that (4.13) is equivalent to

$$\begin{aligned}\lim_{n\to\infty}\int_0^t\int_\Omega [a_{ij}^{kl}(x,\mathbf{u}_n,\nabla\mathbf{u}_n,D^2\mathbf{u}_n)D_j\Delta u_{nl}\\ - a_{ij}^{kl}(x,\mathbf{u}_0,\nabla\mathbf{u}_0,D^2\mathbf{u}_0)D_j\Delta u_{0l}]D_iv_k\,dx\,d\tau = 0.\end{aligned}\tag{4.15}$$

Furthermore, the left part of (4.15) can be rewritten as

$$\begin{aligned}&\lim_{n\to\infty}\int_0^t\int_\Omega [a_{ij}^{kl}(x,\mathbf{u}_n,\nabla\mathbf{u}_n,D^2\mathbf{u}_n)D_j\Delta u_{nl}\\ &- a_{ij}^{kl}(x,\mathbf{u}_0,\nabla\mathbf{u}_0,D^2\mathbf{u}_0)D_j\Delta u_{0l}]D_iv_k\,dx\,d\tau\\ &=\lim_{n\to\infty}\Big\{\int_0^t\int_\Omega [a_{ij}^{kl}(x,\mathbf{u}_n,\nabla\mathbf{u}_n,D^2\mathbf{u}_n)\\ &\quad - a_{ij}^{kl}(x,\mathbf{u}_0,\nabla\mathbf{u}_0,D^2\mathbf{u}_0)]D_j\Delta u_{nl}D_iv_k\,dx\,d\tau\\ &\quad +\int_0^t\int_\Omega a_{ij}^{kl}(x,\mathbf{u}_0,\nabla\mathbf{u}_0,D^2\mathbf{u}_0)[D_j\Delta u_{nl}-D_j\Delta u_{0l}]D_iv_k\,dx\,d\tau\Big\}.\end{aligned}\tag{4.16}$$

Combining assumption (A4), (4.14) and Lemma 2.7, it is clear that

$$\begin{aligned}\lim_{n\to\infty}\int_0^t\int_\Omega [a_{ij}^{kl}(x,\mathbf{u}_n,\nabla\mathbf{u}_n,D^2\mathbf{u}_n)\\ - a_{ij}^{kl}(x,\mathbf{u}_0,\nabla\mathbf{u}_0,D^2\mathbf{u}_0)]D_j\Delta u_{nl}D_iv_k\,dx\,d\tau = 0.\end{aligned}\tag{4.17}$$

Because $\mathbf{u}_n \rightharpoonup \mathbf{u}_0$ in $L^p((0,T),X_2)$ which is uniformly weakly convergent, it is easy to see that the following limit holds

$$\lim_{n\to\infty}\int_0^t\int_\Omega a_{ij}^{kl}(x,\mathbf{u}_0,\nabla\mathbf{u}_0,D^2\mathbf{u}_0)[D_j\Delta u_{nl}-D_j\Delta u_{0l}]D_iv_k\,dx\,d\tau = 0. \tag{4.18}$$

Note that (4.17) and (4.18) imply that (4.13) holds. Hence, $G : X_2\times(0,\infty)\to X_1^*$ is T-weakly continuous.

Consequently, from Lemma 2.8, we can easily obtain that problem (1.1)–(1.4) has one global weak solution

$$\mathbf{u}\in L^p_{\rm loc}((0,\infty),X_2)\cap L^\infty_{\rm loc}((0,\infty),H_2).$$

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Dongpei Zhang
Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, China
E-mail address: zhangdpscu@163.com

Ruikuan Liu
Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, China
E-mail address: liuruikuan2008@163.com