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SPECTRAL PROPERTIES OF A FRANKL TYPE PROBLEM FOR PARABOLIC-HYPERBOLIC EQUATIONS

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Abstract. In this article we study spectral properties of non-local boundary-value problem for an equation of parabolic-hyperbolic type. The non-local condition binds the solution values at points on boundaries of the parabolic and hyperbolic parts of the domain with each other. Nonlocal boundary conditions of such type are called Frankl-type conditions. This problem was first formulated by Kal'menov and Sadybekov who proved the unique strong solvability. In this article we investigate one particular case of this problem, for which we show that the problem does not have eigenvalues.

1. Introduction

The theory of equations of the mixed type is one of the well-developed sections of the modern theory of partial differential equations. This happens because of the appearance of many applied problems such mathematical modeling lead to the studying various types of equations in domains of changing independent variables. Therefore scientists are interested in problems of mixed type.

In 1902 Chaplygin was the first to point out the importance of studying equations of the mixed type in his paper "On gas jet". Researching boundary value problems for equations of the mixed type began from works of Tricomi, Gellerstedt in 20th-30th of the last century. A new stage of developing this theory was founded by papers of Lavrent'ev, Bitsadze, Frankl, Babenko, where the practical significance of some essential issues of this theory was indicated alongside with theoretical researches of these issues. For the most part, these works were devoted to the theoretical and applied aspects of equations of the mixed elliptic-hyperbolic type.

Researching equations of a parabolic-hyperbolic type has gained a rapid development quite recently. These problems are of particular interest due to their application to various problems of mechanics and physics. For example, such problems arise in studying the movement of weak compressible fluid in a channel surrounded by a porous medium: in the channel the pressure of the fluid satisfies the wave equation, but in the porous medium this pressure is described by a diffusion equation.

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Essential contribution to the development of the theory of boundary value problems for parabolic-hyperbolic equations was done by Salakhitdinov, Dzhuraev, Nakhushev. They justified the well-posedness of formulated problems by the method of reduction to integral equations. Issues of well-posed solvability of problems were researched on the basis of solution representation in the form of bilinear series in papers of Moiseev, Kapustin, Sabitov.

Unlike the theory of local boundary value problems, nonlocal boundary value problems are much less researched. In gas dynamics Frankl (in 1945) for the first time set a boundary value problem for the Chaplygin equation

$$k(y)u_{xx} - u_{yy} = 0,$$

where $k(0) = 0$, $k'(y) > 0$. In this problem as a carrier of nonlocal boundary condition ("jump of sealing")

$$u(0, y) - u(0, -y) = f(y)$$

is a part $-a < y < a$ of the boundary $x = 0$ of the domain consisting of parts of the boundary of subdomains of ellipticity and hyperbolicity of the equation [5, 6]. Therefore the nonlocal boundary conditions of such type, that is, binding values of functions on the boundary of domains of equations of various type, are called conditions of the Frankl type.

Pulkin and Lerner (1966) for the general Lavrent'ev-Bitsadze equation formulated and investigated problems in which the Franklle condition is replaced by the Tricomi condition, and different conditions are given on the remaining sections of the boundary [18].

Publications on this subject are quite extensive. From the recent publications related to the theme we can note the papers [13, 14, 19, 20, 21, 22, 24]. However in these papers the nonlocal problems were considered in rectangular domains. But in our formulation of the problem the hyperbolic part coincides with a characteristic triangle. Throughout this note we mainly use techniques from our works [3, 9, 10, 23, 26, 27, 28] .

2. Formulation of the problem and main result on its solvability

Let $\Omega \subset R^2$ be a finite domain bounded for $y > 0$ by the segments AA_0, A_0B_0, B_0B, $A = (0, 0)$, $A_0 = (0, 1)$, $B_0 = (1, 1)$, $B = (1, 0)$ and for $y < 0$ by the characteristics $AC : x + y = 0$ and $BC : x - y = 1$ of an equation of the mixed parabolic-hyperbolic type

$$Lu = \begin{Bmatrix} u_x - u_{yy}, & y > 0 \\ u_{xx} - u_{yy}, & y < 0 \end{Bmatrix} = f(x, y). \tag{2.1}$$

This is an equation of the mixed type. The equation refers to the first kind because the line of change of type $y = 0$ is not a characteristic of the equation.

By $W_2^l(\Omega) = H^l(\Omega)$ we denote the Sobolev space with the norm $\| \cdot \|_l$, $W_2^0(\Omega) = L_2(\Omega)$; $\Omega_1 = \Omega \cap \{y > 0\}$, $\Omega_2 = \Omega \cap \{y < 0\}$.

Consider the following nonlocal boundary value problem being the generalization of an analogue of the Frankl problem for the parabolic-hyperbolic equation (2.1). This problem was first formulated by Kal'menov and Sadybekov [9, 10] .

Problem F. Find a solution to (2.1) satisfying classical boundary conditions

$$u\big|_{AA_0} = 0, \quad u_y\big|_{A_0B_0} = 0, \tag{2.2}$$

and the non-local boundary condition

$$\alpha u(\theta_0(t)) + \beta u(\theta_1(t)) = \gamma u(\theta(t)), \quad 0 \le t \le 1, \tag{2.3}$$

where $\theta(t) = (t, 1)$, $\theta_0(t) = (t/2, -t/2)$, $\theta_1(t) = (\frac{t+1}{2}, \frac{t-1}{2})$; α, β and γ are given numbers.

It is easy to see that $\theta(t) \in A_0B_0$, $\theta_0(t) \in AC$, $\theta_1(t) \in BC$. Therefore the non-local boundary condition (2.3) binds with each other values of the sought-for solution on the parabolic part of the boundary A_0B_0 and on the hyperbolic parts of the boundary of the domain (at the characteristics AC and BC).

Note that for $\gamma = 0$ the boundary conditions in the hyperbolic part of the domain of the form

$$\alpha u(\theta_0(t)) + \beta u(\theta_1(t)) = 0$$

are well-known and called boundary conditions with displacement. They were first introduced by Nakhushev for a wave equation (see [15]). The particular case of Problem F for $\alpha + \beta = 2\gamma$ was considered in [3] and there the unique strong solvability of the problem is proved.

Definition 2.1. A function $u(x, y)$ from the class

$$u \in W = C^1(\overline{\Omega}) \cap C^{1,2}_{x,y}(\overline{\Omega}_1) \cap C^2(\overline{\Omega}_2),$$

satisfying the boundary conditions (2.2)–(2.3) of the problem and turning (2.1) into an identity we will call *a classical solution* to Problem F.

Definition 2.2. A function $u \in L_2(\Omega)$ we will call *a strong solution* to Problem F if there exists a sequence of functions $\{u_n\}$, $u_n \in W$ satisfying the boundary conditions (2.2)–(2.3) of the problem such that sequences u_n and Lu_n reduce in $L_2(\Omega)$ to the functions u and f, respectively.

Kal'menov and Sadybekov [9, 10] proved the unique strong solvability of the problem.

Theorem 2.3 ([9, 10]). *Let $\alpha + \beta \neq 0$. Then*

(a) For any function $f \in L_2(\Omega)$ there exists a unique strong solution $u(x, y)$ to Problem F. This solution belongs to the class $H^1(\Omega) \cap H^{1,2}_{x,y}(\Omega_1) \cap C(\overline{\Omega})$, and satisfies the inequality

$$\|u\|_1 \le C\|f\|_0. \tag{2.4}$$

(b) For any function $f \in C^1(\overline{\Omega})$, $f(A) = 0$, there exists a unique classical solution $u(x, y)$ to Problem F. This solution is stable in the norm

$$\|u\|_{C(\overline{\Omega})} \le C\|f\|_{C(\overline{\Omega})}. \tag{2.5}$$

By L denote a closure in $L_2(\Omega)$ of the differential operator given on functions $\{u_n\}$, $u_n \in W$ satisfying the boundary conditions (2.2)–(2.3). From item (a) of Theorem 2.3 follows that the operator L is invertible and L^{-1} is a compact operator. Therefore the spectrum of the operator L can consist of only eigenvalues. Naturally there arises a question on existence of eigenvalues of the operator L and, consequently, of Problem F.

Unlike the theory of solvability, spectral issues of problems for equations of the mixed type are less studied. The papers by Kal'menov [7, 8], Moiseev [11], Ponomarev [17] have made a sifnificant contribution to this direction. The main bibliography on these issues is given in the monograph of Moiseev [12].

Note that for $\beta = \gamma = 0$ Problem F coincides with the Tricomi problem, and for $\alpha = \gamma = 0$ it coincides with the Tricomi problem with data on an opposite characteristics. The strong solvability of particular cases of the problem for $\alpha = \gamma = 0$ and for $\beta = \gamma = 0$ has been researched in paper by Sadybekov, Toizhanova (Dildabek) [25]. It is shown that for $\beta = \gamma = 0$ the problem is Volterra, and for $\alpha = \gamma = 0$ the problem has an eigenvalue. This method was used in [2] for proving the Volterra property of some problems with the Bitsadze-Samarskii-type conditions for a mixed parabolic-hyperbolic equation.

In the next section we present another particular case of Problem F for $\gamma \neq 0$ which does not have eigenvalues. In virtue of compactness of the operator L^{-1} it will mean that L^{-1} is a Volterra operator. Thus Problem F in this case is Volterra.

3. Absence of eigenvalues

Consider a particular case of Problem F, when $\beta = 0$. In virtue of the condition $\alpha + \beta \neq 0$ from Theorem 2.3 one can consider that $\alpha = 1$.

Problem F_0. Find a solution to (2.1) satisfying the boundary conditions

$$u\big|_{AA_0} = 0, \quad u_y\big|_{A_0B_0} = 0, \tag{3.1}$$

$$u(\theta_0(t)) = \gamma u(\theta(t)), \quad 0 \le t \le 1, \tag{3.2}$$

where $\theta(t) = (t, 1)$, $\theta_0(t) = (t/2, -t/2)$, γ are given numbers.

For proving the Volterra property of Problem F_0 we need to obtain a representation of the inverse operator L^{-1}. A part of the following theorem can be obtained, as a particular case, from [9, 10]. But we prove this result anew because it is important to obtain a form of solution in the integral form.

Theorem 3.1. *For any function $f \in L_2(\Omega)$ there exists a unique strong solution $u(x, y)$ to Problem F_0. This solution belongs to the class $H^1(\Omega) \cap H^{1,2}_{x,y}(\Omega_1) \cap C(\overline{\Omega})$, satisfies inequality (2.4) and can be represented in the form*

$$u(x, y) = \int_\Omega K(x, y; x_1, y_1) f(x_1, y_1) dx_1 dy_1, \tag{3.3}$$

where $K \in L_2(\Omega \times \Omega)$.

Proof. The proof will be given in several stages.

3.1. **Reducing to an integral equation.** At first let $f \in C^1(\overline{\Omega})$, $f(A) = 0$. By the unique solvability of the Cauchy problem for a wave equation, the solution to (2.1) for $y < 0$ is represented according to the d'Alembert formula

$$u(x, y) = -\int_\xi^\eta d\xi_1 \int_{\xi_1}^\eta f_1(\xi_1, \eta_1) d\eta_1 + \frac{1}{2}[\tau(\xi) + \tau(\eta)] - \frac{1}{2}\int_\xi^\eta \nu(s) ds, \tag{3.4}$$

where

$$\tau(x) = u(x, 0), \quad \tau(0) = 0, \quad \xi = x + y, \quad \eta = x - y,$$

$$\nu(x) = \frac{\partial u}{\partial y}(x, 0), \quad f_1(\xi, \eta) = \frac{1}{4} f(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}).$$

Hence, taking into account $\tau(0) = 0$, by direct calculation, we obtain

$$u(\theta_0(t)) = \frac{1}{2}\tau(t) - \frac{1}{2}\int_0^t \nu(s) ds - \int_0^t d\xi_1 \int_{\xi_1}^t f_1(\xi_1, \eta_1) d\eta_1. \tag{3.5}$$

This is the basic relation for $\tau(t)$ and $\nu(t)$ obtained from the hyperbolic part of the domain.

In the parabolic part of the domain we consider a problem with mixed boundary condition:

Find in Ω_1 a solution to the heat equation

$$u_x - u_{yy} = f(x,y), \tag{3.6}$$

satisfying the homogeneous initial-boundary conditions (2.2) and non-homogeneous boundary condition

$$u(x,0) = \tau(x), \quad 0 \le x \le 1. \tag{3.7}$$

It is evident that the natural condition of the sequence $\tau(0) = 0$ is a necessary condition of the solution existence. Further we will assume that this condition holds.

Considering that the function $\tau(x)$ is well-known, we calculate $\nu(x) = \frac{\partial u}{\partial y}(x,0)$.

This is a mixed initial-boundary value problem for the heat equation. Its Green's function has the form [1, p. 198]:

$$\begin{aligned} G(x,y,y_1) == \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{2\sqrt{\pi x}} \Big[\exp\big\{ -\frac{(y-y_1+2n)^2}{4x} \big\} \\ - \exp\big\{ -\frac{(y+y_1+2n)^2}{4x} \big\} \Big]. \end{aligned} \tag{3.8}$$

Therefore for the solution to problem (3.6), (2.2), (3.7) we have the representation

$$\begin{aligned} u(x,y) = \int_0^x dx_1 \int_0^1 G(x-x_1,y,y_1) f(x_1,y_1) dy_1 \\ + \int_0^x G_{y_1}(x-s,y,0)\tau(s) ds. \end{aligned} \tag{3.9}$$

Hence for $y \to 1$ we find

$$\begin{aligned} u(\theta(t)) = \int_0^t dx_1 \int_0^1 G(t-x_1,1,y_1) f(x_1,y_1) dy_1 \\ + \int_0^t G_{y_1}(t-s,1,0)\tau(s) ds, \end{aligned} \tag{3.10}$$

and differentiating with respect to y and letting $y \to 0$, we obtain

$$\nu(x) = \frac{\partial}{\partial y} \int_0^x G_{y_1}(x-s,y,0)\tau(s) ds\big|_{y=0} + \Phi_1(x), \tag{3.11}$$

where

$$\Phi_1(x) = \frac{\partial}{\partial y} \int_0^x dx_1 \int_0^1 G(x-x_1,y,y_1) f(x_1,y_1) dy_1\big|_{y=0}.$$

Formulas (3.10) and (3.11) give the basic relation for $\tau(t)$ and $\nu(t)$ obtained from the parabolic part of the domain.

Substituting (3.5) and (3.10) into the boundary condition (2.3), after differentiating, taking into account $\tau(0) = 0$, we obtain

$$\tau'(t) - \nu(t) - 2\gamma \int_0^t G_{y_1}(t-s,1,0)\tau'(s) ds = 2F(t), \tag{3.12}$$

where $F(t)=F_1(t)+F_2(t)$, with

$$F_1(t)=\gamma\frac{d}{dt}\int_0^t dx_1\int_0^1 G(t-x_1,1,y_1)f(x_1,y_1)dy_1,$$

$$F_2(t)=\int_0^t f_1(\xi_1,t)d\xi_1.$$

We transform the first summand in the right-hand part of (3.11). For this purpose, taking into account $\tau(0)=0$ and the explicit form of the Green's function (3.8), by integrating by parts, we transform

$$\int_0^t G_{y_1}(t-s,y,0)\tau(s)ds=\int_0^t G_1(t-s,y)\tau'(s)ds,$$

where

$$G_1(t-s,y)=-\frac{2}{\sqrt{\pi}}\sum_{n=-\infty}^{-1}(-1)^n\int_{-\infty}^{\frac{y+n}{2\sqrt{(t-s)}}}e^{-z^2}dz+\frac{2}{\sqrt{\pi}}\sum_{n=0}^{+\infty}(-1)^n\int_{\frac{y+n}{2\sqrt{(t-s)}}}^{+\infty}e^{-z^2}dz.$$

Hence it is easy to obtain that

$$\frac{\partial}{\partial y}\int_0^t G_{y_1}(t-s,y,0)\tau(s)ds\Big|_{\,y=0}=-\int_0^t k(t-s)\tau'(s)ds,$$

where

$$k(t-s)=\frac{1}{\sqrt{\pi(t-s)}}\sum_{n=-\infty}^{+\infty}(-1)^n\exp\Big\{-\frac{n^2}{4(t-s)}\Big\},$$

and formula (3.11) takes the form

$$\nu(t)=-\int_0^t k(t-s)\tau'(s)ds+\Phi_1(t). \tag{3.13}$$

Substituting the obtained result from (3.13) into (3.12), we obtain the integral equation

$$\tau'(t)+\int_0^t k(t-s)\tau'(s)ds-2\gamma\int_0^t G_{y_1}(t-s,1,0)\tau'(s)ds=2\Phi(t), \tag{3.14}$$

where

$$\Phi(t)=F(t)+\frac{1}{2}\Phi_1(t). \tag{3.15}$$

3.2. **Constructing a solution to the problem.** It is easy to see that $k(t-s)$ is a kernel with a weak polar peculiarity, and the function $G_{y_1}(t-s,1,0)$ is infinitely continuously differentiable for all $s\le t\le 1$. Therefore (3.14) is an integral Volterra equation of the second kind

$$\tau'(t)-\int_0^t k_1(t-s)\tau'(s)ds=2\Phi(t), \tag{3.16}$$

where

$$k_1(t-s)=-k(t-s)+2\gamma G_{y_1}(t-s,1,0), \tag{3.17}$$

$$\Phi(t)=\gamma\frac{d}{dt}\int_0^t dx_1\int_0^1 G(t-x_1,1,y_1)f(x_1,y_1)dy_1+\int_0^t f_1(\xi_1,t)d\xi_1$$
$$+\Big(\frac{\partial}{\partial y}\int_0^t dx_1\int_0^1 G(t-x_1,y,y_1)f(x_1,y_1)dy_1\Big)\Big|_{y=0},$$

which always has a unique solution. It is easy to see that the kernel $K(x-t)$ is a kernel with weak peculiarity. Therefore there exists the unique strong solution to (3.16) and has the form

$$\tau'(t) = 2\Phi(t) + 2\int_0^t \Gamma(t-s)\Phi(s)ds, \tag{3.18}$$

where $\Gamma(t)$ is a resolvent of (3.16):

$$\Gamma(t) = \sum_{j=1}^{\infty} k_j(t), \quad k_{j+1}(t) = \int_0^t k_1(t-s)k_j(s)ds, \quad j \in N.$$

And the smoothness of this solution depends on the class to which $\Phi(t)$ belongs.

Lemma 3.2 ([9, 10])**.** *Let $f \in C^1(\overline{\Omega})$, $f(A) = 0$, then $\Phi(t) \in C^1[0,1]$ and satisfies estimates*

$$\|\Phi(t)\|_{C[0,1]} \le C\|f\|_{C(\overline{\Omega})}, \tag{3.19}$$

$$\|\Phi(t)\|_{L_2(0,1)} \le C\|f\|_0. \tag{3.20}$$

A proof of the above lemma is obtained by direct calculations, estimating each of summands (3.15) [9] and [10].

On the basis of this lemma there always exists a unique solution $\tau'(t)$ to (3.14). This solution (depending on the smoothness of $\Phi(t)$) belongs to the class $\tau'(t) \in C^1[0,1]$ or $\tau'(t) \in L_2(0,1)$ and by (3.19) and (3.20), it satisfies

$$\|\tau'(t)\|_{C[0,1]} \le C\|f\|_{C(\overline{\Omega})}, \tag{3.21}$$

or

$$\|\tau'(t)\|_{L_2(0,1)} \le C\|f\|_0. \tag{3.22}$$

Taking into account $\tau(0) = 0$, we find a unique $\tau(t)$.

Now the solution to Problem F is reestablished in Ω_1 as a solution to the first initial-boundary value problem by formula (3.9). We find the value of $\nu(x)$ from (3.13). Therefore in the domain Ω_2 the solution to Problem F is uniquely reestablished as the solution to the Cauchy problem by the d'Alembert formula (3.4).

From the solution properties of the first initial-boundary value problem for the heat equation it follows that the solution to Problem F belongs to the classes of smoothness indicated in the Theorem and (by inequalities (3.21) and (3.22)) satisfies estimates (2.4) and (2.5).

Let us show that for $f \in L_2(\Omega)$ the found solution is strong. Since $C_0^1(\overline{\Omega})$ is dense in $L_2(\Omega)$, then for any function $f \in L_2(\Omega)$ there exists a sequence of functions $f_n \in C_0^1(\overline{\Omega})$ such that $\|f_n - f\| \to 0$, $n \to \infty$. By u_n we denote the classical solution to Problem F when the right-hand part is f_n. Such solution exists by virtue of the above-mentioned proof of the theorem and $u_n \in W$ for all $f_n \in C_0^1(\overline{\Omega})$.

By inequality (2.4) we have

$$\|u_n - u\|_1 \le c\|f_n - f\|_0 \to 0, \quad n \to \infty.$$

Consequently, $\{u_n\}$ is the sequence corresponding to the definition of strong solution. Therefore Problem F is strongly solvable for any right-hand part f, and the strong solution belongs to the class $H^1(\Omega) \cap H^{1,2}_{x,y}(\Omega_1) \cap C(\overline{\Omega})$. The existence and uniqueness of the strong solution of Problem F is proved.

Let us obtain now a solution in the form (3.3). From (3.18), taking into account $\tau(0)=0$, after simple transformations, we obtain

$$\tau(x)=\int_0^x \Gamma_1(x-t)\Phi(t)dt, \tag{3.23}$$

where

$$\Gamma_1(x)=2+2\int_0^x \Gamma(t)dt.$$

Substituting the value $\Phi(t)$ into (3.23), after evident transformations, we come to the form

$$\begin{aligned}\tau(x)&=\int_0^x d\xi_1\int_{\xi_1}^x \Gamma_1(x-\xi_1)f_1(\xi_1,\eta_1)d\eta_1\\&\quad+2\gamma\int_0^x dx_1\int_0^1 G(x-x_1,1,y_1)f(x_1,y_1)dy_1\\&\quad+\gamma\int_0^x dx_1\int_0^1\Big(\int_{x_1}^x \Gamma_1(x-t)G(t-x_1,1,y_1)dt\Big)f(x_1,y_1)dy_1\\&\quad+\frac{1}{2}\int_0^x dx_1\int_0^1\Big(\int_{x_1}^x \Gamma_1(x-t)\big(\frac{\partial}{\partial y}G(t-x_1,y,y_1)\big)\big|_{y=0}\,dt\Big)f(x_1,y_1)dy_1.\end{aligned}$$

Substituting this quantity into (3.7) and into (3.13), we obtain formula (3.3), where the detailed form of the kernel $K(x,y;x_1,y_1)$ can be written in the explicit form. We will not show this form here due to its bulkiness.

From the analysis of the kernel representation it is easy to see that

$$K(x,y;x_1,y_1)\in L_2(\Omega\times\Omega).$$

Herewith it is easy to see that the kernel $K(x,y;x_1,y_1)$ can be represented in the form

$$\begin{aligned}K(x,y;x_1,y_1)&=\theta(y)\Big\{\theta(y_1)\theta(x-x_1)G_{11}(x-x_1,y,y_1)\\&\quad+\theta(-y_1)\theta(x-\eta_1)G_{12}(x-x_1,y,y_1)\Big\}\\&\quad+\theta(-y)\Big\{\theta(y_1)\theta(\xi-x_1)G_{21}(x-x_1,y,y_1)\\&\quad+\theta(-y_1)\theta(\eta-\eta_1)G_{22}(x-x_1,y,y_1)\Big\},\end{aligned} \tag{3.24}$$

where $G_{kn}(x-x_1,y,y_1)\in L_2(\Omega_k\times\Omega_n)$, $k,n=1,2$; $\theta(\cdot)$ is a Heaviside step function. The proof is complete. □

3.3. **Theorem on absence of eigenvalues.**

Theorem 3.3. *The inverse operator L^{-1} of Problem F_0 defined by* (3.3) *is Volterra (that is, compact and quasinilpotent).*

Proof. Since $K(x,y;x_1,y_1)\in L_2(\Omega\times\Omega)$, the operator L^{-1} is a Gilbert-Schmidt operator. Consequently, it is compact. For proving the theorem it is sufficient to show that the operator L^{-1} does not have eigenvalues. We need the following definitions and theorem from [16].

Definition 3.4. Let $S\subset\Omega\times\Omega$. The kernel $K(P_1,P_2)$ is called S-kernel, if $K(P_1,P_2)\in L_2(\Omega\times\Omega)$ and $K(P_1,P_2)=0$ for $(P_1,P_2)\in S$.

Definition 3.5. The open set $S \subset \Omega \times \Omega$ is called a set of type V, if any S-kernel does not have eigenvalues.

As in [16], we introduce the notation: $P_1 \overset{S}{\to} P_2$, if $(P_1, P_2) \in S$, and $P_1 \overset{S}{\leftarrow} P_2$, if $(P_1, P_2) \notin S$.

Theorem 3.6 ([16]). *For the set S to be a set of type V it is necessary and sufficient that for any $k \geq 1$ from conditions*

$$P_1 \overset{S}{\to} P_2 \overset{S}{\to} P_3 \overset{S}{\to} \dots \overset{S}{\to} P_k \tag{3.25}$$

it follows that $P_k \overset{S}{\leftarrow} P_1$.

We use this theorem for our operator L^{-1}. Consider the sequence of points $P_i = (x_i, y_i)$, $i = \overline{1, k}$. Let the condition (3.25) hold for any $k \geq 1$. Then for $i < j$ the relation

$$x_i > x_j, \quad i < j. \tag{3.26}$$

follows from the condition $(P_i, P_j) \in S$.

Let us prove that the condition $(P_k, P_1) \notin S$ follows from (3.25). Using the explicit form (3.24) of the kernel $K(x_k, y_k; x_1, y_1)$, by direct calculation we make sure that the fulfillment of the condition (3.26) is sufficient for

$$K(x_k, y_k; x_1, y_1) = 0, \quad \text{if } (P_k, P_1) \notin S.$$

Consequently, (3.24) defines S-kernel not having eigenvalues, and the operator L^{-1} is Volterra. The proof is complete. □

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References

[1] V. M. Babich et al., S. G. Mihlin Ed.; *Linear equations of mathematical physics. Ref. Math. Library*, Nauka, Moscow (1964) (in Russian)

[2] A. S. Berdyshev; The Volterra property of some problems with the Bitsadze-Samarskii-type conditions for a mixed parabolic-hyperbolic equation, *Siberian Mathematical Journal*, 46 (2005), 386–395.

[3] G. Dildabek; On a new nonlocal boundary value problem for an equation of the mixed parabolic-hyperbolic type, *AIP Conference Proceedings*, 1789 (2016), 040018.

[4] G. Dildabek, M. A. Sadybekov, M. B. Saprygina; On a Volterra property of an problem of the Frankl type for an equation of the mixed parabolic-hyperbolic type, *AIP Conference Proceedings*, 1910 (2017), 040004.

[5] F. I. Frankl; On Chaplygin's problems for mixed subsonic and supersonic flows, *Izv. Akad. Nauk SSSR Ser. Mat.*, 9 (1945), 121–142.

[6] F.I . Frankl; Flow around airfoils by a stream of subsonic velocity with supersonic zones terminating in a straight-line condensation shock, *Prikl. Mat. Mekh.*, 20 (1956), 196-202.

[7] T. Sh. Kal'menov; The spectrum of the Tricomi problem for the Lavrent'ev-Bicadze equation, *Differential equations*, 13 (1977), 984–989.

[8] T. Sh. Kal'menov; The spectrum of the Tricomi problem for a fourth-order equation of mixed type, *Differential equations*, 15 (1979), 248–250.

[9] T. S. Kalmenov, M. Sadybekov; On a problem of the Frankl type for an equation of the mixed parabolic-hyperbolic type, *AIP Conference Proceedings*, 1759 (2016), 020001.

[10] T. S. Kal'menov, M. Sadybekov; On a Frankl-type problem for a mixed parabolic-hyperbolic equation, *Siberian Mathematical Journal*, 58 (2017), 227–231.

[11] E. I. Moiseev; Properties of solution of Lavrent'ev-Bitsadze equation, *Mathematical Notes*, 26 (1979), 757–762.

[12] E. I. Moiseev; *Equations of Mixed Type with a Spectral Parameter*, Moscow Univ., Moscow (1988) (in Russian)

[13] E. I. Moiseev, D. A. Gulyaev; The completeness of the eigenfunctions of the Tricomi problem for the Lavrent'ev-Bitsadze equation with the Frankl gluing condition, *Integral Transforms and Special Functions*, 27 (2016), 893–898.

[14] E. I. Moiseev, P. V. Nefedov, A. A. Kholomeeva; Analogs of the Tricomi and Frankl problems for the Lavrent'ev-Bitsadze equation in three-dimensional domains, *Differential Equations*, 50 (2014), 1677–1680.

[15] A. M. Nakhushev; *Problems with displacements for partial differential equations*, Nauka, Moscow (2006) (in Russian)

[16] A. B. Nersesyan; To the theory of integral equations of Volterra type, *Dokl. Akad. Nauk SSSR*, 155 (1964), 1049–1051.

[17] S. M. Ponomarev; Eigenvalue problem for Lavrent'ev-Bitsadze equation, *Doklady Akademii nauk SSSR*, 233 (1977), 39–40.

[18] S. P. Pul'kin, M. E. Lerner; Uniqueness of a solution of problems with Frankl' and Tricomi conditions for the general Lavrent'ev-Bicadze equation *Differential Equations*, 2 (1966), 1255–1263.

[19] K. B. Sabitov; Nonlocal problem for a parabolic-hyperbolic equation in a rectangular domain, *Mathematical Notes*, 89 (2011), 562–567.

[20] K. B. Sabitov; On the theory of the Frankl problem for equations of mixed type, *Izvestiya Mathematics*, 81 (2017), 99–136.

[21] K. B. Sabitov, Y. K. Sabitova; Frankl spectral problem for an operator of mixed type with arbitrary power-law degeneration, *Differential Equations*, 51 (2015), 1145–1156.

[22] Y. K. Sabitova; On location of the spectrum of the Tricomi problem, *Siberian Mathematical Journal*, 56 (2015), 160–176.

[23] M. A. Sadybekov, G. Dildabek, A. Tengayeva; Constructing a basis from systems of eigenfunctions of one not strengthened regular boundary value problem, *Filomat*, 31 (2017), 981–987.

[24] M. Sadybekov, G. Oralsyn Nonlocal initial boundary value problem for the time-fractional diffusion equation *Electronic Journal of Differential Equations*, 2017 (2017), No. 201, 1–7.

[25] M. A. Sadybekov, G. D. Toizhanova; Spectral properties of a class of boundary value problems for a parabolic-hyperbolic equation, *Differential equations*, 28 (1992), 176–179.

[26] M. A. Sadybekov, B.T. Torebek, B. Kh. Turmetov; Representation of the Green's function of the exterior Neumann problem for the Laplace operator, *Siberian Mathematical Journal*, 58 (2017), 153–158.

[27] M. A. Sadybekov, B. Kh. Turmetov, B. T. Torebek; Solvability of nonlocal boundary-value problems for the Laplace equation in the ball, *Electronic Journal of Differential Equations*, 2014 (2014), 1–14.

[28] A. Tengayeva, G. Dildabek; Existence of eigenvalues of problem with shift for an equation of parabolic-hyperbolic type, *AIP Conference Proceedings*, 1759 (2016), 020146.

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