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MULTIPLE SOLUTIONS FOR PERTURBED KIRCHHOFF-TYPE NON-HOMOGENEOUS NEUMANN PROBLEMS THROUGH ORLICZ-SOBOLEV SPACES

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Abstract. We establish the existence of three distinct weak solutions for perturbed Kirchhoff-type non-homogeneous Neumann problems, under suitable assumptions on the nonlinear terms. Our approach is based on recent variational methods for smooth functionals defined on Orlicz-Sobolev spaces.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, ν be the outer unit normal to $\partial\Omega$, $K : [0, +\infty) \to \mathbb{R}$ be a nondecreasing continuous function such that there exist two positive numbers m and M, with $m \leq K(t) \leq M$ for all $t \geq 0$, and $\alpha : (0, \infty) \to \mathbb{R}$ be such that the mapping $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) = \begin{cases} \alpha(|t|)t, & \text{for } t \neq 0, \\ 0, & \text{for } t = 0 \end{cases}$$

is an odd, strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. For the function φ above, let us define

$$\Phi(t) = \int_0^t \varphi(s)\, ds \quad \text{for all } t \in \mathbb{R},$$

on which will be imposed some suitable assumptions later.

Consider the perturbed Kirchhoff-type non-homogeneous Neumann problem

$$\begin{gathered} K\Big(\int_\Omega [\Phi(|\nabla u|) + \Phi(|u|)]dx\Big)\Big(- \operatorname{div}(\alpha(|\nabla u|)\nabla u) + \alpha(|u|)u\Big) \\ = \lambda f(x, u) + \mu g(x, u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where $f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions, $\lambda > 0$ and $\mu \geq 0$ are two parameters.

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It should be mentioned that if $\varphi(t) = p|t|^{p-2}t$, then problem (1.1) becomes the well-known p-Kirchhoff-type Neumann problem

$$\begin{gathered} K\Big(\int_\Omega (|\nabla u|^p + |u|^p)\,dx\Big)\Big(-\Delta_p u + |u|^{p-2}u\Big) = \lambda f(x,u) + \mu g(x,u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

Problem (1.2) is related to the stationary problem

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.3}$$

for $0 < x < L$, $t \geq 0$, where $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E the Young modulus, ρ the mass density, h the cross-section area, L the length and ρ_0 the initial axial tension, proposed by Kirchhoff [35] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. The Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. Some interesting results can be found, for example in [19]. On the other hand, Kirchhoff-type boundary value problems model several physical and biological systems where u describes a process which depend on the average of itself, as for example, the population density. We refer the reader to [5, 31, 48] for some related works. Molica Bisci and Rădulescu [44], applying mountain pass results, studied the existence of solutions to nonlocal equations involving the p-Laplacian. More precisely, they proved the existence of at least one nontrivial weak solution, and under additional assumptions, the existence of infinitely many weak solutions. The existence and multiplicity of stationary higher order problems of Kirchhoff type (in n-dimensional domains, $n \geq 1$) were also treated in some recent papers, via variational methods like the symmetric mountain pass theorem in [23] and via a three critical point theorem in [8]. Moreover, in [7, 6] some evolutionary higher order Kirchhoff problems were treated, mainly focusing on the qualitative properties of the solutions.

In recent years, multiplicity results for Kirchhoff-type elliptic partial differential equations involving the p-Laplacian have been investigated, for instance see [24]. In this paper we consider more general problems, which involve non-homogeneous differential operators. Problems of this type have been intensively studied in the last few years, due to numerous and relevant applications in many fields of mathematics, such as approximation theory, mathematical physics (electrorheological fluids), calculus of variations, nonlinear potential theory, the theory of quasi-conformalmappings, differential geometry, geometric function theory, probability theory and image processing (for instance see [18, 27, 34, 38, 49, 52]). The study of nonlinear elliptic equations involving quasilinear homogeneous type operators is based on the theory of Sobolev spaces $W^{m,p}(\Omega)$ in order to find weak solutions. In the case of non-homogeneous differential operators, the natural setting for this approach is the use of Orlicz-Sobolev spaces. These spaces consists of functions that have weak derivatives and satisfy certain integrability conditions. Many properties of Orlicz-Sobolev spaces come in [1, 26, 28, 29]. Due to these, many researchers have studied the existence of solutions for the eigenvalue problems involving non-homogeneous operators in the divergence form through Orlicz-Sobolev spaces by means of variational methods and critical point theory, monotone operator methods, fixed point theory and degree theory (for instance, see

[2, 3, 4, 13, 14, 15, 17, 20, 21, 22, 30, 33, 36, 40, 41, 42, 50]). For example, Clément et al. [21] discussed the existence of weak solutions in an Orlicz-Sobolev space to the Dirichlet problem

$$\begin{gathered} -\operatorname{div}(\alpha(|\nabla u(x)|)\nabla u(x)) = g(x, u(x)) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, and the function $\varphi(s) = sa(|s|)$ is an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. Under appropriate conditions on φ, g and the Orlicz-Sobolev conjugate Φ^* of $\Phi(s) = \int_0^s \varphi(t)dt$, they investigated the existence of non-trivial solutions of mountain pass type. Moreover Clément et al. in [22] employed Orlicz-Sobolev spaces theory and a variant of the Mountain Pass Lemma of Ambrosetti-Rabinowitz to obtain the existence of a (positive) solution to a semi-linear system of elliptic equations. In addition, by an interpolation theorem of Boyd they found an elliptic regularity result in Orlicz-Sobolev spaces. Halidias and Le in [33] by Brezis-Nirenberg's local linking theorem, investigated the existence of multiple solutions for problem (1.4). Mihăilescu and Rădulescu in [40] by adequate variational methods in Orlicz-Sobolev spaces studied the boundary value problem

$$\begin{gathered} -\operatorname{div}(\log(1 + |\nabla u|^q)|\nabla u|^{p-2}\nabla u) = f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary. They distinguished the cases where either $f(u) = -\lambda|u|^{p-2}u + |u|^{r-2}u$ or $f(u) = \lambda|u|^{p-2}u - |u|^{r-2}u$, with p, $q > 1$, $p + q < \min\{N, r\}$, and $r < (Np - N + p)/(N - p)$. In the first case they showed the existence of infinitely many weak solutions for any $\lambda > 0$ and in the second case they proved the existence of a non-trivial weak solution if λ is sufficiently large. Kristály et al. in [36] by using a recent variational principle of Ricceri, ensured the existence of at least two non-trivial solutions for problem (1.1) in the case $K(t) = 1$ for all $t \geq 0$ and $\mu = 0$, in the Orlicz-Sobolev space $W^1L_\Phi(\Omega)$, while Mihăilescu and Repovš in [42] by combining Orlicz-Sobolev spaces theory with adequate variational methods and a variant of Mountain Pass Lemma established the existence of at least two non-negative and non-trivial weak solutions for the problem

$$\begin{gathered} -\operatorname{div}(\alpha(|\nabla u(x)|)\nabla u(x)) = \lambda f(x, u(x)) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

where α is the same with in problem (1.1), $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and λ is a positive parameter. In [15] Bonanno et al. based on variational methods discussed the existence of infinitely many solutions that converge to zero in the Orlicz-Sobolev space $W^1L_\Phi(\Omega)$ for problem (1.1) in the case $K(t) = 1$ for all $t \geq 0$ and $\mu = 0$, and in [14] they also established a multiplicity result for (1.1). They exploited a recent critical points result for differentiable functionals in order to prove the existence of a determined open interval of positive eigenvalues for which the same problem admits at least three weak solutions in the Orlicz-Sobolev space $W^1L_\Phi(\Omega)$, while in [13] using variational methods, under an appropriate oscillating behavior of the nonlinear term, proved the existence of a determined open interval of positive parameters for which the same problem admits infinitely many weak solutions that strongly converges to zero, in the same Orlicz-Sobolev space. In [20] the author using a three critical points theorem due to Ricceri obtained a multiplicity result for a class of Kirchhoff-type Dirichlet problems in Orlicz-Sobolev spaces. In [3] employing variational methods and critical point theory, in an appropriate Orlicz-Sobolev setting, the existence of infinitely many solutions for Steklov problems associated to non-homogeneous differential operators was established.

Mihăilescu and Rădulescu [39] considered the boundary value problem

$$\begin{gathered} -\operatorname{div}\left((a_1(|\nabla u|) + a_2(|\nabla u|)\nabla u\right) = \lambda |u|^{q(x)-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.5}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary, λ is a positive real number, q is a continuous function and a_1, a_2 are two mappings such that $a_1(|t|)t$, $a_2(|t|)t$ are increasing homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$. They established the existence of two positive constants λ_0 and λ_1 with $\lambda_0 \leq \lambda_1$ such that any $\lambda \in [\lambda_1, \infty)$ is an eigenvalue, while any $\lambda \in (0, \lambda_1)$ is not an eigenvalue of problem (1.5).

Molica Bisci and Rădulescu [43], by using an abstract linking theorem for smooth functionals, established a multiplicity result on the existence of weak solutions for a nonlocal Neumann problem driven by a nonhomogeneous elliptic differential operator. We also refer the reader to [45, 46, 47] in which nonlinear problems with variable exponents were studied.

Motivated by the above facts, in the present paper, employing two kinds of three critical points theorems obtained in [9, 12] which we recall in the next section (Theorems 2.1 and 2.2), we ensure the existence of at least three weak solutions for problem (1.1); see Theorems 3.1 and 3.2. We also list some corollaries in which $K(t) = 1$ for all $t \geq 1$. We point out that our results extend in several directions previous works by relaxing some hypotheses and sharpening the conclusions (see [10, 11, 14]).

To the best of our knowledge, there are just a few contributions to the study of Kirchhoff Neumann problems in Orlicz-Sobolev spaces.

This article is arranged as follows. In Section 2 we present some preliminary knowledge on the Orlicz-Sobolev spaces, while Section 3 is devoted to the existence of multiple weak solutions for problem (1.1).

2. Preliminaries

Our main tools are the following three critical point theorems. In the first one the coercivity of the functional $\Phi - \lambda\Psi$ is required, in the second one a suitable sign hypothesis is assumed.

Theorem 2.1 ([12, Theorem 2.6])**.** *Let X be a reflexive real Banach space, $J : X \to \mathbb{R}$ be a coercive continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $I : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $J(0) = I(0) = 0$. Assume that there exist $r > 0$ and $\overline{v} \in X$, with $r < J(\overline{v})$ such that*

$$\frac{\sup_{J^{-1}(-\infty,r]} I(u)}{r} < \frac{I(\overline{v})}{J(\overline{v})}, \tag{2.1}$$

$$\begin{gathered}\text{for each } \lambda \in \Lambda_r := \Big]\frac{J(\overline{v})}{I(\overline{v})}, \frac{r}{\sup_{J^{-1}(-\infty,r]} I(u)}\Big[\\ \text{the functional } \Phi - \lambda\Psi \text{ is coercive.}\end{gathered} \tag{2.2}$$

Then, for each $\lambda \in \Lambda_r$ the functional $J - \lambda I$ has at least three distinct critical points in X.

Theorem 2.2 ([9, Theorem 3.3]). *Let X be a reflexive real Banach space, $J : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, $I : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

1. $\inf_X J = J(0) = I(0) = 0$;
2. *for each $\lambda > 0$ and for every $u_1, u_2 \in X$ which are local minima for the functional $J - \lambda I$ and such that $I(u_1) \geq 0$ and $I(u_2) \geq 0$, one has*
$$\inf_{s\in[0,1]} I(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are two positive constants r_1, r_2 and $\overline{v} \in X$, with $2r_1 < J(\overline{v}) < \frac{r_2}{2}$, such that

$$\frac{\sup_{u\in J^{-1}(-\infty,r_1)} I(u)}{r_1} < \frac{2}{3}\frac{I(\overline{v})}{J(\overline{v})}; \tag{2.3}$$

$$\frac{\sup_{u\in J^{-1}(-\infty,r_2)} I(u)}{r_2} < \frac{1}{3}\frac{I(\overline{v})}{J(\overline{v})}. \tag{2.4}$$

Then, for each λ in the interval

$$\Big]\frac{3}{2}\frac{J(\overline{v})}{I(\overline{v})},\ \min\Big\{\frac{r_1}{\sup_{u\in J^{-1}(-\infty,r_1)} I(u)},\ \frac{\frac{r_2}{2}}{\sup_{u\in J^{-1}(-\infty,r_2)} I(u)}\Big\}\Big[,$$

the functional $J - \lambda I$ has at least three critical points which lie in $J^{-1}(-\infty, r_2)$.

Theorems 2.1 and 2.2 have been successfully employed to establish the existence of at least three solutions for some boundary value problems in papers [25, 32].

To go further we introduce the functional space setting where problem (1.1) will be studied. In this context we note that the operator in the divergence form is not homogeneous and thus, we introduce an Orlicz-Sobolev space setting for problems of this type.

Let φ and Φ be as introduced at the beginning of the paper. Set

$$\Phi^\star(t) = \int_0^t \varphi^{-1}(s)\,ds, \quad \text{for all } t \in \mathbb{R}.$$

We observe that Φ is a Young function, that is, $\Phi(0) = 0$, Φ is convex, and

$$\lim_{t\to\infty} \Phi(t) = +\infty.$$

Furthermore, since $\Phi(t) = 0$ if and only if $t = 0$,

$$\lim_{t\to 0}\frac{\Phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty}\frac{\Phi(t)}{t} = +\infty,$$

then Φ is called an N-function. The function $\Phi^\star$ is called the complementary function of Φ and it satisfies

$$\Phi^\star(t) = \sup\{st - \Phi(s);\ s \geq 0\}, \quad \text{for all } t \geq 0\,.$$

We observe that $\Phi^\star$ is also an N-function and the following Young's inequality holds true:

$$st \leq \Phi(s) + \Phi^\star(t), \quad \text{for all } s, t \geq 0\,.$$

Assume that Φ satisfies the following structural hypotheses

$$1 < \liminf_{t\to\infty} \frac{t\varphi(t)}{\Phi(t)} \leq p^0 := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \infty; \tag{2.5}$$

$$N < p_0 := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \liminf_{t\to\infty} \frac{\log(\Phi(t))}{\log(t)}. \tag{2.6}$$

The Orlicz space $L_\Phi(\Omega)$ defined by the N-function Φ (see for instance [1] and [37]) is the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_\Phi} := \sup\Big\{ \int_\Omega u(x)v(x)\,dx : \int_\Omega \Phi^\star(|v(x)|)\,dx \leq 1\Big\} < \infty\,.$$

Then $(L_\Phi(\Omega), \|\cdot\|_{L_\Phi})$ is a Banach space whose norm is equivalent to the Luxemburg norm

$$\|u\|_\Phi := \inf\Big\{k > 0 : \int_\Omega \Phi\Big(\frac{u(x)}{k}\Big)\,dx \leq 1\Big\}.$$

We denote by $W^1L_\Phi(\Omega)$ the corresponding Orlicz-Sobolev space for problem (1.1), defined by

$$W^1L_\Phi(\Omega) = \Big\{u \in L_\Phi(\Omega) : \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega),\ i = 1, \dots, N\Big\}.$$

This is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} = \||\nabla u|\|_\Phi + \|u\|_\Phi,$$

see [1, 21].

As mentioned in [13, 15], Assumption (2.5) is equivalent with the fact that Φ and $\Phi^\star$ both satisfy the Δ_2 condition (at infinity), see [1, p. 232]. In particular, (Φ, Ω) and $(\Phi^\star, \Omega)$ are $\Delta-$regular, see [1, p. 232]. Consequently, the spaces $L_\Phi(\Omega)$ and $W^1L_\Phi(\Omega)$ are separable, reflexive Banach spaces, see [1, p. 241 and p. 247].

These spaces generalize the usual spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$, in which the role played by the convex mapping $t \mapsto |t|^p/p$ is assumed by a more general convex function $\Phi(t)$. We recall the following useful properties regarding the norms on Orlicz-Sobolev spaces.

Lemma 2.3 ([36, Lemma 2.2])**.** *On $W^1L_\Phi(\Omega)$ the three norms*

$$\begin{gathered}
\|u\|_{1,\Phi} = \||\nabla u|\|_\Phi + \|u\|_\Phi,\\
\|u\|_{2,\Phi} = \max\{\||\nabla u|\|_\Phi, \|u\|_\Phi\},\\
\|u\| = \inf\Big\{\mu > 0 : \int_\Omega \Big[\Phi\Big(\frac{|u(x)|}{\mu}\Big) + \Phi\Big(\frac{|\nabla u(x)|}{\mu}\Big)\Big]dx \leq 1\Big\},
\end{gathered}$$

are equivalent. More precisely, for every $u \in W^1L_\Phi(\Omega)$ we have

$$\|u\| \leq 2\|u\|_{2,\Phi} \leq 2\|u\|_{1,\Phi} \leq 4\|u\|.$$

The following lemma will be useful in what follows.

Lemma 2.4. *Let $u \in W^1L_\Phi(\Omega)$. Then the following conditions hold*

$$\begin{aligned}
&\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]dx \geq \|u\|^{p^0}, \quad \text{if } \|u\| < 1,\\
&\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]dx \geq \|u\|^{p_0}, \quad \text{if } \|u\| > 1,\\
&\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]dx \leq \|u\|^{p_0}, \quad \text{if } \|u\| < 1,\\
&\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]dx \leq \|u\|^{p^0}, \quad \text{if } \|u\| > 1.
\end{aligned}$$

Proof. The proof of the first two estimates can be carried out as in [36, Lemma 2.3]. Next, arguing as in [39, Lemma 1], assuming that $\|u\| < 1$ we may take $\beta \in (\|u\|, 1)$ and find that for any such β by [22, Lemma C.4-ii] respectively the definition of the Luxemburg-norm that

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]dx \leq \beta^{p_0} \int_\Omega [\Phi\Big(\frac{|u(x)|}{\beta}\Big) + \Phi\Big(\frac{|\nabla u(x)|}{\beta}\Big)]dx \leq \beta^{p_0}.$$

The third estimate in the lemma follows letting $\beta \searrow \|u\|$. For the last estimate in the lemma, for $u \in W^1L_\Phi(\Omega)$ with $\|u\| > 1$, since

$$\frac{\Phi(\sigma t)}{\Phi(t)} \leq \sigma^{p^0}, \quad \forall t > 0 \text{ and } \sigma > 1 \tag{2.7}$$

(see [41, (2.3)]), using the definition of the Luxemburg-norm we deduce

$$\begin{aligned}
&\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla(u(x)|)]\, dx\\
&= \int_\Omega \Big[\Phi\Big(\|u\|\frac{|u(x)|}{\|u\|}\Big) + \Phi\Big(\|u\|\frac{|\nabla(u(x))|}{\|u\|}\Big)\Big]\, dx\\
&\leq \|u\|^{p^0} \int_\Omega \Big[\Phi\Big(\frac{|u(x)|}{\|u\|}\Big) + \Phi\Big(\frac{|\nabla(u(x))|}{\|u\|}\Big)\Big]\, dx\\
&\leq \|u\|^{p^0}.
\end{aligned}$$

□

We also recall the following lemma which will be used in the proofs.

Lemma 2.5 ([14, Lemma 2.2])**.** *Let $u \in W^1L_\Phi(\Omega)$ and assume that*

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]dx \leq r,$$

for some $0 < r < 1$. Then $\|u\| < 1$.

The following lemma which will be used in the proof of Theorem 3.2.

Lemma 2.6. *Let $u \in W^1L_\Phi(\Omega)$ and assume that $\|u\| = 1$. Then*

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]dx = 1.$$

Proof. Arguing as in [16, Remark 2.1], in our hypothesis, there exists a sequence $\{u_n\} \subset W^1L_\Phi(\Omega)$ such that $u_n \to u$ in $W^1L_\Phi(\Omega)$ and $\|u_n\| > 1$ for every $n \in \mathbb{N}$. Using the second and the last estimates in Lemma 2.4 we have

$$\|u_n\|^{p_0} \le \int_\Omega [\Phi(|u_n(x)|) + \Phi(|\nabla u_n(x)|)]dx \le \|u_n\|^{p^0}.$$

Then

$$\lim_{n\to\infty} \int_\Omega [\Phi(|u_n(x)|) + \Phi(|\nabla u_n(x)|)]dx = 1.$$

Therefore, since the map $u \to \int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]dx$ is continuous, we have

$$\lim_{n\to\infty} \int_\Omega [\Phi(|u_n(x)|) + \Phi(|\nabla u_n(x)|)]dx = \int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]dx = 1,$$

and the conclusion is achieved. □

Now from hypothesis (2.6), by [21, Lemma D.2] it follows that $W^1L_\Phi(\Omega)$ is continuously embedded in $W^{1,p_0}(\Omega)$. On the other hand, since we assume $p_0 > N$ we deduce that $W^{1,p_0}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Thus, one has that $W^1L_\Phi(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ and there exists a constant $c > 0$ such that

$$\|u\|_\infty \le c\|u\|_{1,\Phi}, \quad \text{for all } u \in W^1L_\Phi(\Omega), \tag{2.8}$$

where $\|u\|_\infty := \sup_{x\in\overline{\Omega}} |u(x)|$. A concrete estimation of a concrete upper bound for the constant c remains an open question.

A function $u : \overline{\Omega} \to \mathbb{R}$ is a weak solution for problem (1.1) if

$$\begin{aligned}
&K\Big(\int_\Omega [\Phi(|\nabla u(x)|) + \Phi(|u(x)|)]dx\Big)\\
&\times \int_\Omega \Big(\alpha(|\nabla u(x)|)\nabla u(x)\cdot\nabla v(x) + \alpha(|u(x)|)u(x)v(x)\Big)dx\\
&- \lambda \int_\Omega f(x,u(x))v(x)\,dx - \mu \int_\Omega g(x,u(x))v(x)\,dx = 0,
\end{aligned}$$

for every $v \in W^1L_\Phi(\Omega)$.

We need the following proposition in the proof of our main results.

Proposition 2.7. *Let $T : W^1L_\Phi(\Omega) \to (W^1L_\Phi(\Omega))^*$ be the operator defined by*

$$\begin{aligned}
T(u)(v) = {}& K\Big(\int_\Omega [\Phi(|\nabla u(x)|) + \Phi(|u(x)|)]dx\Big)\\
&\times \int_\Omega \Big(\alpha(|\nabla u(x)|)\nabla u(x)\cdot\nabla v(x) + \alpha(|u(x)|)u(x)v(x)\Big)dx
\end{aligned}$$

for every $u, v \in (W^1L_\Phi(\Omega))^*$. Then, T admits a continuous inverse on the space $(W^1L_\Phi(\Omega))^*$, where $(W^1L_\Phi(\Omega))^*$ denotes the dual of $W^1L_\Phi(\Omega)$.

Proof. We will use [51, Theorem 26.A(d)]; namely, it is sufficient to verify that T is coercive, hemicontinuous and strictly convex in the sense of monotone operators. Since

$$p_0 \le \frac{t\varphi(t)}{\Phi(t)}, \quad \forall\, t > 0,$$

by Lemma 2.4 it is clear that for any $u \in X$ with $\|u\| > 1$ we have

$$\frac{T(u)(v)}{\|u\|} = K\Big(\int_\Omega [\Phi(|\nabla u(x)|) + \Phi(|u(x)|)]dx\Big)$$

$$
\begin{aligned}
&\times \int_\Omega \Big(\alpha(|\nabla u(x)|)|\nabla u(x)|^2 + \alpha(|u(x)|)|u(x)|^2\Big)\,dx/\|u\| \\
&\geq K\Big(\int_\Omega [\Phi(|\nabla u(x)|) + \Phi(|u(x)|)]dx\Big) \\
&\quad \times \int_\Omega [\Phi(|\nabla u(x)|) + \Phi(|u(x)|)]dx/\|u\| \\
&\geq \frac{m\|u\|^{2p_0}}{\|u\|} = m\|u\|^{2p_0-1}.
\end{aligned}
$$

Thus,

$$
\lim_{\|u\|\to\infty} \frac{T(u)(v)}{\|u\|} = \infty,
$$

i.e. T is coercive. The fact that T is hemicontinuous can be showed using standard arguments. Using the same arguments as given in the proof of [20, Theorem 2.2] we have that T is strictly convex, and that T is strictly monotone. Thus, by [51, Theorem 26.A(d)], there exists $T^{-1} : X^* \to X$. By a similar method as given in [20], one has that T^{-1} is continuous. $\square$

Corresponding to f, g and K we introduce the functions $F : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $G : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $\tilde{K} : [0, +\infty) \to \mathbb{R}$, respectively, as follows

$$
\begin{gathered}
F(x,t) := \int_0^t f(x,\xi)d\xi \quad \forall (x,t) \in \overline{\Omega} \times \mathbb{R}, \\
G(x,t) := \int_0^t g(x,\xi)d\xi \quad \forall (x,t) \in \overline{\Omega} \times \mathbb{R}, \\
\tilde{K}(t) := \int_0^t K(s)ds \quad \forall t \geq 0.
\end{gathered}
$$

Moreover, we set $G^\theta := \int_\Omega \max_{|t|\leq\theta} G(x,t)dt$ for every $\theta > 0$ and $G_\eta := \inf_{\overline{\Omega}\times[0,\eta]} G$ for every $\eta > 0$. If g is sign-changing, then $G^\theta \geq 0$ and $G_\eta \leq 0$.

3. Main results

To introduce our first result, fixing two positive constants θ and η such that

$$
\frac{\tilde{K}(\Phi(\eta)\,\mathrm{meas}(\Omega))}{\int_\Omega F(x,\eta)dx} < \frac{m\theta^{p^0}}{(2c)^{p^0}\int_\Omega \sup_{|t|\leq\theta} F(x,t)dx},
$$

and taking

$$
\lambda \in \Lambda_1 := \Big]\frac{\tilde{K}(\Phi(\eta)\,\mathrm{meas}(\Omega))}{\int_\Omega F(x,\eta)dx},\ \frac{m\theta^{p^0}}{(2c)^{p^0}\int_\Omega \sup_{|t|\leq\theta} F(x,t)dx}\Big[,
$$

set

$$
\begin{aligned}
\delta_{\lambda,g} \min\Big\{ &\frac{m\theta^{p^0} - (2c)^{p^0}\lambda\int_\Omega \sup_{|t|\leq\theta} F(x,t)dx}{(2c)^{p^0} G^\theta}, \\
&\frac{\tilde{K}(\Phi(\eta)\,\mathrm{meas}(\Omega)) - \lambda\int_\Omega F(x,\eta)dx}{G_\eta\,\mathrm{meas}(\Omega)}\Big\}
\end{aligned} \tag{3.1}
$$

and

$$
\overline{\delta}_{\lambda,g} := \min\Big\{\delta_{\lambda,g},\ \frac{1}{\max\big\{0, \frac{(2c)^{p^0}}{m}\limsup_{|t|\to\infty}\frac{\sup_{x\in\overline{\Omega}} G(x,t)}{t^{p_0}}\big\}}\Big\}, \tag{3.2}
$$

where we read $\rho/0 = +\infty$, so that, for instance, $\overline{\delta}_{\lambda,g} = +\infty$ when

$$\limsup_{|t|\to\infty} \frac{\sup_{x\in\overline{\Omega}} G(x,t)}{t^{p_0}} \le 0,$$

and $G_\eta = G^\theta = 0$. Now, we formulate our first main result.

Theorem 3.1. *Assume that there exist two positive constants θ and η with*

$$\theta < 2c \min\Big\{1, \Big(\frac{\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega))}{m}\Big)^{1/p^0}\Big\}$$

such that

$$\frac{\int_\Omega \sup_{|t|\le\theta} F(x,t)dx}{\theta^{p^0}} < \frac{m}{(2c)^{p^0}} \frac{\int_\Omega F(x,\eta)dx}{\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega))}; \tag{3.3}$$

$$\limsup_{|t|\to+\infty} \frac{\sup_{x\in\overline{\Omega}} F(x,t)}{t^{p_0}} \le 0. \tag{3.4}$$

Then, for each $\lambda \in \Lambda_1$ and for every L^1-Carathéodory function $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\limsup_{|t|\to\infty} \frac{\sup_{x\in\overline{\Omega}} G(x,t)}{t^{p_0}} < +\infty,$$

there exists $\overline{\delta}_{\lambda,g} > 0$ given by (3.2) *such that, for each $\mu \in [0, \overline{\delta}_{\lambda,g}[$, problem* (1.1) *possesses at least three distinct weak solutions in $W^1L_\Phi(\Omega)$.*

Proof. To apply Theorem 2.1 to our problem, we take $X := W^1L_\Phi(\Omega)$ and we introduce the functionals $J, I : X \to \mathbb{R}$ for each $u \in X$, as follows

$$J(u) = \tilde{K}\Big(\int_\Omega [\Phi(|\nabla u(x)|) + \Phi(|u(x)|)]dx\Big),$$

$$I(u) = \int_\Omega (F(x,u(x)) + \frac{\mu}{\lambda} G(x,u(x)))dx.$$

Let us prove that the functionals J and I satisfy the required conditions. It is well known that I is a differentiable functional whose differential at the point $u \in X$ is

$$I'(u)(v) = \int_\Omega (f(x,u(x)) + \frac{\mu}{\lambda} g(x,u(x)))v(x)dx,$$

for every $v \in X$. Moreover, $I' : X \to X^*$ is a compact operator. Indeed, it is enough to show that I' is strongly continuous on X. For this end, for fixed $u \in X$, let $u_n \to u$ weakly in X as $n \to \infty$, then u_n converges uniformly to u on $\overline{\Omega}$ as $n \to \infty$; see [51]. Since f, g are L^1-Carathéodory functions, f, g are continuous in $\mathbb{R}$ for every $x \in \overline{\Omega}$, so

$$f(x,u_n) + \frac{\mu}{\lambda} g(x,u_n) \to f(x,u) + \frac{\mu}{\lambda} g(x,u),$$

as $n \to \infty$. Hence $I'(u_n) \to I'(u)$ as $n \to \infty$. Thus we proved that I' is strongly continuous on X, which implies that I' is a compact operator by [51, Proposition 26.2]. Moreover, J is continuously differentiable whose differential at the point $u \in X$ is

$$J'(u)(v) = K\Big(\int_\Omega [\Phi(|\nabla u(x)|) + \Phi(|u(x)|)]dx\Big)$$

$$\times \int_\Omega \Big(\alpha(|\nabla u(x)|)\nabla u(x)\cdot\nabla v(x)+\alpha(|u(x)|)u(x)v(x)\Big)dx$$

for every $v\in X$. Since $m\le K(t)\le M$ for all $t\ge 0$, we have

$$m\int_\Omega[\Phi(|\nabla u(x)|)+\Phi(|u(x)|)]dx\le J(u)\le M\int_\Omega[\Phi(|\nabla u(x)|)+\Phi(|u(x)|)]dx. \quad (3.5)$$

From the left inequality in (3.5) and Lemma 2.4, we deduce that for any $u\in X$ with $\|u\|>1$ we have $J(u)\ge m\|u\|^{p_0}$ which follows $\lim_{\|u\|\to+\infty}J(u)=+\infty$, namely J is coercive. Moreover, J is sequentially weakly lower semicontinuous. Indeed, let $\{u_n\}\subset X$ be a sequence such that $u_n\to u$ weakly in X. By [39, Lemma 4.3], the the map $u\to\int_\Omega[\Phi(|u(x)|)+\Phi(|\nabla u(x)|)]dx$ is weakly lower semicontinuous, i.e.

$$\int_\Omega[\Phi(|\nabla u(x)|)+\Phi(|u(x)|)]dx\le\liminf_{n\to\infty}\int_\Omega(\Phi(|\nabla u_n(x)|)+\Phi(|u_n(x)|))dx. \quad (3.6)$$

From (3.6) and since $\tilde K$ is continuous and monotone, we have

$$\begin{aligned}\liminf_{n\to\infty}J(u_n)&=\liminf_{n\to\infty}\tilde K\Big(\int_\Omega(\Phi(|\nabla u_n(x)|)+\Phi(|u_n(x)|))dx\Big)\\&\ge\tilde K\Big(\liminf_{n\to\infty}\int_\Omega(\Phi(|\nabla u_n(x)|)+\Phi(|u_n(x)|))dx\Big)\\&\ge\tilde K\Big(\int_\Omega[\Phi(|\nabla u(x)|)+\Phi(|u(x)|)]dx\Big)\\&=J(u),\end{aligned}$$

namely, J is sequentially weakly lower semicontinuous. Furthermore, Proposition 2.7 gives that J' admits a continuous inverse on X^*. Put $r=m\left(\frac{\theta}{2c}\right)^{p^0}$ and $w(x):=\eta$ for all $x\in\Omega$. Clearly $w\in X$. Hence

$$\begin{aligned}J(w)&=\tilde K\Big(\int_\Omega(\Phi(|\nabla w(x)|)+\Phi(|w(x)|))\,dx\Big)\\&=\tilde K\Big(\int_\Omega\Phi(\eta)\,dx\Big)=\tilde K(\Phi(\eta)\operatorname{meas}(\Omega)).\end{aligned} \quad (3.7)$$

Since $\theta<2c\Big(\frac{\tilde K(\Phi(\eta)\operatorname{meas}(\Omega))}{m}\Big)^{1/p^0}$, one has $r<J(w)$. For all $u\in X$, by (2.8) and Lemma 2.3, we have

$$|u(x)|\le\|u\|_\infty\le c\|u\|_{1,\Phi}\le 2c\|u\|,\quad llx\in\Omega.$$

Hence, since $\theta<2c$, taking Lemmas 2.4 and 2.5 into account one has

$$J^{-1}(-\infty,r]\subseteq\{u\in X;\|u\|\le\frac{\theta}{2c}\}\subseteq\{u\in X;|u(x)|\le\theta\ \text{ for all } x\in\Omega\},$$

and it follows that

$$\sup_{u\in J^{-1}(-\infty,r]}I(u)\le\int_\Omega\sup_{|t|\le\theta}F(x,t)+\frac{\mu}{\lambda}G(x,t)\,dx.$$

Therefore, one has

$$\begin{aligned}\sup_{u\in J^{-1}(-\infty,r]}I(u)&=\sup_{u\in J^{-1}(-\infty,r]}\int_\Omega[F(x,u(x))+\frac{\mu}{\lambda}G(x,u(x))]dx\\&\le\int_\Omega\sup_{|t|\le\theta}F(x,t)dx+\frac{\mu}{\lambda}G^\theta.\end{aligned}$$

On the other hand, we have

$$I(w) = \int_\Omega F(x,\eta) + \frac{\mu}{\lambda} G(x,\eta)\,dx.$$

So, we have

$$\begin{aligned}\frac{\sup_{u\in J^{-1}(-\infty,r]} I(u)}{r} &= \frac{\sup_{u\in J^{-1}(-\infty,r]} \int_\Omega [F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))]dx}{r} \\ &\le \frac{\int_\Omega \sup_{|t|\le\theta} F(x,t)dx + \frac{\mu}{\lambda}G^\theta}{m\left(\frac{\theta}{2c}\right)^{p^0}},\end{aligned} \tag{3.8}$$

and

$$\frac{I(w)}{J(w)} \ge \frac{\int_\Omega F(x,\eta)dx + \frac{\mu}{\lambda}\int_\Omega G(x,w(x))dx}{\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega))} \ge \frac{\int_\Omega F(x,\eta)dx + \frac{\mu}{\lambda}G_\eta}{\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega))}. \tag{3.9}$$

Since $\mu < \delta_{\lambda,g}$, one has

$$\mu < \frac{m\theta^{p^0} - (2c)^{p^0}\lambda\int_\Omega \sup_{|t|\le\theta} F(x,t)dx}{(2c)^{p^0}G^\theta},$$

this means

$$\frac{\int_\Omega \sup_{|t|\le\theta} F(x,t)dx + \frac{\mu}{\lambda}G^\theta}{m\left(\frac{\theta}{2c}\right)^{p^0}} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega)) - \lambda\int_\Omega F(x,\eta)dx}{G_\eta \operatorname{meas}(\Omega)},$$

this means

$$\frac{\int_\Omega F(x,\eta)dx + \operatorname{meas}(\Omega)\frac{\mu}{\lambda}G_\eta}{\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega))} > \frac{1}{\lambda}.$$

Then

$$\frac{\int_\Omega \sup_{|t|\le\theta} F(x,t)dx + \frac{\mu}{\lambda}G^\theta}{\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega))} < \frac{1}{\lambda} < \frac{\int_\Omega F(x,\eta)dx + \operatorname{meas}(\Omega)\frac{\mu}{\lambda}G_\eta}{\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega))}. \tag{3.10}$$

Hence from (3.8)-(3.10), we observe that the condition (2.1) of Theorem 2.1 is satisfied. Finally, since $\mu < \overline{\delta}_{\lambda,g}$, we can fix $l > 0$ such that

$$\limsup_{t\to\infty} \frac{\sup_{x\in\overline{\Omega}} G(x,t)}{t^{p_0}} < l,$$

and $\mu l < \frac{m}{c^{p_0}\operatorname{meas}(\Omega)}$. Therefore, there exists a function $h \in L^1(\Omega)$ such that

$$G(x,t) \le l t^{p_0} + h(x), \tag{3.11}$$

for every $x \in \overline{\Omega}$ and $t \in \mathbb{R}$. Now, for $\lambda > 0$, fix ϵ such that

$$0 < \epsilon < \frac{m}{c^{p_0}\operatorname{meas}(\Omega)\lambda} - \frac{\mu l}{\lambda}.$$

From (3.4) there is a function $h_\epsilon \in L^1(\Omega)$ such that

$$F(x,t) \le \epsilon t^{p_0} + h_\epsilon(x), \tag{3.12}$$

for every $x \in \overline{\Omega}$ and $t \in \mathbb{R}$. From (3.11) and (3.12), taking (2.8) into account, it follows that, for each $u \in X$ with $\|u\| > 1$,

$$J(u) - \lambda I(u)$$

$$
\begin{aligned}
&= \tilde{K}\Big(\int_\Omega [\Phi(|\nabla u(x)|) + \Phi(|u(x)|)]dx\Big) - \lambda \int_\Omega [F(x,u(x)) + \frac{\mu}{\lambda} G(x,u(x))]dx \\
&\geq m\|u\|^{p_0} - \lambda\epsilon \int_\Omega |u(x)|^{p_0}\,dx - \lambda\|h_\epsilon\|_{L^1(\Omega)} - \mu l \int_\Omega |u(x)|^{p_0}\,dx - \mu\|h\|_{L^1(\Omega)} \\
&\geq (m - \lambda\epsilon c^{p_0}\operatorname{meas}(\Omega) - \mu l c^{p_0}\operatorname{meas}(\Omega))\|u\|^{p_0} - \lambda\|h_\epsilon\|_{L^1(\Omega)} - \mu\|h\|_{L^1(\Omega)},
\end{aligned}
$$

and thus

$$
\lim_{\|u\|\to+\infty} (J(u) - \lambda I(u)) = +\infty,
$$

which means the functional $J - \lambda I$ is coercive, and the condition (2.2) of Theorem 2.1 is satisfied. From (3.8)-(3.10) one also has

$$
\lambda \in \Big]\frac{J(w)}{I(w)}, \frac{r}{\sup_{J(u)\leq r} I(u)}\Big[.
$$

Finally, since the weak solutions of problem (1.1) are exactly the solutions of the equation $J'(u) - \lambda I'(u) = 0$, Theorem 2.1 (with $\overline{v} = w$) ensures the conclusion. $\square$

Now, we present a variant of Theorem 3.1 in which no asymptotic condition on the nonlinear term g is requested. In such a case f and g are supposed to be nonnegative.

For our goal, let us fix positive constants $\theta_1,\ \theta_2$ and η such that

$$
\begin{aligned}
&\frac{3}{2}\frac{\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega))}{\int_\Omega F(x,\eta)dx} \\
&< \frac{m}{(2c)^{p^0}} \min\Big\{\frac{\theta_1^{p^0}}{\int_\Omega \sup_{|t|\leq\theta_1} F(x,t)dx}, \frac{\theta_2^{p^0}}{2\int_\Omega \sup_{|t|\leq\theta_2} F(x,t)dx}\Big\}
\end{aligned}
$$

and taking λ in the interval

$$
\begin{aligned}
\Lambda_2 := \Big]\frac{3}{2}\frac{\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega))}{\int_\Omega F(x,\eta)dx}, \frac{m}{(2c)^{p^0}} \min\Big\{&\frac{\theta_1^{p^0}}{\int_\Omega \sup_{|t|\leq\theta_1} F(x,t)dx}, \\
&\frac{\theta_2^{p^0}}{2\int_\Omega \sup_{|t|\leq\theta_2} F(x,t)dx}\Big\}\Big[.
\end{aligned}
$$

We formulate our second main result as follows.

Theorem 3.2. *Let $f : \overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ satisfies the condition $f(x,t)\geq 0$ for every $(x,t)\in\overline{\Omega}\times(\mathbb{R}^+\cup\{0\})$. Assume that there exist three positive constants θ_1,θ_2 and η with*

$$
\begin{gathered}
\theta_1 < 2c\min\{1, \Big(\frac{\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega))}{2m}\Big)^{1/p^0}\}, \\
\Big(2\frac{\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega))}{m}\Big)^{1/p^0} < \frac{\theta_2}{2c} < 1
\end{gathered}
$$

such that

$$
\begin{aligned}
&\max\Big\{\frac{\int_\Omega \sup_{|t|\leq\theta_1} F(x,t)dx}{\theta_1^{p^0}}, \frac{2\int_\Omega \sup_{|t|\leq\theta_2} F(x,t)dx}{\theta_2^{p^0}}\Big\} \\
&< \frac{2}{3}\frac{m}{(2c)^{p^0}}\frac{\int_\Omega F(x,\eta)dx}{\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega))}.
\end{aligned} \tag{3.13}
$$

Then for each $\lambda \in \Lambda_2$ *and for every nonnegative* L^1*-Carathéodory function* $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$*, there exists* $\delta^*_{\lambda,g} > 0$ *given by*

$$\delta^*_{\lambda,g} \min\Big\{ \frac{m\theta_1^{p^0} - (2c)^{p^0}\lambda \int_\Omega \sup_{|t|\le\theta_1} F(x,t)dx}{(2c)^{p^0} G^{\theta_1}}, \frac{m\theta_2^{p^0} - 2(2c)^{p^0}\lambda \int_\Omega \sup_{|t|\le\theta_2} F(x,t)dx}{2(2c)^{p^0} G^{\theta_2}} \Big\}$$

such that, for each $\mu \in [0, \delta^*_{\lambda,g}[$*, problem* (1.1) *possesses at least three distinct weak solutions* $u_i \in W^1L_\Phi(\Omega)$ *for* $i = 1,2,3$*, such that*

$$0 \le u_i(x) < \theta_2, \quad \forall x \in \overline{\Omega},\ (i = 1,2,3).$$

Proof. Without loss of generality, we can assume $f(x,t) \ge 0$ for every $(x,t) \in \overline{\Omega} \times \mathbb{R}$. Fix λ, g and μ as in the conclusion and take X, J and I as in the proof of Theorem 3.1. We observe that the regularity assumptions of Theorem 2.2 on J and I are satisfied. Then, our aim is to verify (2.3) and (2.4). To this end, choose $r_1 = m\left(\frac{\theta_1}{2c}\right)^{p^0}$, $r_2 = m\left(\frac{\theta_2}{2c}\right)^{p^0}$ and $w(x) := \eta$ for all $x \in \Omega$. Since

$$\theta_1 < 2c\Big(\frac{\tilde K(\Phi(\eta)\operatorname{meas}(\Omega))}{2m}\Big)^{1/p^0} \quad \text{and} \quad \Big(2\frac{\tilde K(\Phi(\eta)\operatorname{meas}(\Omega))}{m}\Big)^{1/p^0} < \frac{\theta_2}{2c},$$

from (3.7), we get $2r_1 < J(w) < r_2/2$. Since $\mu < \delta^*_{\lambda,g}$ and $G_\eta = 0$, and bearing in mind that $\theta_1 < 2c$ and $\theta_2 < 2c$, one has

$$\begin{aligned}
\frac{\sup_{u\in J^{-1}(-\infty,r_1)} I(u)}{r_1} &= \frac{\sup_{u\in J^{-1}(-\infty,r_1)} \int_\Omega [F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))]dx}{r_1} \\
&\le \frac{\int_\Omega \sup_{|t|\le\theta_1} F(x,t)dx + \frac{\mu}{\lambda}G^{\theta_1}}{m\left(\frac{\theta_1}{2c}\right)^{p^0}} \\
&< \frac{1}{\lambda} < \frac{2}{3}\frac{\int_\Omega F(x,\eta)dx + \operatorname{meas}(\Omega)\frac{\mu}{\lambda}G\eta}{\tilde K(\Phi(\eta)\operatorname{meas}(\Omega))} \\
&\le \frac{2}{3}\frac{I(w)}{J(w)},
\end{aligned}$$

and

$$\begin{aligned}
\frac{2\sup_{u\in J^{-1}(-\infty,r_2)} I(u)}{r_2} &= \frac{2\sup_{u\in J^{-1}(-\infty,r_2)} \int_\Omega [F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))]dx}{r_2} \\
&\le \frac{2\int_\Omega \sup_{|t|\le\theta_2} F(x,t)dx + 2\frac{\mu}{\lambda}G^{\theta_2}}{m\left(\frac{\theta_2}{2c}\right)^{p^0}} \\
&< \frac{1}{\lambda} < \frac{2}{3}\frac{\int_\Omega F(x,\eta)dx + \operatorname{meas}(\Omega)\frac{\mu}{\lambda}G_\eta}{\tilde K(\Phi(\eta)\operatorname{meas}(\Omega))} \\
&\le \frac{2}{3}\frac{I(w)}{J(w)}.
\end{aligned}$$

Therefore, (2.3) and (2.4) of Theorem 2.2 are fulfilled. Finally, we prove that $J - \lambda I$ satisfies the assumption (2) of Theorem 2.2. Let u_1 and u_2 be two local minima for $J - \lambda I$. Then u_1 and u_2 are critical points for $J - \lambda I$, and so, they are weak solutions for problem (1.1). We want to prove that they are nonnegative. Let u_* be a non-trivial weak solution of problem (1.1). Arguing by a contradiction, assume that the set $\mathcal{A} = \{x \in \Omega;\ u_*(x) < 0\}$ is non-empty and of positive measure. Put $u_*^-(x) = \min\{u_*(x), 0\}$. By [30, Remark 5] we deduce that $u_*^- \in W^1L_\Phi(\Omega)$. Suppose that $\|u_*\| < 1$. Using this fact that u_* also is a weak solution of (1.1) and by choosing $v = u_*^-$, since

$$p_0 \le \frac{t\varphi(t)}{\Phi(t)}, \quad \forall t > 0,$$

using the first estimate of Lemma 2.4 and recalling our sign assumptions on the data, we have

$$\begin{aligned}
mp_0\|u_*\|^{p^0}_{W^1L_\Phi(\mathcal{A})} &\le mp_0 \int_{\mathcal{A}} [\Phi(|\nabla u_*(x)|) + \Phi(|u_*(x)|)]dx \\
&\le m \int_{\mathcal{A}} [\varphi(|\nabla u_*(x)|)|\nabla u_*(x)| + \varphi(|u_*(x)|)|u_*(x)|]dx \\
&\le K\Big(\int_{\mathcal{A}} [\Phi(|\nabla u_*(x)|) + \Phi(|u_*(x)|)]dx\Big) \\
&\quad \times \int_{\mathcal{A}} [\alpha(|\nabla u_*(x)|)|\nabla u_*(x)|^2 + \alpha(|u_*(x)|)|u_*(x)|^2]dx \\
&= \lambda \int_{\mathcal{A}} f(x, u_*(x))u_*(x)dx + \mu \int_{\mathcal{A}} g(x, u_*(x))u_*(x)dx \le 0,
\end{aligned}$$

i.e.,

$$\|u_*\|^{p^0}_{W^1L_\Phi(\mathcal{A})} \le 0$$

which contradicts with this fact that u_* is a non-trivial weak solution. Hence, the set $\mathcal{A}$ is empty, and u_* is positive. The proof of the case $\|u_*\| > 1$ is similar to case $\|u_*\| < 1$ (use the second part of Lemma 2.4 instead). For the case $\|u_*\| = 1$, we may assume $\|u_*\|_{W^1L_\Phi(\mathcal{A})} = 1$, and arguing as for the case $\|u_*\| < 1$, using Lemma 2.6 we have

$$\begin{aligned}
mp_0\|u_*\|_{W^1L_\Phi(\mathcal{A})} &= mp_0 \int_{\mathcal{A}} [\Phi(|\nabla u_*(x)|) + \Phi(|u_*(x)|)]dx \\
&\le m \int_{\mathcal{A}} [\varphi(|\nabla u_*(x)|)|\nabla u_*(x)| + \varphi(|u_*(x)|)|u_*(x)|]dx \le 0,
\end{aligned}$$

which also contradicts that u_* is a non-trivial weak solution. Therefore, we deduce $u_1(x) \ge 0$ and $u_2(x) \ge 0$ for every $x \in \overline{\Omega}$. Thus, it follows that $su_1 + (1-s)u_2 \ge 0$ for all $s \in [0,1]$, and that

$$(\lambda f + \mu g)(x, su_1 + (1-s)u_2) \ge 0,$$

and consequently, $J(su_1 + (1-s)u_2) \ge 0$, for every $s \in [0,1]$. By using Theorem 2.2, for every λ in the interval

$$\Big]\frac{3}{2}\frac{J(w)}{I(w)}, \min\Big\{\frac{r_1}{\sup_{u \in J^{-1}(-\infty, r_1)} I(u)}, \frac{r_2/2}{\sup_{u \in J^{-1}(-\infty, r_2)} I(u)}\Big\}\Big[,$$

the functional $J - \lambda I$ has at least three distinct critical points which are the weak solutions of problem (1.1) and the desired conclusion is achieved. □

Remark 3.3. If either $f(x, 0) \ne 0$ for all $x \in \Omega$ or $g(x, 0) \ne 0$ for all $x \in \Omega$, or both are true the solutions of problem (1.1) are nontrivial.

Remark 3.4. A remarkable particular situation of problem 1.1 is the case when $K(t) = a + bt$, $a, b > 0$ for all t in a bounded subset of $\mathbb{R}^+ \cup \{0\}$.

Remark 3.5. If $K(t) = 1$ for all $t \geq 0$ and $\mu = 0$, Theorem 3.1 gives back to [14, Theorem 3.1]. In addition, if $\varphi(t) = |t|^{p-2}t$ with $p > 1$, one has $p_0 = p^0 = p$, and the Orlicz-Sobolev space $W^1L_\Phi(\Omega)$ coincides with the Sobolev space $W^{1,p}(\Omega)$, so, if $p > N$, with this case of φ, Theorems 3.1 and 3.2 extend [10, Theorem 2] by giving the exact collections of the parameter λ.

Here we point out a consequence of Theorem 3.2 in which $K(t) = 1$ for all $t \geq 0$. Let us fix positive constants θ_1, θ_2 and η such that

$$\frac{3}{2}\frac{\Phi(\eta)\operatorname{meas}(\Omega)}{\int_\Omega F(x,\eta)dx} < \frac{1}{(2c)^{p^0}}\min\Big\{\frac{\theta_1^{p^0}}{\int_\Omega \sup_{|t|\leq\theta_1} F(x,t)dx}, \frac{\theta_2^{p^0}}{2\int_\Omega \sup_{|t|\leq\theta_2} F(x,t)dx}\Big\}$$

and taking

$$\begin{aligned}\lambda \in \Lambda_3 := \Big]\frac{3}{2}\frac{\Phi(\eta)\operatorname{meas}(\Omega)}{\int_\Omega F(x,\eta)dx}, \frac{1}{(2c)^{p^0}}\min\Big\{&\frac{\theta_1^{p^0}}{\int_\Omega \sup_{|t|\leq\theta_1} F(x,t)dx},\\ &\frac{\theta_2^{p^0}}{2\int_\Omega \sup_{|t|\leq\theta_2} F(x,t)dx}\Big\}\Big[.\end{aligned}$$

Theorem 3.6. *Let $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ satisfies the condition $f(x,t) \geq 0$ for every $(x,t) \in \overline{\Omega}\times(\mathbb{R}^+\cup\{0\})$. Assume that there exist three positive constants θ_1, θ_2 and η with*

$$\begin{gathered}\theta_1 < 2c\min\Big\{1, \Big(\frac{\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega))}{2}\Big)^{1/p^0}\Big\}\\ \Big(2\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega))\Big)^{1/p^0} < \frac{\theta_2}{2c} < 1\end{gathered}$$

such that

$$\begin{aligned}&\max\Big\{\frac{\int_\Omega \sup_{|t|\leq\theta_1} F(x,t)dx}{\theta_1^{p^0}}, \frac{2\int_\Omega \sup_{|t|\leq\theta_2} F(x,t)dx}{\theta_2^{p^0}}\Big\}\\ &< \frac{2}{3}\frac{1}{(2c)^{p^0}\operatorname{meas}(\Omega)}\frac{\int_\Omega F(x,\eta)dx}{\Phi(\eta)}.\end{aligned} \tag{3.14}$$

Then, for each $\lambda \in \Lambda_3$ and for every nonnegative L^1-Carathéodory function $g : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$, there exists $\delta'^*_{\lambda,g} > 0$ given by

$$\min\Big\{\frac{\theta_1^{p^0} - (2c)^{p^0}\lambda\int_\Omega \sup_{|t|\leq\theta_1} F(x,t)dx}{(2c)^{p^0}G^{\theta_1}}, \frac{\theta_2^{p^0} - 2(2c)^{p^0}\lambda\int_\Omega \sup_{|t|\leq\theta_2} F(x,t)dx}{2(2c)^{p^0}G^{\theta_2}}\Big\}$$

such that, for each $\mu \in [0, \delta'^*_{\lambda,g}[$, the problem

$$\begin{gathered}-\operatorname{div}(\alpha(|\nabla u|)\nabla u) + \alpha(|u|)u = \lambda f(x,u) + \mu g(x,u) \quad \text{in } \Omega,\\ \frac{\partial u}{\partial\nu} = 0 \quad \text{on } \partial\Omega\end{gathered}$$

possesses at least three distinct weak solutions $u_i \in W^1L_\Phi(\Omega)$ for $i = 1,2,3$, such that

$$0 \leq u_i(x) < \theta_2, \quad \forall x \in \overline{\Omega},\ (i = 1,2,3).$$

From now let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Put $F(t) := \int_0^t f(\xi)d\xi$ for each $t \in \mathbb{R}$. A special case of Theorem 3.1 is the following theorem.

Theorem 3.7. *Assume that*

$$\liminf_{t\to 0^+}\frac{F(t)}{t^{p^0}}=\limsup_{|t|\to+\infty}\frac{F(t)}{t^{p_0}}=0.$$

Then, for each $\lambda>\inf_{\eta\in B}\frac{\Phi(\eta)}{F(\eta)}$ *where* $B:=\{\eta>0;\ F(\eta)>0\}$*, and for every nonnegative continuous function* $g:\mathbb{R}\to\mathbb{R}$ *such that*

$$\limsup_{|t|\to+\infty}\frac{\int_0^t g(s)ds}{t^{p_0}}<+\infty, \tag{3.15}$$

there exists $\delta^*>0$ *such that for each* $\mu\in[0,\delta^*[$*, the problem*

$$-\operatorname{div}(\alpha(|\nabla u(x)|)\nabla u(x))+\alpha(|u(x)|)u(x)=\lambda f(u(x))+\mu g(u(x))\quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial \nu}=0\quad \text{on } \partial\Omega$$

possesses at least three distinct nonnegative weak solutions in $W^1L_\Phi(\Omega)$*.*

Proof. Fix $\lambda>\inf_{\eta\in B}\frac{\Phi(\eta)}{F(\eta)}$. Then there exists $\overline{\eta}>0$ such that $F(\overline{\eta})>0$ and $\lambda>\frac{\Phi(\overline{\eta})}{F(\overline{\eta})}$. Recalling that

$$\liminf_{\xi\to 0^+}\frac{F(t)}{t^{p^0}}=0,$$

there is a sequence $\{\theta_n\}\subset]0,+\infty[$ such that $\lim_{n\to\infty}\theta_n=0$ and

$$\lim_{n\to\infty}\frac{\sup_{|t|\le\theta_n}F(t)}{\theta_n^{p^0}}=0.$$

Indeed, one has

$$\lim_{n\to\infty}\frac{\sup_{|t|\le\theta_n}F(t)}{\theta_n^{p^0}}=\lim_{n\to\infty}\frac{F(t_{\theta_n})}{\xi_{\theta_n}^{p^0}}\frac{\xi_{\theta_n}^{p^0}}{\theta_n^{p^0}}=0,$$

where $F(t_{\theta_n})=\sup_{|t|\le\theta_n}F(t)$. Hence, there exists $\overline{\theta}>0$ such that

$$\frac{\sup_{|t|\le\overline{\theta}}F(t)}{\overline{\theta}^{p^0}}<\frac{1}{(2c)^{p^0}\operatorname{meas}(\Omega)}\min\Big\{\frac{F(\eta)}{\Phi(\eta)};\ \frac{1}{\lambda}\Big\},$$
$$\overline{\theta}<2c\min\{1,\Big(\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega))\Big)^{1/p}\}$$

The conclusion follows from Theorem 3.1. $\square$

Here we want to present two existence results as consequences of Theorems 3.7 and 3.6, respectively, by choosing a particular case of $\phi(t)$.

Let $p>N+1$ and define

$$\varphi(t)=\begin{cases}\frac{|t|^{p-2}t}{\log(1+|t|)} & \text{if } t\neq 0\\ 0 & \text{if } t=0.\end{cases}$$

By [22, Example 3] one has

$$p_0=p-1<p^0=p=\liminf_{t\to\infty}\frac{\log(\Phi(t))}{\log(t)}.$$

Thus, the conditions (2.5) and (2.6) are satisfied.

Corollary 3.8. *Assume that*

$$\liminf_{t\to 0^+} \frac{F(t)}{t^p} = \limsup_{|t|\to+\infty} \frac{F(t)}{t^{p-1}} = 0.$$

Then, for each $\lambda > \inf_{\eta\in B} \frac{\Phi(\eta)}{F(\eta)}$ *where* $B := \{\eta > 0; F(\eta) > 0\}$ *and* $\Phi(\eta) := \int_0^\eta \frac{t|t|^3}{\log(1+|t|)}dt$*, and for every nonnegative continuous function* $g : \mathbb{R} \to \mathbb{R}$ *satisfying the condition* (3.15)*, there exists* $\delta'^* > 0$ *such that for each* $\mu \in [0, \delta'^*[$*, the problem*

$$\begin{gathered} -\operatorname{div}\Big(\frac{|\nabla u|^{p-2}}{\log(1+|\nabla u|)}\nabla u\Big) + \frac{|u|^{p-2}}{\log(1+|u|)}u = \lambda f(u) + \mu g(u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.16}$$

possesses at least three distinct nonnegative weak solutions in $W^1L_\Phi(\Omega)$*.*

Corollary 3.9. *Assume that there exist two positive constants* θ *and* η *with*

$$\Big(2\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega))\Big)^{1/p} < \frac{\theta}{2c} < 1\,,$$

where $\Phi(\eta)$ *is as given in Corollary 3.8. Suppose that*

$$\begin{gathered} \lim_{t\to 0^+} \frac{f(t)}{t^{p-1}} = 0, \\ \frac{F(\theta)}{\theta^p} < \frac{1}{3(2c)^p \operatorname{meas}(\Omega)} \frac{F(\eta)}{\Phi(\eta)}. \end{gathered}$$

Then, for every $\lambda \in \big]\frac{3}{2}\frac{\Phi(\eta)}{F(\eta)}, \frac{1}{2(2c)^p\operatorname{meas}(\Omega)}\frac{\theta^p}{F(\theta)}\big[$ *and for every nonnegative continuous function* $g : \mathbb{R} \to \mathbb{R}$ *there exists* $\delta''^* > 0$ *such that, for each* $\mu \in [0, \delta''^*[$*, problem* (3.16) *possesses at least three distinct weak solutions* $u_i \in W^1L_\Phi(\Omega)$ *for* $i = 1, 2, 3$*, such that*

$$0 \le u_i(x) < \theta, \quad \forall x \in \overline{\Omega},\ (i = 1, 2, 3).$$

Proof. Since $\lim_{t\to 0^+} \frac{f(t)}{t^{p-1}} = 0$, one has $\lim_{t\to 0^+} \frac{F(t)}{t^p} = 0$. Then, there exists a positive constant $\overline{\theta} < 2c\min\{1, \Big(\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega))\Big)^{1/p}\}$ such that

$$\frac{F(\overline{\theta})}{\overline{\theta}^p} < \frac{2}{3}\frac{1}{(2c)^p\operatorname{meas}(\Omega)}\frac{F(\eta)}{\Phi(\eta)},$$

and $\frac{\overline{\theta}^p}{F(\overline{\theta})} > \frac{\theta^p}{2F(\theta)}$. Finally, a simple computation shows that all assumptions of Theorem 3.6 are fulfilled, and it follows the conclusion. $\square$

We illustrate Corollary 3.8 by presenting the following example.

Example 3.10. Let $\Omega = \{(x, y, z) \in \mathbb{R}^3;\ x^2+y^2+z^2 < 1\}$, $p > 4$ and let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(t) = \begin{cases} 0, & t < 0, \\ t^p, & 0 \le t < 1, \\ t^{p-3}, & t > 1 \end{cases}$$

and $g(t) = e^{-t}|t|^{p-1}$ for all $t \in \mathbb{R}$. Thus f and g are nonnegative, and

$$F(t) = \begin{cases} 0, & t < 0, \\ \frac{1}{p+1} t^{p+1}, & 0 \le t < 1, \\ \frac{1}{p-2} t^{p-2} - \frac{3}{(p-2)(p+1)}, & t > 1. \end{cases}$$

Therefore,

$$\liminf_{t\to 0^+} \frac{F(t)}{t^p} = \limsup_{|t|\to+\infty} \frac{F(t)}{t^{p-1}} = 0.$$

Then, for each $\lambda > \inf_{\eta\in B} \frac{\Phi(\eta)}{F(\eta)}$ where $B := \{\eta > 0;\ F(\eta) > 0\}$ and $\Phi(\eta) := \int_0^\eta \frac{t|t|^3}{\log(1+|t|)} dt$, there exists $\delta'^* > 0$ such that for each $\mu \in [0, \delta'^*[$, problem (3.16), in this case possesses at least three distinct nonnegative weak solutions in $W^1 L_\Phi(\Omega)$.

Now let $p > N$. Choose $\varphi(t) = \log(1+|t|^\gamma)|t|^{p-2}t$, $t \in \mathbb{R}$, $\gamma > 1$. By [22, Example 2] one has $p_0 = p$ and $p^0 = p + \gamma$, and the conditions (2.5) and (2.6) are satisfied. In this case, Corollaries 3.8 and 3.9 become to the following forms, respectively.

Corollary 3.11. *Assume that*

$$\liminf_{t\to 0^+} \frac{F(t)}{t^{p+\gamma}} = \limsup_{|t|\to+\infty} \frac{F(t)}{t^p} = 0.$$

Then, for each $\lambda > \inf_{\eta\in B} \frac{\Phi(\eta)}{F(\eta)}$ *where* $B := \{\eta > 0;\ F(\eta) > 0\}$ *and*

$$\Phi(\eta) := \int_0^\eta \log(1 + |t|^\gamma)|t|^{p-2} t dt,$$

and for every nonnegative continuous function $g : \mathbb{R} \to \mathbb{R}$ *satisfying the condition* (3.15)*, there exists* $\overline{\delta}'^* > 0$ *such that for each* $\mu \in [0, \overline{\delta}'^*[$*, the problem*

$$\begin{gathered} -\operatorname{div}\Big(\log(1 + |\nabla u(x)|^\gamma)|\nabla u|^{p-2}\nabla u\Big) + \log(1 + |u|^\gamma)|u|^{p-2} \\ = \lambda f(u) + \mu g(u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.17}$$

possesses at least three distinct nonnegative weak solutions in $W^1 L_\Phi(\Omega)$.

Corollary 3.12. *Assume that there exist two positive constants* θ *and* η *with*

$$\Big(2\tilde{K}(\Phi(\eta)\operatorname{meas}(\Omega))\Big)^{1/(p+\gamma)} < \frac{\theta}{2c} < 1$$

where $\Phi(\eta)$ *is as given in Corollary 3.11. Suppose that*

$$\lim_{t\to 0^+} \frac{f(t)}{t^{p+\gamma-1}} = 0,$$
$$\frac{F(\theta)}{\theta^{p+\gamma}} < \frac{1}{3(2c)^{p+\gamma}\operatorname{meas}(\Omega)} \frac{F(\eta)}{\Phi(\eta)}.$$

Then, for every

$$\lambda \in \Big]\frac{3}{2}\frac{\Phi(\eta)}{F(\eta)},\ \frac{1}{2(2c)^{p+\gamma}\operatorname{meas}(\Omega)} \frac{\theta^p}{F(\theta)}\Big[$$

and for every nonnegative continuous function $g : \mathbb{R} \to \mathbb{R}$ there exists $\overline{\delta}''^{*} > 0$ such that, for each $\mu \in [0, \overline{\delta}''^{*}[$, problem (3.17) possesses at least three distinct weak solutions $u_i \in W^1 L_\Phi(\Omega)$ for $i = 1, 2, 3$, such that

$$0 \leq u_i(x) < \theta, \quad \forall x \in \overline{\Omega},\ (i = 1, 2, 3).$$

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