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SOLUTIONS FOR $p(x)$-LAPLACE EQUATIONS WITH CRITICAL FREQUENCY

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Abstract. This article concerns the $p(x)$-Laplace equations with critical frequency

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)+V(x)|u|^{p(x)-2}u=f(x,u)\quad \text{in } \mathbb{R}^N,$$

where $1<p_-\le p(x)\le p_+<N$. We study this equation with the potentials being zero. By using variational method, we obtain the existence of nonnegative solutions. Moreover, if $f(x,t)$ is odd in t, for any $m\in\mathbb{N}$ we derive m pairs of nontrivial solutions.

1. Introduction and statement of main results

Since the variable exponent spaces were thoroughly studied by Kováčik and Rákosník [21], they have been used in the previous decades to model various phenomena. In the studies of a class of non-standard variational problems and PDEs, variable exponent spaces play an important role such as in electrorheological fluids [27, 28, 29], thermorheological fluids [7], image processing [1, 13, 23] and so on. For nonlinear problems with variable growth, there have been a great deal of interests in studying the existence, multiplicity, uniqueness and regularity of solutions, see [2, 3, 5, 6, 10, 12, 18, 19, 20, 21, 24, 25, 26, 31] and references therein.

In this article, we study the $p(x)$-Laplace type equation

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)+V(x)|u|^{p(x)-2}u=f(x,u)\quad \text{in } \mathbb{R}^N. \tag{1.1}$$

We are interested in the critical frequency case in the sense that

$$\min_{x\in\mathbb{R}^N} V(x)=0.$$

It is worth mentioning that the study of Schrödinger equations with the critical frequency was first investigated by Byeon and Wang [8, 9]. In the linear case $p(x)\equiv 2$, Ding and Lin [16] obtained the existence and multiplicity of solutions to a class of Schrödinger equations with critical frequency and critical nonlinearity. For the $p(x)$-Laplace equation (1.1), there have been many papers dealing with the case $\inf_{x\in\mathbb{R}^N} V(x)>0$, which can be found in [4, 5, 18, 19, 24] and references therein.

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Inspired by the works mentioned above, we will study (1.1) with critical frequency in the frame of variable exponent function spaces, the definitions of which will be given in Section 2. More precisely, the aim of this paper is to use variational method to show the existence of solutions for (1.1). The main difficulty is caused by the loss of the compactness for the embedding $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}(\mathbb{R}^N)$, where $1 \le q(x) \ll p^*(x)$. We do not expect that the energy functional satisfies the Palais-Smale condition ((PS) condition for short) at any positive energy level, which makes the study via variational methods rather complicated. We show that (PS) condition holds for energy level less than some positive constant. Then, by using Minimax theorem, we obtain weak solution of (1.1). To this end, we assume the following conditions:

- (A1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\min_{x\in\mathbb{R}^N} V(x) = 0$;
- (A2) There exists $a > 0$ such that the level set $V^a = \{x \in \mathbb{R}^N : V(x) < a\}$ has finite Lebesgue measure;
- (A3) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. There exists $c_0 > 0$, $1 \le q(x) \ll p^*(x)$ such that $|f(x,t)| \le c_0(1 + |t|^{q(x)-1})$ for any $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ is the critical exponent;
- (A4) $\lim_{t\to 0} \frac{f(x,t)}{|t|^{p_+-1}} = 0$ uniformly in $x \in \mathbb{R}^N$, where $p_+ = \sup_{x\in\mathbb{R}^N} p(x)$;
- (A5) There exists $\mu > p_+$ such that $\mu F(x,t) \le f(x,t)t$ for any $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, where $F(x,t) = \int_0^t f(x,s)\,ds$;
- (A6) There exist $c_1 > 0$, $\alpha(x) > 1$ such that $f(x,t) > c_1 t^{\alpha(x)-1}$ for any $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$.

Now we are in a position to give our main results.

Theorem 1.1. *Assume that* (A1)–(A6) *are fulfilled. If $q_- > p_+$, $\alpha_- > p_+$ and $\alpha_+ < p^*_-$, the problem* (1.1) *has a nonnegative nontrivial weak solution u.*

Theorem 1.2. *Assume that* (A1)–(A6) *are fulfilled. If the subcritical nonlinearity $f(x,t)$ is odd in t, problem* (1.1) *has at least m pairs of nontrivial weak solutions for any $m \in \mathbb{N}$.*

The rest of this paper is organized as follows. In Section 2, we collect some basic properties for variable exponent Sobolev spaces which will be used later. We will prove the main result in Section 3.

2. Preliminaries

For the convenience of the readers, we recall some definitions and basic properties of variable exponent spaces. For a deeper treatment on these spaces, we refer to [15].

Let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to [1,\infty)$. For $p \in \mathcal{P}(\Omega)$, we denote

$$p_+ = \sup_{x\in\Omega} p(x), \quad p_- = \inf_{x\in\Omega} p(x).$$

From now on, we only consider the case

$$1 \le p_- \le p(x) \le p_+ < N. \tag{2.1}$$

Define

$$\|u\|_{p(x)} = \inf\big\{t > 0 : \int_\Omega |\frac{u}{t}|^{p(x)}\,dx \le 1\big\}. \tag{2.2}$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of all functions u such that $\int_\Omega |tu(x)|^{p(x)}\,dx < \infty$, for some $t > 0$. $L^{p(x)}(\Omega)$ is a Banach space equipped with the norm (2.2).

Theorem 2.1. *For any $u \in L^{p(x)}(\Omega)$, we have*

(1) *if $\|u\|_{p(x)} \ge 1$, then $\|u\|_{p(x)}^{p_-} \le \int_\Omega |u|^{p(x)}\,dx \le \|u\|_{p(x)}^{p_+}$;*
(2) *if $\|u\|_{p(x)} < 1$, then $\|u\|_{p(x)}^{p_+} \le \int_\Omega |u|^{p(x)}\,dx \le \|u\|_{p(x)}^{p_-}$.*

Theorem 2.2. *The dual space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ if and only if p satisfies* (2.1)*, where $p'(x) = \frac{p(x)}{p(x)-1}$. The space $L^{p(x)}(\Omega)$ is reflexive if and only if*

$$1 < p_- \le p(x) \le p_+ < \infty. \tag{2.3}$$

Theorem 2.3. *For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$,*

$$\int_\Omega |uv|\,dx \le 2\|u\|_{p(x)}\|v\|_{p'(x)}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is the class of all functions $u \in L^{p(x)}(\Omega)$ such that $|\nabla u| \in L^{p(x)}(\Omega)$. $W^{1,\,p(x)}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{1,\,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}. \tag{2.4}$$

By $W_0^{1,p(x)}(\Omega)$ we denote the subspace of $W^{1,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.4). Under the assumption (2.3), $W^{1,\,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are reflexive. And we denote the dual space of $W_0^{1,p(x)}(\Omega)$ by $W^{-1,p'(x)}(\Omega)$.

For any $p_1, p_2 \in \mathcal{P}(\Omega)$, we denote by $p_1(x) \ll p_2(x)$ the fact that

$$\inf_{x\in\Omega}\,(p_2(x) - p_1(x)) > 0.$$

Theorem 2.4. *Let Ω be a bounded domain with the cone property. If $p \in C(\bar\Omega)$ satisfying* (2.1) *and q is a measurable function defined on Ω with*

$$1 \le q(x) \ll p^*(x) := \frac{Np(x)}{N - p(x)} \quad \text{a.e. } x \in \Omega,$$

then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Theorem 2.5. *Let Ω be a bounded domain with the cone property. If p is Lipschitz continuous and satisfies* (2.1)*, q is a measurable function defined on Ω with*

$$p(x) \le q(x) \le p^*(x) \quad \text{a.e. } x \in \Omega,$$

then there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

In this paper, we use the following subspace of $W^{1,\,p(x)}(\mathbb{R}^N)$:

$$X = \Big\{u \in W^{1,\,p(x)}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^{p(x)}\,dx < \infty\Big\}$$

with the norm

$$\|u\|_X = \inf\big\{t > 0 : \int_{\mathbb{R}^N} \frac{|\nabla u|^{p(x)} + V(x)|u|^{p(x)}}{t^{p(x)}}\,dx \le 1\big\},$$

where $V(x) \ge 0$. By standard arguments, it is clear that X is a uniformly convex Banach space. The norm $\|u\|_X$ has the following properties:

Theorem 2.6. *For any $u \in X$, we have*

(1) *if $\|u\|_X \ge 1$, then $\|u\|_X^{p_-} \le \int_{\mathbb{R}^N}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,dx \le \|u\|_X^{p_+}$;*
(2) *if $\|u\|_X < 1$, then $\|u\|_X^{p_+} \le \int_{\mathbb{R}^N}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,dx \le \|u\|_X^{p_-}$.*

Lemma 2.7. *Assume that hypothesis* (A2) *is fulfilled. Then the embedding $X \hookrightarrow W^{1,p(x)}(\mathbb{R}^N)$ is continuous.*

Proof. For any $u \in X$, we obtain

$$\int_{\mathbb{R}^N} \big|\nabla u\big|^{p(x)}\,dx + \int_{\mathbb{R}^N} V(x)|u|^{p(x)}\,dx < \infty.$$

Thus $\int_{\mathbb{R}^N} \big|\nabla u\big|^{p(x)}dx < \infty$, which implies $\int_{\mathbb{R}^N} \big|u\big|^{p^*(x)}dx < \infty$ (see [15, Theorem 8.3.1]). As

$$\int_{V^a} \big|u\big|^{p(x)}dx \le \int_{V^a} 1\,dx + \int_{V^a} \big|u\big|^{p^*(x)}dx,$$

we obtain

$$\int_{V^a} \big|u\big|^{p(x)}dx < \infty.$$

Note that

$$\begin{aligned} &\int_{\mathbb{R}^N} \big|\nabla u\big|^{p(x)}\,dx + \int_{\mathbb{R}^N} V(x)|u|^{p(x)}\,dx + a\int_{V^a} \big|u\big|^{p(x)}\,dx \\ &\ge \int_{\mathbb{R}^N} \big|\nabla u\big|^{p(x)}dx + a\int_{\mathbb{R}^N\setminus V^a} \big|u\big|^{p(x)}\,dx + a\int_{V^a} \big|u\big|^{p(x)}\,dx, \end{aligned}$$

we derive that

$$\int_{\mathbb{R}^N} (\big|\nabla u\big|^{p(x)} + \big|u\big|^{p(x)})dx < \infty.$$

Then $u \in W^{1,p(x)}(\mathbb{R}^N)$, i.e. $X \subset W^{1,p(x)}(\mathbb{R}^N)$. Define $i : X \to W^{1,p(x)}(\mathbb{R}^N)$ as follows:

$$i(u) = u.$$

Take $\{u_n\} \subset X$ such that $\|u_n - u\|_X \to 0$ as $n \to \infty$. We get that $\int_{\mathbb{R}^N} \big|\nabla u_n - \nabla u\big|^{p(x)}dx \to 0$ and $\int_{\mathbb{R}^N} V(x)\big|u_n - u\big|^{p(x)}dx \to 0$. Then, $\int_{\mathbb{R}^N} \big|u_n - u\big|^{p^*(x)}dx \to 0$, which implies
$|u_n - u|^{p(x)}\|_{p^*(x)/p(x)} \to 0$.

By Hölder inequality,

$$\int_{V^a} \big|u_n - u\big|^{p(x)}dx \le 2\|1\|_{\frac{p^*(x)}{p^*(x)-p(x)},V^a} \|u_n - u|^{p(x)}\|_{p^*(x)/p(x)}.$$

As V^a has finite Lebesgue measure, we derive that

$$\int_{V^a} \big|u_n - u\big|^{p(x)}dx \to 0.$$

From,

$$\int_{\mathbb{R}^N\setminus V^a} \big|u_n - u\big|^{p(x)}dx \le \frac{1}{a}\int_{\mathbb{R}^N} V(x)\big|u_n - u\big|^{p(x)}dx \to 0,$$

we obtain that $u_n \to u$ in $W^{1,p(x)}(\mathbb{R}^N)$. Thus i is continuous, which implies that X embeds continuously in $W^{1,p(x)}(\mathbb{R}^N)$. ∎

3. Proof of main results

Throughout this section, we assume that conditions (A1)–(A6) are satisfied. Without loss of generality, we assume that $V(0) = \min_{x\in\mathbb{R}^N} V(x) = 0$. Now we give the definition of weak solutions for problem (1.1):

Definition 3.1. We say that u is a weak solution of (1.1) if for any $v \in X$,

$$\int_{\mathbb{R}^N} (|\nabla u|^{p(x)-2}\nabla u\nabla v + V(x)uv)\,dx = \int_{\mathbb{R}^N} f(x,u)v\,dx.$$

The energy functional associated with (1.1) on X is defined as

$$I(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx.$$

It is easy to check that $I \in C^1(X, \mathbb{R})$ and the critical point for I is the weak solution of problem (1.1).

In the following, let $\{u_n\}$ be a $(\mathrm{PS})_{\mathrm{c}}$ sequence for functional I, i.e. $I(u_n) \to c$ and $I'(u_n) \to 0$ in X^*, as $n \to \infty$, where X^* is the dual space of X. Using (A5) and standard arguments we derive that $\{u_n\}$ is bounded in X. Passing to a subsequence, still denoted by $\{u_n\}$, we assume that $u_n \to u$ weakly in X, $u_n \to u$ in $L^{p(x)}_{loc}(\mathbb{R}^N)$, $L^{q(x)}_{loc}(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$, as $n \to \infty$. It is easy to verify that $I'(u) = 0$ and $I(u) \geq 0$.

Firstly, we will give some results to show that the $(\mathrm{PS})_{\mathrm{c}}$ condition holds for energy level c below some positive constant.

Lemma 3.2. *There is a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that for any $\sigma > 0$, there exists $r_\sigma > 0$, which satisfies*

$$\limsup_{j\to\infty} \int_{B_j\setminus B_r} |u_{n_j}|^s\,dx \leq \sigma \tag{3.1}$$

for any $r \geq r_\sigma$, where $s = p(x)$ or $s = q(x)$, $B_r = \{x \in \mathbb{R}^N : |x| < r\}$.

Proof. From Theorem 2.4, $\int_{B_r} |u_n|^s\,dx \to \int_{B_r} |u|^s\,dx$ as $n \to \infty$, for any $r > 0$. Then, there exists $n_j \in \mathbb{N}$ with $n_{j+1} > n_j$ such that

$$\int_{B_j} |u_{n_j}|^s\,dx - \int_{B_j} |u|^s\,dx < \frac{1}{j}.$$

For any $\sigma > 0$, there exists $r_\sigma > 0$ such that for any $r \geq r_\sigma$,

$$\int_{\mathbb{R}^N\setminus B_r} |u|^s\,dx < \sigma.$$

If $j > r_\sigma$, we have

$$\begin{aligned}
\int_{B_j\setminus B_r} |u_{n_j}|^s\,dx &= \int_{B_j} |u_{n_j}|^s\,dx - \int_{B_j} |u|^s\,dx + \int_{B_j\setminus B_r} |u|^s\,dx \\
&\quad + \int_{B_r} |u|^s\,dx - \int_{B_r} |u_{n_j}|^s\,dx \\
&< \frac{1}{j} + \sigma + \int_{B_r} |u|^s\,dx - \int_{B_r} |u_{n_j}|^s\,dx
\end{aligned}$$

for any $r \geq r_\sigma$, which implies $\limsup_{j\to\infty} \int_{B_j\setminus B_r} |u_{n_j}|^s\,dx \leq \sigma$. ∎

Take $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \varphi \le 1$; $\varphi(x) = 1$ for any $|x| \le 1$ and $\varphi(x) = 0$ for any $|x| \ge 2$. For any $j \in \mathbb{N}$, define $\varphi_j(x) = \varphi(\frac{2x}{j})$. Denote

$$\widehat{u}_j(x) = \varphi_j(x)u(x),$$

we have the following conclusion.

Lemma 3.3. *$\|\widehat{u}_j - u\|_X \to 0$ as $j \to \infty$.*

Proof. We have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla \widehat{u}_j - \nabla u|^{p(x)}\,dx \\
&= \int_{\mathbb{R}^N} |\nabla \varphi_j \cdot u + \varphi_j \cdot \nabla u - \nabla u|^{p(x)}\,dx \\
&\le 2^{p_+} \int_{\mathbb{R}^N} |\nabla \varphi_j|^{p(x)} |u|^{p(x)}\,dx + 2^{p_+} \int_{\mathbb{R}^N} |\varphi_j - 1|^{p(x)} |\nabla u|^{p(x)}\,dx.
\end{aligned}$$

As $|\nabla \varphi_j(x)| = |\frac{2}{j}\nabla\varphi(\frac{2x}{j})| \le \frac{C}{j}$, we obtain

$$\int_{\mathbb{R}^N} |\nabla \varphi_j|^{p(x)} |u|^{p(x)}\,dx \to 0.$$

Note that $|\varphi_j - 1|^{p(x)} |\nabla u|^{p(x)} \le 2^{p_+} |\nabla u|^{p(x)} \in L^1(\mathbb{R}^N)$ and $|\varphi_j - 1|^{p(x)} |\nabla u|^{p(x)} \to 0$ a.e. in $\mathbb{R}^N$, as $j \to \infty$, it follows from Lebesgue dominated convergence theorem that

$$\int_{\mathbb{R}^N} |\varphi_j - 1|^{p(x)} |\nabla u|^{p(x)}\,dx \to 0,$$

as $j \to \infty$. Then $\int_{\mathbb{R}^N} |\nabla \widehat{u}_j - \nabla u|^{p(x)}\,dx \to 0$.

Similarly, we obtain that as $j \to \infty$,

$$\int_{\mathbb{R}^N} V(x)|\widehat{u}_j - u|^{p(x)}\,dx = \int_{\mathbb{R}^N} V(x)|\varphi_j - 1|^{p(x)} |u|^{p(x)}\,dx \to 0.$$

Thus, $\|\widehat{u}_j - u\|_X \to 0$ as $j \to \infty$. □

Lemma 3.4. *For any $m > 1$, $\delta \in (0,1)$, there exists $L > 0$ such that for any ξ, $\eta \in \mathbb{R}^N$, if $|\xi - \eta| \ge \delta(|\xi| + |\eta|)$, then*

$$(|\xi|^{m-2}\xi - |\eta|^{m-2}\eta)(\xi - \eta) \ge L|\xi - \eta|^m.$$

Proof. From algebraic inequalities (see [11]) it follows that

$$(|\xi|^{m-2}\xi - |\eta|^{m-2}\eta)(\xi - \eta) \ge C\frac{|\xi - \eta|^2}{(|\xi| + |\eta|)^{2-m}}, \tag{3.2}$$

if $1 < m < 2$, and

$$(|\xi|^{m-2}\xi - |\eta|^{m-2}\eta)(\xi - \eta) \ge C|\xi - \eta|^m, \tag{3.3}$$

if $2 \le m < \infty$, for any $\xi, \eta \in \mathbb{R}^N$. We obtain that if $|\xi - \eta| \ge \delta(|\xi| + |\eta|)$, then

$$(|\xi|^{m-2}\xi - |\eta|^{m-2}\eta)(\xi - \eta) \ge L|\xi - \eta|^m.$$

The result follows. □

Lemma 3.5. *For any $R > 0$, $\nabla u_{n_j} \to \nabla u$ in $(L^{p(x)}(B_R))^N$ as $j \to \infty$.*

Proof. Take $R>0$. Let $v\in C_0^\infty(B_{2R})$ such that $0\le v\le 1$ in B_{2R} and $v\equiv 1$ in B_R, we obtain

$$\begin{aligned}
0&\le \int_{B_R}(|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_{n_j}-\nabla u)\,dx\\
&\le \int_{B_{2R}}(|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_{n_j}-\nabla u)v\,dx\\
&=\langle I'(u_{n_j}),u_{n_j}v\rangle-\langle I'(u_{n_j}),uv\rangle-\int_{B_{2R}}|\nabla u|^{p(x)-2}\nabla u(\nabla u_{n_j}-\nabla u)v\,dx\\
&\quad-\int_{B_{2R}}\big(|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}\nabla v\cdot u_{n_j}+V(x)|u_{n_j}|^{p(x)}v-f(x,u_{n_j})u_{n_j}v\big)\,dx\\
&\quad+\int_{B_{2R}}\big(|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}\nabla v\cdot u+V(x)|u_{n_j}|^{p(x)-2}u_{n_j}uv\\
&\quad-f(x,u_{n_j})uv\big)\,dx.
\end{aligned}$$

Note that $I'(u_{n_j})\to 0$ in X^*, we have $\langle I'(u_{n_j}),u_{n_j}v\rangle\to 0$ and $\langle I'(u_{n_j}),uv\rangle\to 0$. As $u_{n_j}\to u$ weakly in X, $\nabla u_{n_j}\to\nabla u$ weakly in $(L^{p(x)}(\mathbb{R}^N))^N$ as $j\to\infty$. Then

$$\int_{B_{2R}}|\nabla u|^{p(x)-2}\nabla u(\nabla u_{n_j}-\nabla u)v\,dx\to 0.$$

Using Theorem 2.3, we obtain

$$\begin{aligned}
&\Big|\int_{B_{2R}}(|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}\nabla v\cdot u_{n_j}-|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}\nabla v\cdot u)\,dx\Big|\\
&\le C\int_{B_{2R}}|\nabla u_{n_j}|^{p(x)-1}|u_{n_j}-u|\,dx\\
&\le C\||\nabla u_{n_j}|^{p(x)-1}\|_{p'(x)}\|u_{n_j}-u\|_{p(x),B_{2R}},
\end{aligned}$$

which implies

$$\int_{B_{2R}}|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}\nabla v\cdot u_{n_j}\,dx-\int_{B_{2R}}|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}\nabla v\cdot u\,dx\to 0,$$

as $j\to\infty$. Similarly, we obtain

$$\begin{gathered}
\int_{B_{2R}}V(x)|u_{n_j}|^{p(x)}v\,dx-\int_{B_{2R}}V(x)|u_{n_j}|^{p(x)-2}u_{n_j}uv\,dx\to 0,\\
\int_{B_{2R}}f(x,u_{n_j})u_{n_j}v\,dx-\int_{B_{2R}}f(x,u_{n_j})uv\,dx\to 0,
\end{gathered}$$

thus

$$\int_{B_R}(|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_{n_j}-\nabla u)\,dx\to 0.$$

Similarly to the proof of [12, Theorem 3.1], B_R is divided into two parts:

$$B_{R,1}=\{x\in B_R:p(x)<2\},\quad B_{R,2}=\{x\in B_R:p(x)\ge 2\}.$$

We could verify that

$$\begin{gathered}
\int_{B_{R,1}}|\nabla u_{n_j}-\nabla u|^{p(x)}\,dx\to 0,\\
\int_{B_{R,2}}|\nabla u_{n_j}-\nabla u|^{p(x)}\,dx\to 0.
\end{gathered}$$

Thus, we obtain
$$\int_{B_R} |\nabla u_{n_j} - \nabla u|^{p(x)}\,dx \to 0,$$
i.e. $\nabla u_{n_j} \to \nabla u$ in $(L^{p(x)}(B_R))^N$ for any $R > 0$. Moreover, up to a subsequence, we assume that $\nabla u_{n_j} \to \nabla u$ a.e. in $\mathbb{R}^N$. □

Lemma 3.6. *(1)*
$$|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j} - |\nabla u_{n_j} - \nabla \widehat{u}_j|^{p(x)-2}\nabla(u_{n_j} - \widehat{u}_j) - |\nabla \widehat{u}_j|^{p(x)-2}\nabla \widehat{u}_j \to 0$$
in $(L^{p'(x)}(\mathbb{R}^N))^N$*;*
(2)
$$V(x)^{\frac{1}{p'(x)}}\big(|u_{n_j}|^{p(x)-2}u_{n_j} - |u_{n_j} - \widehat{u}_j|^{p(x)-2}(u_{n_j} - \widehat{u}_j) - |\widehat{u}_j|^{p(x)-2}\widehat{u}_j\big) \to 0$$
in $L^{p'(x)}(\mathbb{R}^N)$*, as* $j \to \infty$*.*

Proof. For any $\xi \in (L^{p(x)}(\mathbb{R}^N))^N$, define
$$\begin{aligned} I_j(\xi) = \int_{\mathbb{R}^N} \Big(&|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j} - |\nabla u_{n_j} - \nabla \widehat{u}_j|^{p(x)-2}\nabla(u_{n_j} - \widehat{u}_j) \\ &- |\nabla \widehat{u}_j|^{p(x)-2}\nabla \widehat{u}_j\Big)\xi\,dx. \end{aligned}$$
Then, I_j belongs to $(L^{p'(x)}(\mathbb{R}^N))^N$, which is the dual space of $(L^{p(x)}(\mathbb{R}^N))^N$. To derive (1), it suffices to verify that $I_j \to 0$ in $(L^{p'(x)}(\mathbb{R}^N))^N$, as $j \to \infty$.

As $|\nabla u|^{p(x)} \in L^1(\mathbb{R}^N)$, it follows that for any $\varepsilon \in (0,1)$, there exists $R_0 > 0$ such that if $R \geq R_0$,
$$\int_{\mathbb{R}^N \setminus B_R} |\nabla u|^{p(x)}\,dx \leq \min\{\varepsilon^{p_+}, \varepsilon^{\frac{p_-}{p_- - 1}}\} < 1.$$
Also, $\widehat{u}_j \to u$ in X implies that $\int_{\mathbb{R}^N} |\nabla \widehat{u}_j - \nabla u|^{p(x)}\,dx \to 0$. There exists $j_0 > 0$ such that for any $j \geq j_0$,
$$\int_{\mathbb{R}^N \setminus B_{R_0}} |\nabla \widehat{u}_j|^{p(x)}\,dx \leq \min\big\{\varepsilon^{p_+}, \varepsilon^{\frac{p_-}{p_- - 1}}\big\} < 1.$$
Using Theorem 2.1, we obtain $\|\nabla \widehat{u}_j\|_{p(x),\mathbb{R}^N \setminus B_{R_0}} < \varepsilon$ and $|\nabla \widehat{u}_j|^{p(x)-1}\|_{p'(x),\mathbb{R}^N \setminus B_{R_0}} < \varepsilon$.

For any $\delta > 0$, denote
$$\begin{aligned} C_{\delta,j} = \Big\{x \in \mathbb{R}^N \setminus B_{R_0} :& \big|\,|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j} - |\nabla u_{n_j} - \nabla \widehat{u}_j|^{p(x)-2}\nabla(u_{n_j} - \widehat{u}_j)\big| \\ &\geq \delta(|\nabla u_{n_j}|^{p(x)-1} + |\nabla u_{n_j} - \nabla \widehat{u}_j|^{p(x)-1})\Big\} \end{aligned}$$
and
$$A_j = \mathbb{R}^N \setminus (B_{R_0} \cup C_{\delta,j}),$$
we have $\|\nabla \widehat{u}_j\|_{p(x),C_{\delta,j}} < \varepsilon$, $\|\nabla \widehat{u}_j\|_{p(x),A_j} < \varepsilon$ and $|\nabla \widehat{u}_j|^{p(x)-1}\|_{p'(x),C_{\delta,j}} < \varepsilon$.

Next, we split the integral in $I_j(\xi)$ on the sets B_{R_0}, $C_{\delta,j}$ and A_j and denote
$$\begin{aligned} I_j^1(\xi) = \int_{B_{R_0}} \Big(&|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j} - |\nabla u_{n_j} - \nabla \widehat{u}_j|^{p(x)-2}\nabla(u_{n_j} - \widehat{u}_j) \\ &- |\nabla \widehat{u}_j|^{p(x)-2}\nabla \widehat{u}_j\Big)\xi\,dx, \end{aligned}$$

$$
\begin{aligned}
I_j^2(\xi)=\int_{C_{\delta,j}}&\Big(|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}-|\nabla u_{n_j}-\nabla\widehat{u}_j|^{p(x)-2}\nabla(u_{n_j}-\widehat{u}_j)\\
&-|\nabla\widehat{u}_j|^{p(x)-2}\nabla\widehat{u}_j\Big)\xi\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
I_j^3(\xi)=\int_{A_j}&\Big(|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}-|\nabla u_{n_j}-\nabla\widehat{u}_j|^{p(x)-2}\nabla(u_{n_j}-\widehat{u}_j)\\
&-|\nabla\widehat{u}_j|^{p(x)-2}\nabla\widehat{u}_j\Big)\xi\,dx.
\end{aligned}
$$

By Holder's inequality, we obtain

$$
\begin{aligned}
|I_j^1(\xi)|\le 2\|\,&|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}-|\nabla u_{n_j}-\nabla\widehat{u}_j|^{p(x)-2}\nabla(u_{n_j}-\widehat{u}_j)\\
&-|\nabla\widehat{u}_j|^{p(x)-2}\nabla\widehat{u}_j\|_{p'(x),B_{R_0}}\|\xi\|_{p(x),B_{R_0}}.
\end{aligned}
$$

By Lemma 3.5, we have $\nabla u_{n_j}\to\nabla u$ in $(L^{p(x)}(B_{R_0}))^N$ and $\nabla u_{n_j}\to\nabla u$ a.e. in B_{R_0}. Then

$$
\big\{\big||\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}-|\nabla u_{n_j}-\nabla\widehat{u}_j|^{p(x)-2}\nabla(u_{n_j}-\widehat{u}_j)-|\nabla\widehat{u}_j|^{p(x)-2}\nabla\widehat{u}_j\big|^{p'(x)}\big\}
$$

is equi-integrable in $L^1(B_{R_0})$. Using Vitali's theorem, we derive

$$
\begin{aligned}
\int_{B_{R_0}}&\big||\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}-|\nabla u_{n_j}-\nabla\widehat{u}_j|^{p(x)-2}\nabla(u_{n_j}-\widehat{u}_j)\\
&-|\nabla\widehat{u}_j|^{p(x)-2}\nabla\widehat{u}_j\big|^{p'(x)}\,dx\to 0,
\end{aligned}
$$

as $j\to\infty$, which implies

$$
\big\|\,|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}-|\nabla u_{n_j}-\nabla\widehat{u}_j|^{p(x)-2}\nabla(u_{n_j}-\widehat{u}_j)-|\nabla\widehat{u}_j|^{p(x)-2}\nabla\widehat{u}_j\big\|_{p'(x),B_{R_0}}\to 0.
$$

Thus, for any $\varepsilon>0$, there exists $j_1\ge j_0$ such that if $j\ge j_1$,

$$
|I_j^1(\xi)|\le C\varepsilon\|\xi\|_{p(x),B_{R_0}}.
$$

Note that

$$
\begin{aligned}
|I_j^2(\xi)|\le 2\|&\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}-|\nabla u_{n_j}-\nabla\widehat{u}_j|^{p(x)-2}\nabla(u_{n_j}-\widehat{u}_j)\|_{p'(x),C_{\delta,j}}\\
&\times\|\xi\|_{p(x),C_{\delta,j}}+2\big\||\nabla\widehat{u}_j|^{p(x)-1}\big\|_{p'(x),C_{\delta,j}}\|\xi\|_{p(x),C_{\delta,j}}.
\end{aligned}
$$

Using Lemma 3.4, we obtain

$$
\begin{aligned}
&\int_{C_{\delta,j}}\big||\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}-|\nabla u_{n_j}-\nabla\widehat{u}_j|^{p(x)-2}\nabla(u_{n_j}-\widehat{u}_j)\big|^{p'(x)}\,dx\\
&\le C\int_{C_{\delta,j}}(|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}-|\nabla u_{n_j}-\nabla\widehat{u}_j|^{p(x)-2}\nabla(u_{n_j}-\widehat{u}_j))\nabla\widehat{u}_j\,dx\\
&\le C\big\||\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}-|\nabla u_{n_j}-\nabla\widehat{u}_j|^{p(x)-2}\nabla(u_{n_j}-\widehat{u}_j)\big\|_{p'(x)}\|\nabla\widehat{u}_j\|_{p(x),C_{\delta,n}}\\
&\le C\varepsilon\big\||\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}-|\nabla u_{n_j}-\nabla\widehat{u}_j|^{p(x)-2}\nabla(u_{n_j}-\widehat{u}_j)\big\|_{p'(x)}.
\end{aligned}
$$

As $\{|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}-|\nabla u_{n_j}-\nabla\widehat{u}_j|^{p(x)-2}\nabla(u_{n_j}-\widehat{u}_j)\}$ is bounded in the space $(L^{p'(x)}(\mathbb{R}^N))^N$, we obtain

$$
|I_j^2(\xi)|\le C\varepsilon\|\xi\|_{p(x),C_{\delta,j}}.
$$

It follows from Theorem 2.3 that

$$\begin{aligned}|I_j^3(\xi)| &\le \int_{A_j} \big(\delta(|\nabla u_{n_j}|^{p(x)-1} + |\nabla u_{n_j} - \nabla \widehat{u}_j|^{p(x)-1})|\xi| + |\nabla \widehat{u}_j|^{p(x)-1}|\xi|\big)\,dx \\ &\le C\delta\big\||\nabla u_{n_j}|^{p(x)-1} + |\nabla u_{n_j} - \nabla \widehat{u}_j|^{p(x)-1}\big\|_{p'(x)}\|\xi\|_{p(x)} \\ &\quad + C\|\,|\nabla \widehat{u}_j|^{p(x)-1}\|_{p'(x),\,A_j}\|\xi\|_{p(x)} \\ &\le C\delta\|\xi\|_{p(x)} + C\varepsilon\|\xi\|_{p(x)}.\end{aligned}$$

Taking $\delta = \varepsilon$, we obtain

$$|I_j^3(\xi)| \le C\varepsilon\|\xi\|_{p(x)}.$$

From the above discussion, we obtain

$$|I_j(\xi)| \le C\varepsilon\|\xi\|_{p(x)},$$

which implies $I_j \to 0$ in $(L^{p'(x)}(\mathbb{R}^N))^N$, as $j \to \infty$. Thus, conclusion (1) follows. Similarly, we can get the proof of (2). $\square$

Lemma 3.7. *For any $j \in \mathbb{N}$, denote $u^1_{n_j} = u_{n_j} - \widehat{u}_j$. Then $I(u^1_{n_j}) \to c - I(u)$ and $I'(u^1_{n_j}) \to 0$ in X^*, as $j \to \infty$.*

Proof. As $u_{n_j} \to u$ weakly in X and $\widehat{u}_j \to u$ in X, we obtain

$$\int_{\mathbb{R}^N} |\nabla \widehat{u}_j|^{p(x)-2}\nabla \widehat{u}_j(\nabla u_{n_j} - \nabla \widehat{u}_j)\,dx \to 0.$$

Note that the set $\{|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}\}$ is bounded in $(L^{p'(x)}(\mathbb{R}^N))^N$, and that $|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j} \to |\nabla u|^{p(x)-2}\nabla u$ a.e. in $\mathbb{R}^N$, thus

$$|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j} \to |\nabla u|^{p(x)-2}\nabla u \quad \text{weakly in } (L^{p'(x)}(\mathbb{R}^N))^N,$$

which implies $\int_{\mathbb{R}^N} |\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}\nabla \widehat{u}_j\,dx \to \int_{\mathbb{R}^N} |\nabla u|^{p(x)}\,dx$. It follows from Lemma 3.6 that

$$\begin{aligned}&\int_{\mathbb{R}^N} \big(|\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j} - |\nabla u_{n_j} - \nabla \widehat{u}_j|^{p(x)-2}(\nabla u_{n_j} - \nabla \widehat{u}_j) - |\nabla \widehat{u}_j|^{p(x)-2}\nabla \widehat{u}_j\big) \\ &\times (\nabla u_{n_j} - \nabla \widehat{u}_j)\,dx \to 0,\end{aligned}$$

which implies

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla u_{n_j} - \nabla \widehat{u}_j|^{p(x)}\,dx \\ &= \int_{\mathbb{R}^N} |\nabla u_{n_j}|^{p(x)}\,dx - \int_{\mathbb{R}^N} |\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j}\nabla \widehat{u}_j\,dx + o(1) \\ &= \int_{\mathbb{R}^N} |\nabla u_{n_j}|^{p(x)}\,dx - \int_{\mathbb{R}^N} |\nabla u|^{p(x)}\,dx + o(1).\end{aligned}$$

Similarly, we obtain

$$\int_{\mathbb{R}^N} V(x)|u_{n_j} - \widehat{u}_j|^{p(x)}\,dx = \int_{\mathbb{R}^N} V(x)|u_{n_j}|^{p(x)}\,dx - \int_{\mathbb{R}^N} V(x)|u|^{p(x)}\,dx + o(1).$$

Similar to the proof of Brezis-Lieb lemma, it is easy to get

$$\int_{\mathbb{R}^N} (F(x, u_{n_j}) - F(x, u_{n_j} - \widehat{u}_j) - F(x, \widehat{u}_j))\,dx \to 0,$$

as $j \to \infty$. We obtain

$$I(u^1_{n_j}) = I(u_{n_j}) - I(u) + o(1) = c - I(u) + o(1)$$

Thus, $I(u^1_{n_j}) \to c - I(u)$, as $j \to \infty$.

Note that

$$\big\|I'(u^1_{n_j}) - I'(u_{n_j}) + I'(\widehat{u}_j)\big\| = \sup_{\|v\|_X=1, v\in X} \big|\langle I'(u^1_{n_j}) - I'(u_{n_j}) + I'(\widehat{u}_j), v\rangle\big|.$$

Take $v \in X$ with $\|v\|_X = 1$. It follows from Hölder inequality that

$$\begin{aligned}
&\big|\langle I'(u^1_{n_j}) - I'(u_{n_j}) + I'(\widehat{u}_j), v\rangle\big| \\
&= \Big|\int_{\mathbb{R}^N} \big((|\nabla u^1_{n_j}|^{p(x)-2}\nabla u^1_{n_j} - |\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j} + |\nabla \widehat{u}_j|^{p(x)-2}\nabla \widehat{u}_j)\nabla v \\
&\quad + V(x)(|u^1_{n_j}|^{p(x)-2}u^1_{n_j} - |u_{n_j}|^{p(x)-2}u_{n_j} + |\widehat{u}_j|^{p(x)-2}\widehat{u}_j)v\big)\, dx \\
&\quad + \int_{\mathbb{R}^N} (f(x,u_{n_j}) - f(x,u_{n_j} - \widehat{u}_j) - f(x,\widehat{u}_j))v\big)\, dx\Big| \\
&\le 2\big\||\nabla u^1_{n_j}|^{p(x)-2}\nabla u^1_{n_j} - |\nabla u_{n_j}|^{p(x)-2}\nabla u_{n_j} + |\nabla \widehat{u}_j|^{p(x)-2}\nabla \widehat{u}_j\big\|_{p'(x)}\|\nabla v\|_{p(x)} \\
&\quad + 2\big\|V(x)^{\frac{1}{p'(x)}}(|u^1_{n_j}|^{p(x)-2}u^1_{n_j} - |u_{n_j}|^{p(x)-2}u_{n_j} + |\widehat{u}_j|^{p(x)-2}\widehat{u}_j)\big\|_{p'(x)}\|V^{\frac{1}{p(x)}}v\|_{p(x)} \\
&\quad + \int_{\mathbb{R}^N} |(f(x,u_{n_j}) - f(x,u_{n_j} - \widehat{u}_j) - f(x,\widehat{u}_j))v|\, dx.
\end{aligned}$$

We have

$$\begin{aligned}
&|\int_{\mathbb{R}^N}(f(x,u_{n_j}) - f(x,u_{n_j} - \widehat{u}_j) - f(x,\widehat{u}_j))v\, dx| \\
&\le \int_{B_r} |f(x,u_{n_j}) - f(x,u_{n_j} - \widehat{u}_j) - f(x,\widehat{u}_j)| \cdot |v|\, dx \\
&\quad + \int_{\mathbb{R}^N\setminus B_r} |f(x,u_{n_j}) - f(x,u_{n_j} - \widehat{u}_j) - f(x,\widehat{u}_j(x))|\, |v|\, dx
\end{aligned}$$

for any $r \ge r_\sigma$, where r_σ is from Lemma 3.2.

By (A3) and (A4), for any $t > 0$ we obtain

$$|f(x,t)| \le C(|t|^{p(x)-1} + |t|^{q(x)-1}).$$

Note that $\widehat{u}_j \to u$ and $u_{n_j} \to u$ in $L^{p(x)}(B_r)$ and $L^{q(x)}(B_r)$, respectively, we could verify that

$$\int_{B_r} |f(x,u_{n_j}) - f(x,u_{n_j} - \widehat{u}_j) - f(x,\widehat{u}_j)| \cdot |v|\, dx \to 0 \tag{3.4}$$

uniformly in $v \in X$ with $\|v\|_X \le 1$. Also,

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus B_r} |f(x,u_{n_j}) - f(x,u_{n_j} - \widehat{u}_j) - f(x,\widehat{u}_j)| \cdot |v|\, dx \\
&= \int_{B_j\setminus B_r} |f(x,u_{n_j}) - f(x,u_{n_j} - \widehat{u}_j) - f(x,\widehat{u}_j)| \cdot |v|\, dx \\
&\le C\int_{B_j\setminus B_r} (|u_{n_j}|^{p(x)-1} + |\widehat{u}_j|^{p(x)-1} + |u_{n_j}|^{q(x)-1} + |\widehat{u}_j|^{q(x)-1})\, |v|\, dx.
\end{aligned}$$

For any $\sigma > 0$, by (3.1) and the Hölder inequality we obtain

$$\begin{aligned}
&\limsup_{j\to\infty}\int_{B_j\setminus B_r}(|u_{n_j}|^{p(x)-1}+|u_{n_j}|^{q(x)-1})\cdot|v|\,dx\\
&\le \limsup_{j\to\infty}2\big\||u_{n_j}|^{p(x)-1}\big\|_{p'(x),B_j\setminus B_r}\|v\|_{p(x),B_j\setminus B_r}\\
&\quad+\limsup_{j\to\infty}2\big\||u_{n_j}|^{q(x)-1}\big\|_{q'(x),B_j\setminus B_r}\|v\|_{q(x),B_j\setminus B_r}\\
&\le C\Big(\sigma^{\frac{p_--1}{p_-}}+\sigma^{\frac{q_--1}{q_-}}\Big).
\end{aligned}\tag{3.5}$$

As $\widehat{u}_j \to u$ in X, $\widehat{u}_j \to u$ in $L^{p(x)}(\mathbb{R}^N)$ and $L^{q(x)}(\mathbb{R}^N)$. Then

$$\begin{aligned}
&\limsup_{j\to\infty}\int_{\mathbb{R}^N\setminus B_r}(|\widehat{u}_j|^{p(x)-1}+|\widehat{u}_j|^{q(x)-1})\cdot|v|\,dx\\
&=\int_{\mathbb{R}^N\setminus B_r}(|u|^{p(x)-1}+|u|^{q(x)-1})\cdot|v|\,dx\\
&\le 2\big\||u|^{p(x)-1}\big\|_{p'(x),\mathbb{R}^N\setminus B_r}\|v\|_{p(x),\mathbb{R}^N\setminus B_r}\\
&\quad+2\big\||u|^{q(x)-1}\big\|_{q'(x),\mathbb{R}^N\setminus B_r}\|v\|_{q(x),\mathbb{R}^N\setminus B_r}\\
&\le C\Big(\sigma^{\frac{p_--1}{p_-}}+\sigma^{\frac{q_--1}{q_-}}\Big).
\end{aligned}\tag{3.6}$$

From (3.4)-(3.6), we have

$$\begin{aligned}
&\limsup_{j\to\infty}\int_{\mathbb{R}^N}|f(x,u_{n_j})-f(x,u_{n_j}-\widehat{u}_j)-f(x,\widehat{u}_j)|\cdot|v|\,dx\\
&\le C\Big(\sigma^{\frac{p_--1}{p_-}}+\sigma^{\frac{q_--1}{q_-}}\Big)
\end{aligned}$$

uniformly in $v \in X$ with $\|v\|_X \le 1$. Let $\sigma \to 0$,

$$\limsup_{j\to\infty}\int_{\mathbb{R}^N}|f(x,u_{n_j})-f(x,u_{n_j}-\widehat{u}_j)-f(x,\widehat{u}_j)|\cdot|v|\,dx=0.$$

As $I'(u_{n_j}) \to 0$ and $I'(\widehat{u}_j) \to I'(u) = 0$, we obtain $I'(u^1_{n_j}) \to 0$, as $j \to \infty$. ☐

Next, we show that I satisfies the $(\mathrm{PS})_{\mathrm{c}}$ condition for energy level c below some positive constant.

Theorem 3.8. *There exists $\widetilde{c} > 0$ such that for any $c \in (0, \widetilde{c})$, $u_{n_j} \to u$ in X as $j \to \infty$.*

Proof. From (A3), we have

$$\begin{aligned}
\Big(\frac{f(x,t)}{|t|^{p_+-1}}\Big)^{\frac{q(x)}{q(x)-p_+}}\frac{1}{f(x,t)|t|}&\le c_0(1+|t|^{q(x)-1})^{\frac{p_+}{q(x)-p_+}}|t|^{-(p_+-1)\frac{q(x)}{q(x)-p_+}-1}\\
&\le C+C|t|^{-(p_+-1)\frac{q(x)}{q(x)-p_+}-1}.
\end{aligned}\tag{3.7}$$

By (A4), for any $\delta > 0$, there exists $t_\delta \in (0,1)$ such that for any $|t| < t_\delta$, $|f(x,t)/|t|^{p_+-1}| < \delta$; Also, for any $|t| > t_\delta$, from (3.7) we obtain that

$$\Big|\frac{f(x,t)}{t^{p_+-1}}\Big|^{\frac{q(x)}{q(x)-p_+}} \le C_\delta|f(x,t)t|.$$

We have

$$\begin{aligned} &\int_{\mathbb{R}^N} f(x,u^1_{n_j})u^1_{n_j}\,dx \\ &= \int_{\{x\in\mathbb{R}^N:|u^1_{n_j}|<t_\delta\}} f(x,u^1_{n_j})u^1_{n_j}\,dx + \int_{\{x\in\mathbb{R}^N:|u^1_{n_j}|\ge t_\delta\}} f(x,u^1_{n_j})u^1_{n_j}\,dx. \end{aligned}$$

For the first term on the right-hand side, we obtain

$$\begin{aligned} \Big|\int_{\{x\in\mathbb{R}^N:|u^1_{n_j}|<t_\delta\}} f(x,u^1_{n_j})u^1_{n_j}\,dx\Big| &\le \int_{\{x\in\mathbb{R}^N:|u^1_{n_j}|<t_\delta\}} \Big|\frac{f(x,u^1_{n_j})}{|u^1_{n_j}|^{p_+-1}}\Big|\,|u^1_{n_j}|^{p_+}\,dx \\ &< \delta\int_{\{x\in\mathbb{R}^N:|u^1_{n_j}|<t_\delta\}} |u^1_{n_j}|^{p_+}\,dx \\ &< \delta\int_{\{x\in\mathbb{R}^N:|u^1_{n_j}|<t_\delta\}} |u^1_{n_j}|^{p(x)}\,dx. \end{aligned} \tag{3.8}$$

For the second term,

$$\begin{aligned} &\Big|\int_{\{x\in\mathbb{R}^N:|u^1_{n_j}|\ge t_\delta\}} f(x,u^1_{n_j})u^1_{n_j}\,dx\Big| \\ &\le \int_{\{x\in\mathbb{R}^N:|u^1_{n_j}\ge t_\delta\}} \Big|\frac{f(x,u^1_{n_j})}{|u^1_{n_j}|^{p_+-1}}\Big||u^1_{n_j}|^{p_+}\,dx \\ &\le \Big\|\frac{f(x,u^1_{n_j})}{|u^1_{n_j}|^{p_+-1}}\Big\|_{\frac{q(x)}{q(x)-p_+}}\,\||u^1_{n_j}|^{p_+}\|_{\frac{q(x)}{p_+}}. \end{aligned} \tag{3.9}$$

Note that

$$\begin{aligned} &I(u^1_{n_j}) - \frac{1}{p_+}\langle I'(u^1_{n_j}),u^1_{n_j}\rangle \\ &= c - I(u) + o(1) \\ &= \int_{\mathbb{R}^N} \Big(\frac{1}{p(x)}-\frac{1}{p_+}\Big)|\nabla u^1_{n_j}|^{p(x)} + \Big(\frac{1}{p(x)}-\frac{1}{p_+}\Big)V(x)|u^1_{n_j}|^{p(x)} \\ &- \int_{\mathbb{R}^N}(F(x,u^1_{n_j}) - \frac{1}{p_+}f(x,u^1_{n_j})u^1_{n_j})\,dx \\ &\ge \big(\frac{1}{p_+}-\frac{1}{\mu}\big)\int_{\mathbb{R}^N} f(x,u^1_{n_j})u^1_{n_j}\,dx, \end{aligned}$$

which implies

$$\begin{aligned} &\int_{\{x\in\mathbb{R}^N:|u^1_{n_j}|\ge t_\delta\}} \Big|\frac{f(x,u^1_{n_j})}{|u^1_{n_j}|^{p_+-1}}\Big|^{\frac{q(x)}{q(x)-p_+}}\,dx \\ &\le C_\delta\int_{\{x\in\mathbb{R}^N:|u^1_{n_j}|\ge t_\delta\}} |f(x,u^1_{n_j})u^1_{n_j}|\,dx \\ &\le C_\delta\frac{\mu p_+}{\mu-p_+}(c-I(u)+o(1)). \end{aligned} \tag{3.10}$$

As

$$\langle I'(u^1_{n_j}),u^1_{n_j}\rangle = \int_{\mathbb{R}^N}(|\nabla u^1_{n_j}|^{p(x)} + V(x)|u^1_{n_j}|^{p(x)})\,dx - \int_{\mathbb{R}^N} f(x,u^1_{n_j})u^1_{n_j}\,dx$$

$$= o(1)$$

and

$$\big\||u_{n_j}^1|^{p_+}\big\|_{\frac{q(x)}{p_+}} = \|u_{n_j}^1\|_{q(x)}^{p_+} \le C\|u_{n_j}^1\|_X^{p_+},$$

by (3.8)-(3.10), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} (|\nabla u_{n_j}^1|^{p(x)} + V(x)|u_{n_j}^1|^{p(x)})\,dx \\
&= \int_{\mathbb{R}^N} f(x, u_{n_j}^1)u_{n_j}^1\,dx + o(1) \\
&\le \delta \int_{\{x\in\mathbb{R}^N:|u_{n_j}^1|<t_\delta\}} |u_{n_j}^1|^{p(x)}\,dx + C\|u_{n_j}^1\|_X^{p_+}\Big[(C_\delta \frac{\mu p_+}{\mu - p_+}(c - I(u) + o(1)))^{\frac{q_- - p_+}{q_-}} \\
&\quad + (C_\delta \frac{\mu p_+}{\mu - p_+}(c - I(u) + o(1)))^{\frac{q_+ - p_+}{q_+}}\Big].
\end{aligned}$$

As V^a has finite Lebesgue measure, we obtain $\lim_{R\to\infty}|V^a \setminus B_R| = 0$. Then, for any $\eta \in (0,1)$, there exists $R_1 > 0$ such that $|V^a \setminus B_R| < \eta$ for any $R \ge R_1$. We have

$$\begin{aligned}
\int_{\mathbb{R}^N} (a - V(x))|u_{n_j}^1|^{p(x)}\,dx &\le \int_{V^a} (a - V(x))|u_{n_j}^1|^{p(x)}\,dx \\
&= \int_{V^a\setminus B_{R_1}} (a - V(x))|u_{n_j}^1|^{p(x)}\,dx \\
&\quad + \int_{V^a\bigcap B_{R_1}} (a - V(x))|u_{n_j}^1|^{p(x)}\,dx.
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
\Big|\int_{V^a\setminus B_{R_1}} (a - V(x))|u_{n_j}^1|^{p(x)}\,dx\Big| &\le \int_{V^a\setminus B_{R_1}} 2a|u_{n_j}^1|^{p(x)}\,dx \\
&\le 2a\big\||u_{n_j}^1|^{p(x)}\big\|_{\frac{q(x)}{q(x)-p_+}}\|1\|_{\frac{q(x)}{p_+},V^a\setminus B_{R_1}} \\
&\le C\eta^{\frac{p_+}{q_+}}.
\end{aligned} \tag{3.11}$$

As $u_{n_j}^1 \to 0$ weakly in X, $u_{n_j}^1 \to 0$ in $L^{p(x)}(B_{R_1})$, as $j \to \infty$. Then, for the above $\eta > 0$, there exists $j_2 \in \mathbb{N}$ such that for any $j \ge j_2$,

$$\Big|\int_{V^a\cap B_{R_1}} (a - V(x))|u_{n_j}^1|^{p(x)}\,dx\Big| \le 2a\int_{B_{R_1}} |u_{n_j}^1|^{p(x)}\,dx \le 2a\eta. \tag{3.12}$$

Then,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big(|\nabla u_{n_j}^1|^{p(x)} + \frac{1}{2}V(x)|u_{n_j}^1|^{p(x)} + \frac{a}{2}|u_{n_j}^1|^{p(x)}\big)\,dx \\
&= \int_{\mathbb{R}^N} \big(f(x, u_{n_j}^1)u_{n_j}^1 + \frac{a}{2}|u_{n_j}^1|^{p(x)} - \frac{1}{2}V(x)|u_{n_j}^1|^{p(x)}\big)\,dx + o(1) \\
&\le \delta \int_{\{x\in\mathbb{R}^N:|u_{n_j}^1|<t_\delta\}} |u_{n_j}^1|^{p(x)}\,dx + C\|u_{n_j}^1\|_X^{p_+}\Big[(C_\delta \frac{\mu p_+}{\mu - p_+}(c - I(u) + o(1)))^{\frac{q_- - p_+}{q_-}} \\
&+ (C_\delta \frac{\mu p_+}{\mu - p_+}(c - I(u) + o(1)))^{\frac{q_+ - p_+}{q_+}}\Big] + C\eta^{\frac{p_+}{q_+}} + a\eta + o(1).
\end{aligned}$$

Letting $\eta \to 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big(|\nabla u_{n_j}^1|^{p(x)} + \frac{1}{2}V(x)|u_{n_j}^1|^{p(x)} + \frac{a}{2}|u_{n_j}^1|^{p(x)}\big)\,dx \\
&\le \delta \int_{\{x\in\mathbb{R}^N:|u_{n_j}^1|<t_\delta\}} |u_{n_j}^1|^{p(x)}\,dx + C\|u_{n_j}^1\|_X^{p_+}\Big[(C_\delta \frac{\mu p_+}{\mu - p_+}(c - I(u) + o(1)))^{\frac{q_- - p_+}{q_-}} \\
&\quad + (C_\delta \frac{\mu p_+}{\mu - p_+}(c - I(u) + o(1)))^{\frac{q_+ - p_+}{q_+}}\Big] + o(1).
\end{aligned}$$

Taking $\delta = a/2$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big(|\nabla u_{n_j}^1|^{p(x)} + \frac{1}{2}V(x)|u_{n_j}^1|^{p(x)}\big)\,dx \\
&\le C\|u_{n_j}^1\|_X^{p_+}\Big[(C_\delta \frac{\mu p_+}{\mu - p_+}(c - I(u) + o(1)))^{\frac{q_- - p_+}{q_-}} \\
&\quad + (C_\delta \frac{\mu p_+}{\mu - p_+}(c - I(u) + o(1)))^{\frac{q_+ - p_+}{q_+}}\Big] + o(1).
\end{aligned}$$

Suppose that $u_{n_j}^1 \nrightarrow 0$ in X. If $\|u_{n_j}^1\|_X < 1$, then

$$\begin{aligned}
\|u_{n_j}^1\|_X^{p_+} &\le C\|u_{n_j}^1\|_X^{p_+}\Big[(C_\delta \frac{\mu p_+}{\mu - p_+}(c - I(u) + o(1)))^{\frac{q_- - p_+}{q_-}} \\
&\quad + (C_\delta \frac{\mu p_+}{\mu - p_+}(c - I(u) + o(1)))^{\frac{q_+ - p_+}{q_+}} + o(1)\Big].
\end{aligned}$$

Then

$$\begin{aligned}
1 &\le C\Big[(C_\delta \frac{\mu p_+}{\mu - p_+}(c - I(u) + o(1)))^{\frac{q_- - p_+}{q_-}} \\
&\quad + (C_\delta \frac{\mu p_+}{\mu - p_+}(c - I(u) + o(1)))^{\frac{q_+ - p_+}{q_+}} + o(1)\Big].
\end{aligned} \tag{3.13}$$

Let $j \to \infty$. From $I(u) \ge 0$ we have

$$\begin{aligned}
1 &\le C\Big[(C_\delta \frac{\mu p_+}{\mu - p_+}(c - I(u)))^{\frac{q_- - p_+}{q_-}} + (C_\delta \frac{\mu p_+}{\mu - p_+}(c - I(u)))^{\frac{q_+ - p_+}{q_+}}\Big] \\
&\le C\Big[(C_\delta \frac{\mu p_+}{\mu - p_+}c)^{\frac{q_- - p_+}{q_-}} + (C_\delta \frac{\mu p_+}{\mu - p_+}c)^{\frac{q_+ - p_+}{q_+}}\Big] \\
&:= A_1\Big(c^{\frac{q_- - p_+}{q_-}} + c^{\frac{q_+ - p_+}{q_+}}\Big)
\end{aligned} \tag{3.14}$$

If $\|u_{n_j}^1\|_X \ge 1$ and

$$\begin{aligned}
\|u_{n_j}^1\|_X^{p_-} &\le C\|u_{n_j}^1\|_X^{p_+}\Big[(C_\delta \frac{\mu p_+}{\mu - p_+}(c - I(u) + o(1)))^{\frac{q_- - p_+}{q_-}} \\
&\quad + (C_\delta \frac{\mu p_+}{\mu - p_+}(c - I(u) + o(1)))^{\frac{q_+ - p_+}{q_+}} + o(1)\Big],
\end{aligned} \tag{3.15}$$

then

$$\begin{aligned}
1 &\le C\|u_{n_j}^1\|_X^{p_+ - p_-}\Big[(C_\delta \frac{\mu p_+}{\mu - p_+}(c + o(1)))^{\frac{q_- - p_+}{q_-}} \\
&\quad + (C_\delta \frac{\mu p_+}{\mu - p_+}(c + o(1)))^{\frac{q_+ - p_+}{q_+}} + o(1)\Big].
\end{aligned}$$

Besides,

$$I(u_{n_j}^1) - \frac{1}{\mu}\langle I'(u_{n_j}^1), u_{n_j}^1\rangle \geq \frac{\mu - p_+}{\mu p_+}\|u_{n_j}^1\|_X^{p_-},$$

which implies $\|u_{n_j}^1\|_X^{p_-} \leq \frac{\mu p_+}{\mu - p_+}(c - I(u) + o(1))$. We get

$$\begin{aligned} 1 \leq C\Big[\frac{\mu p_+}{\mu - p_+}(c - I(u) + o(1))\Big]^{\frac{p_+ - p_-}{p_-}} \Big[(C_\delta \frac{\mu p_+}{\mu - p_+}(c - I(u) + o(1)))^{\frac{q_- - p_+}{q_-}} \\ + (C_\delta \frac{\mu p_+}{\mu - p_+}(c - I(u) + o(1)))^{\frac{q_+ - p_+}{q_+}} + o(1)\Big]. \end{aligned} \tag{3.16}$$

Let $j \to \infty$,

$$\begin{aligned} 1 &\leq C\Big(\frac{\mu p_+}{\mu - p_+}c\Big)^{\frac{p_+ - p_-}{p_-}} \Big[(C_\delta \frac{\mu p_+}{\mu - p_+}c)^{\frac{q_- - p_+}{q_-}} + (C_\delta \frac{\mu p_+}{\mu - p_+}c)^{\frac{q_+ - p_+}{q_+}}\Big] \\ &:= A_2\Big(c^{\frac{p_+ - p_-}{p_-} + \frac{q_- - p_+}{q_-}} + c^{\frac{p_+ - p_-}{p_-} + \frac{q_+ - p_+}{q_+}}\Big) \end{aligned} \tag{3.17}$$

Choosing $\widetilde{c} > 0$ such that for any $c \in (0, \widetilde{c})$,

$$A_1\Big(c^{\frac{q_- - p_+}{q_-}} + c^{\frac{q_+ - p_+}{q_+}}\Big) \leq 1,$$

$$A_2\Big(c^{\frac{p_+ - p_-}{p_-} + \frac{q_- - p_+}{q_-}} + c^{\frac{p_+ - p_-}{p_-} + \frac{q_+ - p_+}{q_+}}\Big) \leq 1.$$

From (3.14) and (3.17), if $c \in (0, \widetilde{c})$, we obtain a contradiction, which implies $u_{n_j}^1 \to 0$ in X. $\square$

The next result shows that I has a Mountain Pass geometry.

Theorem 3.9. *For any $\delta > 0$, there exists $t_\delta > 0$ and $\phi_\delta \in X$ such that $I(t_\delta \phi_\delta) < 0$.*

Proof. Recall that

$$\inf\Big\{\int_{\mathbb{R}^N} |\nabla \phi|^{p_-}\,dx : \phi \in C_0^\infty(\mathbb{R}^N), \|\phi\|_{\alpha_+} = 1\Big\} = 0. \tag{3.18}$$

Then, for any $\delta > 0$ we can choose $\phi_\delta \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \phi_\delta \leq 1$, $\|\phi_\delta\|_{\alpha_+} = 1$, $\int_{\mathbb{R}^N} |\nabla \phi_\delta|^{p_-}\,dx < \delta$ and $\operatorname{supp}\phi_\delta \subset B_{r_\delta}$.

Taking $t > 1$, from (A6) we obtain

$$\begin{aligned} I(t\phi_\delta) &= \int_{\mathbb{R}^N} \frac{t^{p(x)}}{p(x)}(|\nabla \phi_\delta|^{p(x)} + V(x)|\phi_\delta|^{p(x)})\,dx - \int_{\mathbb{R}^N} F(x, t\phi_\delta)\,dx \\ &< \frac{t^{p_+}}{p_-}\int_{\mathbb{R}^N}(|\nabla \phi_\delta|^{p(x)} + V(x)|\phi_\delta|^{p(x)})\,dx - c_1 t^{\alpha_-}\int_{\mathbb{R}^N} |\phi_\delta|^{\alpha(x)}\,dx. \end{aligned} \tag{3.19}$$

As $\alpha_- > p_+$, there exists $t_\delta > 0$ such that $I(t_\delta \phi_\delta) < 0$. $\square$

Theorem 3.10. *There exist $r > 0$ and $0 < \rho < \|t_\delta \phi_\delta\|_X$ such that $I(u) \geq r$ for any $u \in X$ with $\|u\|_X = \rho$ and $I(u) > 0$ for any $u \in X$ with $\|u\|_X < \rho$, where t_δ and ϕ_δ are from Theorem 3.9.*

Proof. By (A3) and (A4), for any $\lambda' > 0$, there exists $C > 0$ such that

$$|F(x,t)| \leq \lambda'|t|^{p_+} + C|t|^{q(x)}. \tag{3.20}$$

We take $c_2, c_3 > 0$, which are the embedding constant of $X \hookrightarrow L^{p_+}(\mathbb{R}^N), L^{q(x)}(\mathbb{R}^N)$, respectively. For any $u \in X$ with $\|u\|_X \le \min\{1, \frac{1}{c_2}, \frac{1}{c_3}\}$, we derive

$$\begin{aligned} I(u) &\ge \int_{\mathbb{R}^N} \frac{1}{p_+}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,dx - \int_{\mathbb{R}^N} (\lambda'|u|^{p_+} + C|u|^{q(x)})\,dx \\ &\ge \frac{1}{p_+}\|u\|_X^{p_+} - \lambda'\|u\|_{p_+}^{p_+} - C\|u\|_{q(x)}^{q_-} \\ &\ge \frac{1}{p_+}\|u\|_X^{p_+} - \lambda' c_2\|u\|_X^{p_+} - C\|u\|_X^{q_-}. \end{aligned}$$

Taking $\lambda' < 1/(2c_2p_+)$, we obtain

$$I(u) \ge \frac{1}{2p_+}\|u\|_X^{p_+} - C\|u\|_X^{q_-}.$$

As $q_- > p_+$, there exist $r > 0$ and $0 < \rho < \min\{1, \frac{1}{c_2}, \frac{1}{c_3}, \|t_\delta\phi_\delta\|_X\}$ such that $I(u) \ge r$ for any $u \in X$ with $\|u\|_X = \rho$ and $I(u) > 0$ for any $u \in X$ with $\|u\|_X < \rho$. □

Next, we verify that, problem (1.1) has a nonnegative solution.

Proof of Theorem 1.1. (1) For any $t > 0$, it follows from (3.19) that

$$\begin{aligned} I(t\phi_\delta) &< \frac{t^{p_+}}{p_-}\Big(\delta + \int_{\mathbb{R}^N} V(x)|\phi_\delta|^{p(x)}\,dx\Big) - c_1t^{\alpha_-} \\ &\le \Big(\frac{p_+}{\alpha_-}\Big)^{\frac{p_+}{\alpha_- - p_+}}\Big(\frac{1}{p_-}\Big)^{\frac{\alpha_-}{\alpha_- - p_+}}\frac{\alpha_- - p_+}{\alpha_-}\frac{(\delta + \int_{\mathbb{R}^N} V(x)|\phi_\delta|^{p(x)}\,dx)^{\frac{\alpha_-}{\alpha_- - p_+}}}{c_1^{\frac{p_+}{\alpha_- - p_+}}} \end{aligned}$$

As $V(0) = \min_{x\in\mathbb{R}^N} V(x) = 0$, there exists $x_0 > 0$ such that for any $|x| < x_0$, $|V(x)| \le \frac{\delta}{\|\phi_\delta\|_{p_-}^{p_-}}$. Then

$$\int_{\mathbb{R}^N} V(x)|\phi_\delta|^{p(x)}\,dx \le \int_{\mathbb{R}^N} V(x)|\phi_\delta|^{p_-}\,dx < \delta.$$

We have

$$I(t\phi_\delta) \le \Big(\frac{p_+}{\alpha_-}\Big)^{\frac{p_+}{\alpha_- - p_+}}\Big(\frac{1}{p_-}\Big)^{\frac{\alpha_-}{\alpha_- - p_+}}\frac{\alpha_- - p_+}{\alpha_-}c_1^{\frac{p_+}{p_+ - \alpha_-}}(2\delta)^{\frac{\alpha_-}{\alpha_- - p_+}}.$$

Denote

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = t_\delta\phi_\delta\}$. Then

$$\inf_{\|u\|_X=\rho} I(u) > I(0) > I(t_\delta\phi_\delta)$$

and

$$\begin{aligned} 0 < r \le c &\le I(tt_\delta\phi_\delta) \\ &\le \Big(\frac{p_+}{\alpha_-}\Big)^{\frac{p_+}{\alpha_- - p_+}}\Big(\frac{1}{p_-}\Big)^{\frac{\alpha_-}{\alpha_- - p_+}}\frac{\alpha_- - p_+}{\alpha_-}c_1^{\frac{p_+}{p_+ - \alpha_-}}(2\delta)^{\frac{\alpha_-}{\alpha_- - p_+}}. \end{aligned} \tag{3.21}$$

By the Mountain Pass Theorem, there exists $\{u_n\}_n \subset X$ such that

$$I(u_n) \to c, \quad I'(u_n) \to 0 \quad \text{in } X^*, \text{ as } n \to \infty.$$

For any $\delta > 0$ with

$$(\frac{p_+}{\alpha_-})^{\frac{p_+}{\alpha_- - p_+}}(\frac{1}{p_-})^{\frac{\alpha_-}{\alpha_- - p_+}}\frac{\alpha_- - p_+}{\alpha_-}c_1^{\frac{p_+}{p_+ - \alpha_-}}(2\delta)^{\frac{\alpha_-}{\alpha_- - p_+}} < \widetilde{c},$$

by Theorem 3.8, there is a subsequence $\{u_{n_j}\}$ such that $u_{n_j} \to u$ in X. Thus $I(u) = c$ and $I'(u) = 0$, i.e. u is a nontrivial weak solution of problem (1.1).

(2) u is nonnegative. In fact, it suffices to consider the following functionals on X:

$$I^+(u) = I(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,dx - \int_{\mathbb{R}^N} F(x, u^+)\,dx.$$

Similar to the discussion in (1), we could verify that problem (1.1) has a nonnegative weak solution u. ☐

Now we establish the existence of m pairs of solutions of (1.1) using the Lusternik-Schnirelman theory of critical points. Let $\Sigma(X)$ be the family of sets $F \subseteq \Sigma(X)\setminus\{0\}$ such that F is closed in X and symmetric with respect to 0, i.e. $x \in F$ implies $-x \in F$. For $F \in \Sigma(X)$, we define the genus of F to be k, denoted by $\text{gen}(F) = k$, if there is a continuous and odd map $\psi : F \to \mathbb{R}^k \setminus \{0\}$ and k is the smallest integer with this property. The definition of genus here, which was introduced by Coffman [14], is equivalent with the the Krasnoselski original genus.

Denote by Γ_* the set of all odd homeomorphisms $g \in C(X, X)$ such that $g(0) = 0$ and $g(B_1) \subseteq \{u \in X : I_\varepsilon(u) \geq 0\}$. We denote by Γ_m the set of all compact subsets F of X which are symmetric with respect to the origin and satisfies $\text{gen}(F \cap g(\partial B_1)) \geq m$ for any $g \in \Gamma_*$. We refer to [11] for more details.

Proof of Theorem 1.2. From (3.18), for any $m \in \mathbb{N}$, we take $\phi_\delta^j \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\phi_\delta^j \subset B_{r_{m,\delta}}$, $\|\phi_\delta^j\|_{\alpha_+} = 1$,

$$\int_{\mathbb{R}^N} |\nabla \phi_\delta^j|^{p_-}\,dx < \delta,$$

for any $j = 1, 2, \dots, m$ and $\operatorname{supp}\phi_\delta^i \cap supp\phi_\delta^j = \emptyset$ for any $i \neq j$.

Define the m-dimensional subspace $F_m^\delta = \text{span}\{\phi_\delta^j : j = 1, 2, \dots, m\}$. For any $\delta > 0$ with

$$\Big(\frac{p_+}{\alpha_-}\Big)^{\frac{p_+}{\alpha_- - p_+}}\Big(\frac{1}{p_-}\Big)^{\frac{\alpha_-}{\alpha_- - p_+}}\frac{\alpha_- - p_+}{\alpha_-}c_1^{\frac{p_+}{p_+ - \alpha_-}}(2\delta)^{\frac{\alpha_-}{\alpha_- - p_+}} < \widetilde{c},$$

where $\widetilde{c}$ is from Theorem 3.8. Then, for any $u \in F_m^\delta$ with $u = \sum_{j=1}^m t_j\phi_\delta^j$, by (A6) we obtain

$$\begin{aligned}
I(u) &\leq \frac{1}{p_-}\int_{\mathbb{R}^N}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,dx - c_1\int_{\mathbb{R}^N}|u|^{\alpha(x)}\,dx \\
&= \sum_{j=1}^m \Big(\frac{1}{p_-}\int_{\mathbb{R}^N} t_j^{p(x)}(|\nabla\phi_\delta^j|^{p(x)} + V(x)|\phi_\delta^j|^{p(x)})\,dx \\
&\quad - c_1\int_{\mathbb{R}^N} t_j^{\alpha(x)}|\phi_\delta^j|^{\alpha(x)}\,dx\Big) \\
&\leq m\Big(\frac{p_+}{\alpha_-}\Big)^{\frac{p_+}{\alpha_- - p_+}}\Big(\frac{1}{p_-}\Big)^{\frac{\alpha_-}{\alpha_- - p_+}}\frac{\alpha_- - p_+}{\alpha_-}c_1^{\frac{p_+}{p_+ - \alpha_-}}(2\delta)^{\frac{\alpha_-}{\alpha_- - p_+}}.
\end{aligned} \tag{3.22}$$

As F_m^δ is a finite-dimensional space and $\|\cdot\|_{\alpha_+}$ is also a norm on F_m^δ, $\|\cdot\|_{\alpha_+}$ and $\|\cdot\|_X$ are equivalent. By the first inequality in (3.22), we obtain $I(u)\to-\infty$, as $u\in F_m^\delta$ with $\|u\|_X\to\infty$. Then, there exists $R_m>\rho$ such that for any $u\in F_m^\delta$: if $\|u\|_X\geq R_m$, $I(u)<0$.

For any $1\leq j\leq m$, let

$$c^j=\inf_{F\in\Gamma_m}\max_{u\in F}I(u),$$

we have $r\leq c^1\leq c^2\leq\dots\leq c^m\leq\sup_{u\in F_m^\delta}I(u)\leq\widetilde{c}$. From Theorem 3.8, I satisfies $(PS)_{c^j}$ condition. Thus, c^j is a critical value of I and u_j is a critical point of I with $I(u_j)=c^j$. As $f(x,t)$ is odd in t, we derive that $-u_j$ is also a critical point of I. Then I has at least m pairs of nontrivial solutions. □

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