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GROUND AND BOUND STATES OF PERIODIC SCHRÖDINGER EQUATIONS WITH SUPER OR ASYMPTOTICALLY LINEAR TERMS

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Abstract. This paper is concerned with existence of ground and bound states for a class of nonlinear Schrödinger equation with periodic potential. We impose general assumptions on the nonlinearity with super or asymptotically linear growth, and find some refinements of known results and new results by using the perturbation method and a mountain pass argument. In particular, a critical point theory is established for the asymptotically linear growth case.

1. Introduction and statement of main results

We consider the nonlinear stationary Schrödinger equation

$$\begin{gathered} -\triangle u + V(x)u = f(x,u), \quad \text{for } x \in \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{gathered} \tag{1.1}$$

where $V \in C(\mathbb{R}^N)$ and $f \in C(\mathbb{R}^N \times \mathbb{R})$ depend periodically on x. Equation (1.1) is a fundamental equation of quantum mechanics which appears in many applications, such as studies of Bose-Einstein condensates in condensed matter physics and gap solitons in photonic crystals [18]. Solutions of (1.1) can also be interpreted as stationary states of the corresponding reaction-diffusion equation which models phenomena from chemical dynamics [3].

The principal aim of this paper is to establish the existence of ground and bound states of (1.1) when f is asymptotically linear or superlinear as $|u| \to \infty$. Denote by Φ the energy functional associated with (1.1), see (1.7). A nontrivial solution u_0 of (1.1) is called a ground state if $\Phi(u_0)$ is the lowest level for Φ at which there are nontrivial solutions of (1.1), i.e., u_0 satisfies

$$\Phi(u_0) = \inf_{\mathcal{K}} \Phi, \quad \text{where } \mathcal{K} := \{u \in E \setminus \{0\} : \Phi'(u) = 0\}, \tag{1.2}$$

it will be seen later that ground state u_0 can be characterized as the minimizer of Φ on the Nehari-Pankov manifold $\mathcal{N}^-$ which has infinite dimension and infinite co-dimension, i.e.,

$$\begin{gathered} \Phi(u_0) = \inf_{\mathcal{N}^-} \Phi, \\ \text{where } \mathcal{N}^- = \{u \in E \setminus E^- : \langle \Phi'(u), u\rangle = \langle \Phi'(u), v\rangle = 0, \ \forall v \in E^-\}, \end{gathered} \tag{1.3}$$

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$\mathcal{N}^-$ is first introduced by Pankov [18], and E, E^- are working spaces defined later in Section 2. Since $V(x)$ is periodic in x, the operator $\mathcal{A} := -\triangle + V$ has purely continuous spectrum $\sigma(\mathcal{A})$ which is bounded below and consists of closed disjoint intervals (see [24, Theorem XIII.100]). Depending on the location of 0 in $\sigma(\mathcal{A})$, there are many results of existence and multiplicity solutions to equation (1.1) with periodic potential V, see, e.g., [1, 3, 7, 13, 17, 18, 19, 23, 26, 29, 30, 36, 37, 38] for superlinear case and [6, 8, 9, 10, 15, 16, 12, 21, 27, 31, 33, 34] for asymptotically linear case.

In this article, we consider mainly the case when 0 is a boundary point of the spectrum $\sigma(\mathcal{A})$, i.e. the potential $V(x)$ satisfies the assumption

(H1) $V \in C(\mathbb{R}^N)$ is 1-periodic in x_i, $i = 1, 2, \dots, N$, $0 \in \sigma(\mathcal{A})$, and there exists $b_0 > 0$ such that $(0, b_0] \cap \sigma(\mathcal{A}) = \emptyset$.

Different from the case $0 \notin \sigma(\mathcal{A})$, the working space is only a Banach space, not a Hilbert space. Another difficulty to overcome is the lack of a priori bounds for the Palais-Smale sequences or Cerami sequences. Existence of nontrivial solutions were obtained in [3, 37] with the aid of an approximation argument, and in [36] using an improved generalized linking theorem. In recent paper [30], a new variational setting which is more suitable for this case was established by Tang. Moreover, ground state solutions satisfying (1.2) were obtained under following general assumptions:

(H2) $f \in C(\mathbb{R}^N \times \mathbb{R})$ is 1-periodic in each of $x_1, x_2, \dots, x_N$, and there exist constants c_1, $c_2 > 0$ and $2 < \varrho \le p < 2^*$ such that

$$c_1 \min\left\{|t|^\varrho, |t|^2\right\} \le tf(x,t) \le c_2(|t|^\varrho + |t|^p), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}; \tag{1.4}$$

(H3) $\lim_{|t|\to\infty} F(x,t)/t^2 = \infty$ uniformly in $x \in \mathbb{R}^N$, where $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$.

A similar result can be found in [17] where following Nehari type condition was used,

(H4) $t \mapsto f(x,t)/|t|$ is strictly increasing on $(-\infty, 0) \cup (0, \infty)$.

The generalized Nehari manifold method used in [17, 26] depends heavily on (H4), and it seems not valid by weakening (H4) to following weak version:

(H5) $t \mapsto f(x,t)/|t|$ is non-decreasing on $(-\infty, 0) \cup (0, \infty)$.

Therefore it is interesting to obtain a ground state solution of (1.1) without (H4). Using the non-Nehari manifold method introduced in [31], existence of a ground state solution satisfying (1.3) was proved in [19] under (H2)–(H3) and (H5), however, it is still unknown whether the same result can be found by using the generalized Nehari manifold method. So some new tricks are looked forward to being introduced which is the right issue this paper intends to address. When the nonlinearity f is asymptotically linear at infinity, the variational framework and approaches used in [3, 17, 36, 37] seem no more applicable, there seem only two papers in the literature dealing with this case, they are references [20, 21] where ground state solutions satisfying (1.2) and (1.3), were obtained using the concentration compactness arguments and the non-Nehari manifold method. In particular, (H2), (H5) and following assumption on asymptotical behavior of f were used there.

(H6) $f(x,t) = V_\infty(x)t + f_\infty(x,t)$, where $V_\infty \in C(\mathbb{R}^N)$ is 1-periodic in x_1, x_2, $\dots$, x_N, $f_\infty(x,t) = o(|t|)$ as $|t| \to \infty$ uniformly in $x \in \mathbb{R}^N$, and $\inf V_\infty(x) > \overline{\Lambda} := \inf[\sigma(\mathcal{A}) \cap (0,\infty)]$.

Therefore, it is nature to ask whether the asymptotically linear case can be studied further via a new method different from those used in [20, 21]?

Motivated by above works, in the paper, we shall introduce new tricks to solve above intriguing problems. Using the idea introduced in [22], we first establish a critical point theory for asymptotically linear growth case and show that $\inf_{\mathcal{N}^-}\Phi$ can be attained by a nontrivial solution of (1.1). Particularly, following weaker condition than (H4) is used, which allows to introduce the Nehari-Pankov manifold and to define a ground state as minimizer of the energy functional on it.

(H7) $t \mapsto f(x,t)/|t|$ is non-decreasing on $(-\infty,0)\cup(0,\infty)$, and strictly increasing on $[-\tau_0,0)\cup(0,\tau_0]$ for some $\tau_0>0$.

Afterwards, we consider the superlinear growth case and introduce a abstract critical point theory on manifold $\mathcal{N}^-$ and following manifold studied in [5]:

$$\mathcal{M} := \{u \in E : \Phi'(u)|_{E^-} = 0\}. \tag{1.5}$$

Based on the perturbation method used in [32], we first show that $\inf_{\mathcal{N}^-}\Phi$ is achieved under (H2), (H3) and (H5). Subsequently, we obtain a nontrivial solution of (1.1) via a mountain pass argument on the constraint $\mathcal{M}$, provided the following classic Ambrosetti-Rabinowitz condition and convexity condition (H9) are satisfied. Minimax characterization of the corresponding critical value is also given if (H5) is additionally satisfied.

(H8) there exists $2<\mu<2^*$ such that

$$tf(x,t) \ge \mu F(x,t), \quad \forall x \in \mathbb{R}^N,\ t \in \mathbb{R}; \tag{1.6}$$

(H9) $F(x,t)$ is convex with respect to $t\in\mathbb{R}$ for all $x\in\mathbb{R}^N$, or, equivalently, $F(x,t+s)-F(x,t)-f(x,t)s\ge 0$ for all $t,s\in\mathbb{R}$, $x\in\mathbb{R}^N$.

By Lemma 3.4 we see that (H9) is satisfied if (H5) holds.

Let E be the Banach space defined in Section 2. Under assumptions (H1) and (H2), the following functional

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x, \tag{1.7}$$

is well defined for all $u\in E$, moreover $\Phi\in C^1(E,\mathbb{R})$ (see Lemma 2.2). A standard argument [35] shows that critical points of Φ are the solutions of (1.1).

Before stating result for asymptotically linear growth case, we introduce following condition:

(H10) $f(x,t) = V_\infty(x)t + f_\infty(x,t)$, where $V_\infty\in C(\mathbb{R}^N)$ is 1-periodic in each of $x_1,x_2,\dots,x_N$ with $\inf V_\infty>0$, $f_\infty(x,t)=o(|t|)$ as $|t|\to\infty$ uniformly in $x\in\mathbb{R}^N$, and there exists a $u_0\in E^+\setminus\{0\}$ such that

$$\|u_0\|_*^2 - \|v\|_*^2 - \int_{\mathbb{R}^N} V_\infty(x)(u_0+v)^2\mathrm{d}x < 0, \quad \forall v \in E^-. \tag{1.8}$$

The norm $\|\cdot\|_*$ is defined later in (2.3). Condition (H10) is weaker than (H6). Indeed, let $\{\mathcal{E}(\lambda) : -\infty<\lambda<+\infty\}$ be the spectral family of operator $\mathcal{A}$, if $\inf V_\infty > \overline{\Lambda}$, then we have $\bar\Lambda\|\bar u\|_2^2 \le \|\bar u\|_*^2 \le \bar\mu\|\bar u\|_2^2$ for every $\bar u\in(\mathcal{E}(\bar\mu)-\mathcal{E}(0))E\subset E^+$ by taking $\bar\mu\in\big(\bar\Lambda,\inf V_\infty\big)$. Thus for any $v\in E^-$,

$$\begin{aligned}
&\|\bar u\|_*^2 - \|v\|_*^2 - \int_{\mathbb{R}^N} V_\infty(x)(\bar u+v)^2\mathrm{d}x\\
&\le \|\bar u\|_*^2 - \|v\|_*^2 - \inf V_\infty\left(\|\bar u\|_2^2 + \|v\|_2^2\right)\\
&\le -\big[(\inf V_\infty - \bar\mu)\|\bar u\|_2^2 + \inf V_\infty\|v\|_2^2\big] < 0,
\end{aligned}$$

this shows (1.8).

Theorem 1.1. *Let* (H1), (H2), (H7), (H10) *be satisfied. Then* (1.1) *has a ground state solution* $u_0 \in E \setminus \{0\}$ *satisfying* $\Phi(u_0) = \inf_{\mathcal{N}^-} \Phi = \inf_{\mathcal{K}} \geq \kappa$, *where* κ *is a positive constant. Moreover,*

$$\int_{\mathbb{R}^N} \left[|\nabla u_0|^2 + (V(x) - V_\infty(x))\, u_0^2\right] \mathrm{d}x < 0.$$

Corollary 1.2. *Let* (H1), (H2), (H6), (H7) *be satisfied. Then there is a ground state solution* $u_0 \in E \setminus \{0\}$ *of* (1.1) *such that* $\Phi(u_0) = \inf_{\mathcal{N}^-} \Phi = \inf_{\mathcal{K}} \Phi \geq \kappa > 0$ *and*

$$\int_{\mathbb{R}^N} \left[|\nabla u_0|^2 + (V(x) - V_\infty(x))\, u_0^2\right] \mathrm{d}x < 0.$$

Note that, $\mathcal{N}^-$ contains all nontrivial critical points of Φ, i.e. the critical set $\mathcal{K}$ is a very small subset of $\mathcal{N}^-$. Therefore $\inf_{\mathcal{N}^-} \Phi \leq \inf_{\mathcal{K}} \Phi$. If there is a nontrivial solution u_0 such that $\Phi(u_0) = \inf_{\mathcal{N}^-} \Phi$, then $\inf_{\mathcal{K}} \Phi$ is attained at u_0 and $\Phi(u_0) = \inf_{\mathcal{N}^-} \Phi = \inf_{\mathcal{K}} \Phi$ which implies that u_0 is a ground state solution.

For superlinear growth case, we have following results.

Theorem 1.3. *Let* (H1), (H2) *be satisfied.*

(a) *If* (H3), (H5) *hold, then* (1.1) *has a solution* $u_0 \in E\setminus\{0\}$ *satisfying* $\Phi(u_0) = \inf_{\mathcal{N}^-} \Phi = \inf_{\mathcal{K}} \Phi \geq \kappa_0$, *where* κ_0 *is a positive constant.*

(b) (H8), (H9) *hold, then* (1.1) *has a nontrivial solution. Moreover if* (H5) *is satisfied, then* (1.1) *has a solution* $u_0 \in E \setminus \{0\}$ *such that*

$$\Phi(u_0) = \inf_{\mathcal{N}^-} \Phi = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \Phi(\gamma(t)) \geq \kappa_0 > 0, \tag{1.9}$$

where $\Gamma = \{\gamma \in C([0,1], \mathcal{M}) : \gamma(0) = 0,\ \|\gamma(1)^+\| > r,\ \Phi(\gamma(1)) < 0\}$, *and* $r > 0$ *satisfying* $\inf_{u\in E^+:\|u\|=r} \Phi(u) > 0$.

Theorems 1.1 and 1.3 give a positive answer to the problems mentioned in the introduction. In Section 3, we give the proof of Theorem 1.1 by improving the generalized Nehari manifold method [4, 26], see Theorem 3.1, as a consequence of which the least energy value $c_0 := \inf_{\mathcal{N}^-} \Phi$ has a minimax characterization given by

$$c_0 = \Phi(u_0) = \inf_{v\in E_0^+\setminus\{0\}} \max_{u\in E^-\oplus\mathbb{R}^+v} \Phi(u), \tag{1.10}$$

where E_0^+ is defined later by (3.1). Theorem 1.1 complements the results in [21] where Corollary 1.2 was obtained via the concentration compactness arguments. Based on the generalized Nehari manifold method [26] and a perturbation argument, we find the Palais-Smale sequences in Section 4 and give the proof of Theorem 1.3 in Section 5, respectively. It is easy to see that Theorem 1.3-(a) improves the related results in [17, 37] and Theorem 1.3-(b) seems to be new. Compared with [3, Theorem 1.1], the following necessary condition (H11) used in [3, 36, 37] is not needed in Theorem 1.3-(b) with the additional convexity condition (H9).

(H11) there exist constants $C > 0$, $2 < \varrho < 2^*$ such that

$$tf(x,t) \geq C|t|^\varrho, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Under condition (H5), characterization of minimal energy can be derived from Theorem 1.3-(b), i.e.,

$$c_0 = \Phi(u_0) = \inf_{v \in E^+ \setminus \{0\}} \max_{u \in E^- \oplus \mathbb{R}^+ v} \Phi(u) = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \Phi(\gamma(t)).$$

Note that these minimax principles are much simpler than the usual characterizations related to the concept of linking.

Remark 1.4. Under assumptions of Theorem 1.1, there exists a $\alpha_0 > 0$ such that

$$\begin{gathered} f(x,t) f_\infty(x,t) \le 0, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, \quad \text{and} \\ f(x,t) f_\infty(x,t) < 0 \quad \text{for } 0 < |t| \le \alpha_0. \end{gathered} \tag{1.11}$$

Indeed, by (H2), (H5) and (H10), one sees that $f_\infty(x,t)/t$ is non-decreasing on $t \in (0,\infty)$ and non-increasing on $t \in (-\infty, 0)$, and $f_\infty(x,t)/t \to -V_\infty(x) < 0$ as $|t| \to 0$. These together with $f_\infty(x,t) = o(|t|)$ as $|t| \to \infty$ imply that $t f_\infty(x,t) \le 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Together with the fact $t f(x,t) > 0$ for $t \neq 0$ using (1.4), we deduce that there exists a $\alpha_0 > 0$ satisfying (1.11).

Before proceeding to the proof of main results, we give some nonlinear examples. Functions listed in Examples 1.5, 1.6 satisfy all assumptions of Corollary 1.2. For function f defined by Example 1.7, it satisfies all the assumptions of Theorem 1.3, but it does not satisfy (H11).

Example 1.5. $f(x,t) = V_\infty(x) \min\{|t|^\nu, 1\} t$, where $\nu \in (0, 2^* - 2)$, $V_\infty \in C(\mathbb{R}^N)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$ and $\inf V_\infty > \overline{\Lambda}$.

Example 1.6. $f(x,t) = V_\infty(x) t [1 - \frac{1}{\ln(e + |t|^\nu)}] t$, where $\nu \in (0, 2^* - 2)$, $V_\infty \in C(\mathbb{R}^N)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$ and $\inf V_\infty > \overline{\Lambda}$.

Example 1.7. $f(x,t) = h(x) \min\{\frac{1}{\varrho_1} |t|^{\varrho_1 - 2}, \frac{1}{\varrho_2} |t|^{\varrho_2 - 2}\} t$, where $2 < \varrho_1 < \varrho_2 < 2^*$ and $h \in C(\mathbb{R}^N)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$ with $\inf h > 0$.

This article is organized as follows. In Section 2, we briefly introduce the variational framework setting established in [30]. Equation (1.1) with asymptotically linear growth nonlinearity is considered in Section 3 where a critical point theory is established and Theorem 1.1 is proved. In Section 4, we consider the superlinear growth case and find the Palais-Smale sequences on the manifolds $\mathcal{M}$ and $\mathcal{N}^-$, respectively. Theorem 1.3 is showed in the last Section by studying a perturbed problem associated with (1.1).

2. Variational setting

In this section, as in [30], we introduce the variational framework associated with problem (1.1). Throughout this paper, we denote by $\|\cdot\|_s$ the usual $L^s(\mathbb{R}^N)$ norm for $s \in [1, \infty)$ and C_i, $i \in \mathbb{N}$ for different positive constants. Note that operator $\mathcal{A} = -\Delta + V$ is self-adjoint in $L^2(\mathbb{R}^N)$ with domain $\mathfrak{D}(\mathcal{A}) = H^2(\mathbb{R}^N)$. Let $\{\mathcal{E}(\lambda) : -\infty < \lambda < +\infty\}$ be the spectral family of $\mathcal{A}$, and $|\mathcal{A}|^{1/2}$ be the square root of $|\mathcal{A}|$. Set $\mathcal{U} = id - \mathcal{E}(0) - \mathcal{E}(0-)$. Then $\mathcal{U}$ commutes with $\mathcal{A}$, $|\mathcal{A}|$ and $|\mathcal{A}|^{1/2}$, and $\mathcal{A} = \mathcal{U}|\mathcal{A}|$ is the polar decomposition of $\mathcal{A}$ (see [11, Theorem 4.3.3]). Let $E_* = \mathfrak{D}(|\mathcal{A}|^{1/2})$, the domain of $|\mathcal{A}|^{1/2}$, then $\mathcal{E}(\lambda) E_* \subset E_*$ for all $\lambda \in \mathbb{R}$. E_* is Hilbert space with the inner product

$$(u,v)_0 = \big(|\mathcal{A}|^{1/2} u, |\mathcal{A}|^{1/2} v\big)_{L^2} + (u,v)_{L^2}, \quad \forall u, v \in E_*,$$

and the norm

$$\|u\|_0 = \sqrt{(u,v)_0}, \quad \forall u \in E_*,$$

here and in the sequel, $(\cdot,\cdot)_{L^2}$ denotes the usual $L^2(\mathbb{R}^N)$ inner product. Clearly, $C_0^\infty(\mathbb{R}^N)$ is dense in E_*.

By (H1), there exists a $a_0 > 0$ such that

$$V(x) + a_0 > 0, \quad \forall x \in \mathbb{R}^N. \tag{2.1}$$

The argument of [30, (3.2)-(3.3)] shows that

$$\frac{1}{1+a_0}\|u\|^2_{H^1(\mathbb{R}^N)} \le \|u\|_0^2 \le (1+2a_0+M)\|u\|^2_{H^1(\mathbb{R}^N)}, \quad \forall u \in E_* = H^1(\mathbb{R}^N). \tag{2.2}$$

Denote

$$E_*^- = \mathcal{E}(0)E_*, \quad E^+ = [\mathcal{E}(+\infty) - \mathcal{E}(0)]E_*,$$

and

$$(u,v)_* = \left(|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v\right)_{L^2}, \quad \|u\|_* = \sqrt{(u,u)_*}, \quad \forall u,v \in E_*. \tag{2.3}$$

Lemma 2.1 ([30, Lemma 3.1])**.** *Suppose that* (H1) *is satisfied. Then* $E_* = E_*^- \oplus E^+$*,*

$$(u,v)_* = (u,v)_{L^2} = 0, \quad \forall u \in E_*^-,\ v \in E^+, \tag{2.4}$$

and

$$\|u^+\|_*^2 \ge \overline{\Lambda}\|u^+\|_2^2, \quad \|u^-\|_*^2 \le a_0\|u^-\|_2^2, \quad \forall u = u^- + u^+ \in E_* = E_*^- \oplus E^+, \tag{2.5}$$

where a_0 *is given by* (2.1)*.*

It is easy to see that $\|\cdot\|_*$ and $\|\cdot\|_{H^1(\mathbb{R}^N)}$ are equivalent norms on E^+, and if $u \in E_*$ then $u \in E^+ \Leftrightarrow \mathcal{E}(0)u = 0$. Thus E^+ is a closed subset of $(E_*, \|\cdot\|_0) = H^1(\mathbb{R}^N)$. Define a new norm on E_*^- by setting

$$\|u\|_- = \left(\|u\|_*^2 + \|u\|_\varrho^2\right)^{1/2}, \quad \forall u \in E_*^-. \tag{2.6}$$

Let E^- be the completion of E_*^- with respect to $\|\cdot\|_-$. Then E^- is separable and reflexive,

$$E^- \cap E^+ = \{0\}, \quad (u,v)_* = 0, \quad \forall u \in E^-,\ v \in E^+. \tag{2.7}$$

Set $E = E^- \oplus E^+$ and define norm $\|\cdot\|$ as follows

$$\|u\| = \left(\|u^-\|_-^2 + \|u^+\|_*^2\right)^{1/2}, \quad \forall u = u^- + u^+ \in E = E^- \oplus E^+. \tag{2.8}$$

It is easy to verify that $(E, \|\cdot\|)$ is a Banach space, and

$$\sqrt{\overline{\Lambda}}\|u^+\|_2 \le \|u^+\|_* = \|u^+\|, \quad \|u^+\|_s \le \gamma_s\|u^+\|, quad \forall u \in E,\ s \in [2, 2^*], \tag{2.9}$$

where $\gamma_s \in (0, +\infty)$ is imbedding constant.

Lemma 2.2 ([30, Lemma 3.2])**.** *Suppose that* (H1) *is satisfied. Then the following statements hold:*

(i) $E^- \hookrightarrow L^s(\mathbb{R}^N)$ *for* $\varrho \le s \le 2^*$*;*
(ii) $E^- \hookrightarrow H^1_{loc}(\mathbb{R}^N)$ *and* $E^- \hookrightarrow\hookrightarrow L^s_{loc}(\mathbb{R}^N)$ *for* $2 \le s < 2^*$*;*
(iii) *For* $\varrho \le s \le 2^*$*, there exists a constant* $C_s > 0$ *such that*

$$\|u\|_s^s \le C_s\Big[\|u\|_*^s + \Big(\int_\Omega |u|^\varrho \,\mathrm{d}x\Big)^{s/\varrho} + \Big(\int_{\Omega^c} |u|^2 \,\mathrm{d}x\Big)^{s/2} Big], \quad \forall u \in E^-, \tag{2.10}$$

where $\Omega \subset \mathbb{R}^N$ *is any measurable set,* $\Omega^c = \mathbb{R}^N \setminus \Omega$*.*

Since E^- is a separable and reflective subspace of E, it follows that $(E^-)^*$ is also separable. Thus one may choose a dense subset $\{\mathfrak{f}_k\}_{k\in\mathbb{N}} \subset (E^-)^*$ with $\|\mathfrak{f}_k\|_{(E^-)^*} = 1$. Now define a new norm

$$\|u\|_\tau := \max\Big\{\|u^+\| : \sum_{k=1}^{\infty} \frac{1}{2^k}|\langle \mathfrak{f}_k, u^-\rangle|\Big\}, \quad \forall u \in E. \tag{2.11}$$

The topology generated by $\|\cdot\|_\tau$ will be denoted by τ and all topological notions related to it will include the symbol. It is clear that

$$\|u^+\| \le \|u\|_\tau \le \|u\|, \quad \forall u \in E. \tag{2.12}$$

If $\{u_n\} \subset E^-$ is bounded, then by [30, Lemma 2.1],

$$u_n \xrightarrow{\tau} u \Longleftrightarrow u_n \rightharpoonup u. \tag{2.13}$$

By (H2) and Lemma 2.2, the functional Φ defined by (1.7) is of class C^1, moreover

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)\,\mathrm{d}x - \int_{\mathbb{R}^N} f(x,u)v\,\mathrm{d}x, \quad \forall u, v \in E. \tag{2.14}$$

By (2.3), (2.4) and (2.8), it holds

$$\begin{aligned} \Phi(u) &= \frac{1}{2}(\|u^+\|_*^2 - \|u^-\|_*^2) - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x \\ &= \frac{1}{2}(\|u^+\|^2 - \|u^-\|_*^2) - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x, \quad \forall u = u^+ + u^- \in E, \end{aligned} \tag{2.15}$$

and

$$\langle \Phi'(u), v\rangle = (u^+, v^+)_* - (u^-, v^-)_* - \int_{\mathbb{R}^N} f(x,u)v\,\mathrm{d}x, \quad \forall u, v \in E. \tag{2.16}$$

3. Critical point theory for asymptotically linear problems

Different from the superlinear linear case [4, 17, 26], the Nehari-Pankov manifold $\mathcal{N}^-$ is not homeomorphic to the unit sphere S^+ in E^+ for asymptotically linear problem (1.1). In this section, following the idea used in [22], a critical point theory for elliptic problem like or similar to (1.1) is established in a Banach space.

To explain this in detail, we define a set

$$E_0^+ = \Big\{u \in E^+ \setminus \{0\} : \|u\|^2 - \|v^-\|^2 - \int_\Omega V_\infty(x)|u+v|^2 \mathrm{d}x < 0,\ \forall v \in E^-\Big\}. \tag{3.1}$$

Clearly, E_0^+ is nonempty by (H10) and it is a cone since for $v \in E_0^+$ and $\alpha \in \mathbb{R}\setminus\{0\}$ it holds $\alpha v \in E_0^+$. For any $u \in E \setminus E^-$, let

$$E(u) = E^- \oplus \mathbb{R}u, \quad \hat{E}(u) = E^- \oplus \mathbb{R}^+ u = E^- \oplus \mathbb{R}^+ u^+.$$

Define

$$I(u) = \frac{1}{2}\|u^-\|_*^2 + \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x, \tag{3.2}$$

then (2.15) can be rewritten as

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - I(u), \quad \forall u = u^+ + u^- \in E. \tag{3.3}$$

It will be shown later if (H2) and (H7) and (H10) are satisfied, then Φ possesses the following properties:

(H12) $I \in C^1(E, \mathbb{R})$ and $I(u) \ge I(0) = 0$ for all $u \in E$;

(H13) I is $\tau-$sequentially lower semicontinuous: $u_n \xrightarrow{\tau} u \Rightarrow \liminf_{n\to\infty} I(u_n) \geq I(u)$;
(H14) if $u_n \xrightarrow{\tau} u$ and $I(u_n) \to I(u)$ then $u_n \to u$;
(H15) there exists a $r > 0$ such that $\kappa := \inf_{u\in E^+, \|u\|=r} \Phi(u) > 0$;
(H16) For each $u \in (E_0^+ \oplus E^-) \setminus E^-$, there exists a unique nontrivial critical point $\hat{n}(u)$ of $J|_{\hat{E}(u)}$. Moreover $\hat{n}(u)$ is the unique global maximum of $J|_{\hat{E}(u)}$;
(H17) For each compact subset $\mathcal{W} \subset (E_0^+ \oplus E^-) \setminus E^-$ there exists a constant $C_{\mathcal{W}} > 0$ such that $\|\hat{n}(u)\| \leq C_{\mathcal{W}}$ for all $u \in \mathcal{W}$.

By (H16), we can define the mappings:
$$\hat{n} : (E_0^+ \oplus E^-) \setminus E^- \to \mathcal{N}^-, \quad u \mapsto \hat{n}(u) \quad \text{and} \quad n := \hat{n}|_{S_0^+} : S_0^+ \to \mathcal{N}^-, \tag{3.4}$$
where
$$S_0^+ := \{u \in E_0^+ : \|u\| = 1\}.$$
For any $u \in (E_0^+ \oplus E^-) \setminus E^-$, we deduce from (3.3), (H12) and (H15)–(H17) that $\hat{E}(u) \cap \mathcal{N}^- = \{\hat{n}(u)\}$ and
$$\frac{1}{2}\|\hat{n}(u)^+\|^2 \geq \Phi(\hat{n}(u)) \geq \Phi(ru^+/\|u^+\|) \geq \kappa \Longrightarrow \|\hat{n}(u)^+\| \geq \sqrt{2\kappa}. \tag{3.5}$$

Theorem 3.1. *Suppose that* (H12)–(H17) *are satisfied. Then following statements hold.*

(a) $c_0 := \inf_{\mathcal{N}^-} \Phi \geq \kappa > 0$ *and* Φ *has a* $(PS)_{c_0}$ *sequence in* $\mathcal{N}^-$.
(b) *If* Φ *satisfies the* $(PS)_{c_0}^\tau$ *condition in* $\mathcal{N}^-$, *i.e., every* $(PS)_c$ *sequence in* $\mathcal{N}^-$ *has a subsequence which converges in* τ, *then* c_0 *is achieved by a critical point of* Φ.

Proof. First, we claim that

(i) n is a homeomorphism with inverse $\check{n} : \mathcal{N}^- \to S_0^+$, $z \mapsto z^+/\|z^+\|$;
(ii) $\Phi \circ n \in C^1(S_0^+, \mathbb{R})$ and
$$\langle(\Phi \circ n)'(u), \phi\rangle = \|n(u)^+\|\langle\Phi'(n(u)), \phi\rangle,$$
for all $\phi \in T_u(S_0^+) = \{\xi \in E^+ \mid (u, \xi)_* = 0\}$, where $T_u\left(S_0^+\right)$ is the tangent space of S_0^+ at u.
(iii) if $\{u_n\} \subset S_0^+$ is a Palais-Smale sequence for $\Phi \circ n$, then $\{n(u_n)\}$ is a Palais-Smale sequence for Φ in $\mathcal{N}^-$. If $\{z_n\} \subset \mathcal{N}^-$ is a bounded Palais-Smale sequence for Φ, then $\left\{n^{-1}(z_n)\right\}$ is a Palais-Smale sequence for $\Phi \circ n$ in S_0^+;
(iv) $u \in S_0^+$ is a critical point of $\Phi \circ n$ if and only if $n(u)$ is a nontrivial critical point of Φ. Moreover, the corresponding critical values coincide and $\inf_{S_0^+} \Phi \circ n = \inf_{\mathcal{N}^-} \Phi$;
(v) if Φ is even, then so is $\Phi \circ n$.

Statements (i)–(v) have been proved in [22, Lemmas 3.4, 3.5 and Corollary 3.6] in a Hilbert space. Since the working space E considered here is only a Banach space, the argument should be modified. We postpone the proof to the Appendix.

(a) By (i) and (3.5) we have $\Phi(z) \geq \Phi(rz^+/\|z^+\|) \geq \kappa$ for all $z \in \mathcal{N}^- \Longrightarrow \|z^+\| \geq \sqrt{2\kappa}$. Thus $c_0 \geq \kappa > 0$. Existence of a $(\mathrm{PS})_{c_0}$ sequence for Φ in $\mathcal{N}^-$ follows from (ii), (iii) and the Ekeland variational principle [35, Theorem 2.4] since $c_0 = \inf_{\mathcal{N}^-} \Phi = \inf_{S_0^+} \Phi \circ n$.

To prove (b) we consider a $(\mathrm{PS})_c$ sequence $\{u_n\} \subset S_0^+$ for $\Phi \circ n$. Then $\{n(u_n)\}$ is a Palais-Smale sequence for Φ in $\mathcal{N}^-$ by (iii), hence $n(u_n) \xrightarrow{\tau} z$ in $\mathcal{N}^-$ after passing to a subsequence. This implies that $u_n = n(u_n)^+/\|n(u_n)^+\| \to z^+/\|z^+\|$ and we have proved:

(vi) if Φ satisfies the $(\mathrm{PS})_c^\tau$ condition in $\mathcal{N}^-$ for some $c > 0$ then $\Phi \circ n$ satisfies the $(\mathrm{PS})_c$ condition.

Next observe that if Φ satisfies the $(\mathrm{PS})_{c_0}^\tau$ condition in $\mathcal{N}^-$, then c_0 is achieved by a critical point $u \in S_0^+$ of $\Phi \circ n$. It follows from (iv) that $n(u) \in \mathcal{N}^-$ is a critical point of Φ and $\Phi(n(u)) = c_0$. This proves (b). □

Note that, Theorem 3.1 can also be used to consider other nonlinear problems for asymptotically linear case, such as p-Laplacian problems, Dirac equation, Hamiltonian system and elliptic system.

Lemma 3.2 ([30, Lemma 3.3])**.** *Suppose that* (H1), (H2) *are satisfied. Then* $\Phi \in C^1(E,\mathbb{R})$ *is* τ*-upper semi-continuous and* $\Phi' : (\Phi_a, \|\cdot\|_\tau) \to (E^*, \mathcal{T}_{w^*})$ *is continuous for every* $a \in \mathbb{R}$*, i.e.,*

$$u_n, u \in E,\ \|u_n - u\|_\tau \to 0\ \Rightarrow\ \Phi(u) \geq \liminf_{n\to\infty} \Phi(u_n),$$
$$u_n, u \in \Phi_a,\ \|u_n - u\|_\tau \to 0\ \Rightarrow\ \lim_{n\to\infty} \langle \Phi'(u_n), v\rangle = \langle \Phi'(u), v\rangle,\ \forall\ v \in E,$$

where $\Phi_a := \{u \in E :\ \Phi(u) \geq a\}$ *and* T_{w^*} *denotes the weak** *topology on* E^**.*

Lemma 3.3. *Let* (H1), (H2) *be satisfied. Then* (H12)–(H15) *hold.*

Proof. It follows from (H2) and Lemma 3.2 that (H12) and (H13) hold. Proof of (H15) is standard, see [19, Lemma 3.1]. Next, we prove (H14). Assume that $u_n \xrightarrow{\tau} u$ and $I(u_n) \to I(u)$ for $u_n,\ u \in E$. Then $u_n^+ \to u^+$ in E^+ and

$$\frac{1}{2}\|u_n^-\|_*^2 + \int_{\mathbb{R}^N} F(x,u_n)\,\mathrm{d}x \to \frac{1}{2}\|u^-\|_*^2 + \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x. \tag{3.6}$$

By the same argument as in [30, Lemma 3.3], we can show that $\{u_n^-\} \subset E^-$ is bounded. By (2.12), $u_n^- \rightharpoonup u^-$ in E^- and passing to a subsequence it holds $u_n \to u$ a.e. on $\mathbb{R}^N$. It follows from Fatou's Lemma and weakly lower semicontinuity of the norm that

$$\|u_n^-\|_*^2 \to \|u^-\|_*^2 \quad \text{and} \quad \int_{\mathbb{R}^N} F(x,u_n)\,\mathrm{d}x \to \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x. \tag{3.7}$$

Therefore $\|u_n^- - u^-\|_*^2 = \|u_n^-\|_*^2 + \|u^-\|_*^2 - 2\langle u_n^-, u^-\rangle_* \to 0$. Observe that

$$\begin{aligned}\int_{\mathbb{R}^N} (F(x,u_n) - F(x,u_n-u))\,\mathrm{d}x &= \int_{\mathbb{R}^N}\int_0^1 \frac{d}{dt}F(x,u_n-u+tu)\mathrm{d}t\,\mathrm{d}x \\ &= \int_0^1\int_{\mathbb{R}^N} f(x,u_n-u+tu)u\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.8}$$

By (H2) and Hölder inequality, for any $\Omega \subset \mathbb{R}^N$ it holds

$$\begin{aligned}&\int_\Omega |f(x,u_n-u+tu)u|\,\mathrm{d}x \\ &\leq c_2\int_{\mathbb{R}^N} \big(|u_n-u+tu|^{\varrho-1} + |u_n-u+tu|^{p-1}\big)\,|u\chi_\Omega|\,\mathrm{d}x \\ &\leq c_2\|u_n-u+tu\|_\varrho^{\varrho-1}\|u\chi_\Omega\|_\varrho + c_2\|u_n-u+tu\|_p^{p-1}\|u\chi_\Omega\|_p,\end{aligned} \tag{3.9}$$

where χ_Ω denotes the characteristic function on Ω. By Lemma 2.2, $\{\|u_n-u+tu\|_\varrho\}_n$ and $\{\|u_n-u+tu\|_p\}_n$ are bounded. Then for any $\epsilon>0$ there is $\delta>0$ such that for any Ω with Lebesgue measure $|\Omega|<\delta$ it holds

$$\int_\Omega |f(x,u_n-u+tu)u|\,\mathrm{d}x<\epsilon,\quad \forall n\in\mathbb{N}. \tag{3.10}$$

Moreover, for any $\epsilon>0$ there is $\Omega\in\mathbb{R}^N$ with $|\Omega|<\infty$ such that

$$\int_{\mathbb{R}^N\setminus\Omega} |f(x,u_n-u+tu)u|\,\mathrm{d}x<\epsilon,\quad \forall n\in\mathbb{N}. \tag{3.11}$$

In light of (3.10), (3.11) and Vitali convergence theorem, one sees that $f(x,tu)u\in L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} f(x,u_n-u+tu)u\,\mathrm{d}x\to\int_{\mathbb{R}^N} f(x,tu)u\,\mathrm{d}x,\quad \text{as } n\to\infty. \tag{3.12}$$

By (3.8), we have

$$\int_{\mathbb{R}^N}\left(F(x,u_n)-F(x,u_n-u)\right)\mathrm{d}x\to\int_0^1\int_{\mathbb{R}^N} f(x,tu)u\,\mathrm{d}x\mathrm{d}t=\int_{\mathbb{R}^N}F(x,u)\,\mathrm{d}x, \tag{3.13}$$

as $n\to\infty$, this together with (3.7) implies

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}F(x,u_n-u)\,\mathrm{d}x=0. \tag{3.14}$$

By (H2),

$$\begin{aligned} o(1)&=\int_{\mathbb{R}^N}F(x,u_n-u)\,\mathrm{d}x\\ &\ge\frac{c_1}{\varrho}\Big(\int_{|u_n-u|\le1}|u_n-u|^\varrho\,\mathrm{d}x+\int_{|u_n-u|\ge1}|u_n-u|^2\,\mathrm{d}x\Big), \end{aligned} \tag{3.15}$$

then we deduce from (2.9), (2.10) and the fact $u_n^+\to u^+$ in E^+ that

$$\begin{aligned} &\|u_n^- - u^-\|_\varrho^\varrho\\ &\le C_1\Big[\|u_n^- - u^-\|_*^\varrho+\int_{|u_n-u|\le1}|u_n^- - u^-|^\varrho\,\mathrm{d}x+\Big(\int_{|u_n-u|\ge1}|u_n^- - u^-|^2\,\mathrm{d}x\Big)^{\varrho/2}\Big]\\ &\le C_2\Big[\|u_n^- - u^-\|_*^\varrho+\int_{|u_n-u|\le1}|u_n^+ - u^+|^\varrho\,\mathrm{d}x+\int_{|u_n-u|\le1}|u_n-u|^\varrho\,\mathrm{d}x\\ &\quad+\Big(\int_{|u_n-u|\ge1}|u_n^+ - u^+|^2\,\mathrm{d}x+\int_{|u_n-u|\ge1}|u_n-u|^2\,\mathrm{d}x\Big)^{\varrho/2}\Big]=o(1). \end{aligned}$$

This shows that $\|u_n-u\|\to0$ as $n\to\infty$.

Before proving (H16)–(H17), we introduce a useful result for functions satisfying (H5) or (H7).

Lemma 3.4. *Suppose that $h(x,t)$ is non-decreasing in $t\in\mathbb{R}$.*

(i) *If $h(x,0)=0$ for any $x\in\mathbb{R}^N$, then*

$$\Big(\frac{1-\theta^2}{2}\tau-\theta\sigma\Big)h(x,\tau)|\tau|\ge\int_{\theta\tau+\sigma}^{\tau}h(x,s)|s|\mathrm{d}s,\quad\forall\theta\ge0,\ \tau,\sigma\in\mathbb{R}. \tag{3.16}$$

(ii) *If $h(x,t)=0 \Leftrightarrow t=0$ for any $x\in\mathbb{R}^N$, then*

$$\Big(\frac{1-\theta^2}{2}\tau-\theta\sigma\Big)h(x,\tau)|\tau|>\int_{\theta\tau+\sigma}^{\tau}h(x,s)|s|\mathrm{d}s, \tag{3.17}$$

for all $\theta\geq 0$, $\tau\in\mathbb{R}$, $\sigma\in\mathbb{R}\setminus\{0\}$; moreover if $h(x,t)$ is strictly increasing on $[-\tau_0,0)\cup(0,\tau_0]$ for some $\tau_0>0$, then

$$\frac{1-\theta^2}{2}|\tau|h(x,\tau)\tau>\int_{\theta\tau}^{\tau}h(x,s)|s|\mathrm{d}s, \quad \forall\theta\in(0,1),\ |\tau|\in(0,\tau_0]. \tag{3.18}$$

Proof. Since $h(x,t)$ is non-decreasing in $t\in\mathbb{R}$, then for any $x\in\mathbb{R}^N$,

$$h(x,s)\leq h(x,\tau)\ \forall s\leq\tau;\quad h(x,s)\geq h(x,\tau),\ \forall s\geq\tau. \tag{3.19}$$

To show (3.17), we consider five possible cases. Since $sh(x,s)>0$ for $s\neq 0$, it follows from (3.19) that

Case 1. $\tau=0$,

$$\int_{\theta\tau+\sigma}^{\tau}h(x,s)|s|\mathrm{d}s=\int_{\sigma}^{0}h(x,s)|s|\mathrm{d}s<\Big(\frac{1-\theta^2}{2}\tau-\theta\sigma\Big)h(x,\tau)|\tau|=0.$$

Case 2. $0\leq\theta\tau+\sigma\leq\tau$ or $\theta\tau+\sigma\leq\tau<0$,

$$\begin{aligned}\int_{\theta\tau+\sigma}^{\tau}h(x,s)|s|\mathrm{d}s&\leq h(x,\tau)\int_{\theta\tau+\sigma}^{\tau}|s|\mathrm{d}s\\&=\Big(\frac{1-\theta^2}{2}\tau^2-\theta\tau\sigma-\frac{\sigma^2}{2}\Big)h(x,\tau)\ \mathrm{sign}\tau\\&<\Big(\frac{1-\theta^2}{2}\tau-\theta\sigma\Big)h(x,\tau)|\tau|;\end{aligned}$$

Case 3. $\theta\tau+\sigma<0<\tau$,

$$\begin{aligned}\int_{\theta\tau+\sigma}^{\tau}h(x,s)|s|\mathrm{d}s&<\int_{0}^{\tau}h(x,s)|s|\mathrm{d}s\leq h(x,\tau)\int_{0}^{\tau}|s|\mathrm{d}s\\&\leq\frac12\big(\tau^2-\theta\tau(\theta\tau+\sigma)-\theta\tau\sigma\big)h(x,\tau)\\&=\Big(\frac{1-\theta^2}{2}\tau-\theta\sigma\Big)h(x,\tau)|\tau|;\end{aligned}$$

Case 4. $\tau<0<\theta\tau+\sigma$,

$$\begin{aligned}\int_{\tau}^{\theta\tau+\sigma}h(x,s)|s|\mathrm{d}s&>\int_{\tau}^{0}h(x,s)|s|\mathrm{d}s\geq h(x,\tau)\int_{\tau}^{0}|s|\mathrm{d}s\\&\geq\frac12\Big(\tau^2-\theta\tau(\theta\tau+\sigma)-\theta\tau\sigma\Big)h(x,\tau)\\&=-\Big(\frac{1-\theta^2}{2}\tau-\theta\sigma\Big)h(x,\tau)|\tau|;\end{aligned}$$

Case 5. $0<\tau<\theta\tau+\sigma$ or $\tau<\theta\tau+\sigma\leq 0$,

$$\int_{\tau}^{\theta\tau+\sigma}h(x,s)|s|\mathrm{d}s\geq h(x,\tau)\int_{\tau}^{\theta\tau+\sigma}|s|\mathrm{d}s$$

$$
\begin{aligned}
&= \Big(\frac{\theta^2-1}{2}\tau^2+\theta\tau\sigma+\frac{\sigma^2}{2}\Big)h(x,\tau)\ \mathrm{sign}\tau \\
&> -\Big(\frac{1-\theta^2}{2}\tau-\theta\sigma\Big)h(x,\tau)|\tau|.
\end{aligned}
$$

The above five cases show that (3.17) holds. By the same argument, one can prove (3.16).

Since $h(x,t)$ is strictly increasing on $[-\tau_0,0)\cup(0,\tau_0]$, then for $\tau\in(0,\tau_0]$,

$$
\int_{\theta\tau}^{\tau} h(x,s)|s|\mathrm{d}s < h(x,\tau)\int_{\theta\tau}^{\tau}|s|\mathrm{d}s = \frac{1-\theta^2}{2}|\tau|h(x,\tau)\tau, \quad \forall\theta\in(0,1);
$$

Similarly for $\tau\in[-\tau_0,0)$ we have

$$
\begin{aligned}
&\int_{\tau}^{\theta\tau} h(x,s)|s|\mathrm{d}s > h(x,\tau)\int_{\tau}^{\theta\tau}|s|\mathrm{d}s \\
&= \frac{1-\theta^2}{2}\tau^2 h(x,\tau) = -\frac{1-\theta^2}{2}|\tau|h(x,\tau)\tau, \quad \forall\theta\in(0,1).
\end{aligned}
$$

Both cases show that (3.18) holds. □

Lemma 3.5. *Let* (H1), (H2) *be satisfied.*

(i) *If* (H5) *holds, then for any* $u\in E$,

$$
\Phi(u)\ge\Phi(tu+v)+\frac12\|v\|_*^2+\frac{1-t^2}{2}\langle\Phi'(u),u\rangle-t\langle\Phi'(u),v\rangle, \tag{3.20}
$$

for all $t\ge0$, $v\in E^-$, *and the inequality is strict for* $v\in E^-\setminus\{0\}$, *moreover if* $u\in\mathcal{N}^-$ *then*

$$
\Phi(u)>\Phi(tu+v)+\frac12\|v\|_*^2, \quad \forall t\ge0,\ v\in E^-\setminus\{0\}. \tag{3.21}
$$

(ii) *If* (H7) *holds, then (i) holds, and for any* $u\in E\setminus\{0\}$,

$$
\Phi(u)>\Phi(tu)+\frac{1-t^2}{2}\langle\Phi'(u),u\rangle, \quad \forall t\in(0,1),\ |u(x)|\le\tau_0,
$$

moreover if $u\in\mathcal{N}^-$ *then*

$$
\Phi(u)>\Phi(tu), \quad \forall t\in(0,1),\ |u(x)|\le\tau_0. \tag{3.22}
$$

Proof. For any $x\in\mathbb{R}^N$, from (H2), (H5) and (3.17) it follows that

$$
\Big(\frac{1-\theta^2}{2}\tau-\theta\sigma\Big)f(x,\tau)>\int_{\theta\tau+\sigma}^{\tau}f(x,s)\mathrm{d}s, \tag{3.23}
$$

for all $\theta\ge0$, $\tau\in\mathbb{R}$, $\sigma\in\mathbb{R}\setminus\{0\}$, Similarly, by (H7) and (3.18) we have (3.23) and

$$
\frac{1-\theta^2}{2}\tau f(x,\tau)>\int_{\theta\tau}^{\tau}f(x,s)\mathrm{d}s, \quad \forall\theta\in(0,1),\ |\tau|\in(0,\tau_0]. \tag{3.24}
$$

Let $u \in E \setminus \{0\}$. Since $u \in L^2(\mathbb{R}^N)$, $\operatorname{meas}\{x \in \mathbb{R}^N : |u(x)| \le \tau_0\} = \infty$. Then we deduce from (2.16), (3.2)–(3.3) and (3.23)–(3.24) that

$$
\begin{aligned}
&\Phi(u) - \Phi(tu+v) - \frac{1-t^2}{2}\langle \Phi'(u), u\rangle + t\langle \Phi'(u), v\rangle \\
&= -\big[I(u) - I(tu+v) - \frac{1-t^2}{2}\langle I'(u), u\rangle + t\langle I'(u), v\rangle\big] \\
&= -\frac{1}{2}\|u^-\|_*^2 + \frac{1}{2}\|tu^- + v\|_*^2 + \frac{1-t^2}{2}\|u^-\|_*^2 - t\langle u^-, v\rangle_* \\
&\quad + \int_{\mathbb{R}^N} \Big(\frac{1-t^2}{2} f(x,u)u - tf(x,u)v - \int_{tu+v}^{u} f(x,s)\,\mathrm{d}s\Big)\,\mathrm{d}x \\
&= \frac{1}{2}\|v\|_*^2 + \int_{\mathbb{R}^N} \Big(\frac{1-t^2}{2} f(x,u)u - tf(x,u)v - \int_{tu+v}^{u} f(x,s)\,\mathrm{d}s\Big)\,\mathrm{d}x
\end{aligned} \tag{3.25}
$$

which is positive when

$$
\begin{gathered}
t \ge 0 \text{ and } v \in E^- \setminus \{0\}, \text{ or when} \\
t \in (0,1),\ v = 0 \text{ and } |u(x)| \le \tau_0.
\end{gathered} \tag{3.26}
$$

By the definition of $\mathcal{N}^-$ and (3.26), we have (i) and (ii). □

Lemma 3.6. *Let* (H1), (H10), (H2), (H7) *be satisfied. Then (H16)–(H17) hold.*

Proof. By the same argument as in [21, Lemma 3.2], we can show that

(H18) for any compact set $\mathcal{W} \subset (E_0^+ \oplus E^-) \setminus E^-$, there is a $R > 0$ such that $\Phi(\cdot) \le 0$ on $\hat{E}(u) \setminus B_R(0)$ for every $u \in \mathcal{W}$;

To prove (H16), we first prove that $\mathcal{N}^- \cap \hat{E}(u) \ne \emptyset$ for each $u \in (E_0^+ \oplus E^-) \setminus E^-$. Since $\hat{E}(u) = \hat{E}(u^+)$ for each $u \in (E_0^+ \oplus E^-) \setminus E^-$, we may assume that $u \in E_0^+$ and $\|u\| = 1$. By (3.2), (3.3) and (H18) above, there exists $R > 0$ such that $\Phi(\cdot) \le 0$ on $E^- \cup (\hat{E}(u) \setminus B_R(0))$. [19, Lemma 3.1] (i.e. (H15)) yields that $\Phi(ru^+) \ge \kappa > 0$ for some $r > 0$, thus $0 < \sup_{\hat{E}(u)} \Phi < \infty$. By Lemma 3.2, Φ is weakly upper semi-continuous on $\hat{E}(u)$, therefore $\Phi(u_0) = \sup_{\hat{E}(u)} \Phi$ for some $u_0 := t_0 u + v_0 \in \hat{E}(u) \setminus \{0\}$. This u_0 is a critical point of $\Phi|_{\hat{E}(u)}$, so

$$
\langle \Phi'(u_0), u_0\rangle = \langle \Phi'(u_0), v\rangle = 0, \quad \forall v \in E^-.
$$

Consequently, $u_0 \in \mathcal{N}^- \cap \hat{E}(u)$, and $t_0 > 0$ (otherwise $\Phi(u_0) \le 0$, a contradiction).

Suppose that u_1, u_2 are two critical points of $\Phi|_{\hat{E}(u)}$. Then $u_i \in \mathcal{N}^- \cap \hat{E}(u)$ for $i = 1, 2$ and we deduce from (3.20) that

$$
\Phi(u_1) = \max_{\hat{E}(u)} \Phi(\cdot) = \Phi(u_2). \tag{3.27}
$$

Moreover, it holds $u_1 = t_1 u_2 + v_1$, $u_2 = t_2 u_1 + v_2$ with $v_i \in E^-$ and $t_i > 0$. By (3.21) we see that $v_1 = v_2 = 0$, i.e. $u_1 = t_1 u_2$. Let $\Omega_i := \{x \in \mathbb{R}^N : |u_i(x)| \le \tau_0\}$, $i = 1, 2$. Since $u_i \in L^2(\mathbb{R}^N)$, the measure $|\Omega_i| = \infty$. If $t_1 < 1$, then by (3.22) we have

$$
\Phi(u_2) > \Phi(t_1 u_2) = \Phi(u_1).
$$

which contradicts with (3.27). Similarly $t_1 > 1$ leads to $\Phi(u_2) < \Phi(u_1)$. Then we get $t_1 = 1$, i.e. $u_1 = u_2$. Hence for each $u \in (E_0^+ \oplus E^-) \setminus E^-$, $\mathcal{N}^- \cap \hat{E}(u)$ consists of precisely one point denoted by $\hat{n}(u)$ which is the unique global maximum of $\Phi|_{\hat{E}(u)}$.

Condition (H17) follows immediately form (H15)–(H16) and (H18). □

Lemma 3.7. *Let* (H1), (H2), (H7), (H10) *be satisfied. Then each Palais-Smale sequence of* $\Phi|_{\mathcal{N}^-}$ *is bounded, i.e. any sequence* $\{u_n\} \subset \mathcal{N}^-$ *satisfying*

$$\|\Phi'(u_n)\|_{E^*} \to 0, \quad \Phi(u_n) \le d \quad \textit{for some } d \in [c_0, \infty) \tag{3.28}$$

is bounded in E

Proof. First we prove that $\{\|u_n\|_*\}$ is bounded. To this end, arguing by contradiction, suppose that $\|u_n\|_* \to \infty$. Let $v_n = u_n/\|u_n\|_*$, then $\|v_n\|_* = 1$. If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |v_n^+|^2 \mathrm{d}x = 0,$$

then by Lions's concentration compactness principle ([14] or [35, Lemma 1.21]), $v_n^+ \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. Fix $R > [2(1+d)]^{1/2}$. It follows from (H2) that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} F(x, Rv_n^+)\mathrm{d}x \le \lim_{n\to\infty} c_2 \left(R^\varrho \|v_n^+\|_\varrho^\varrho + R^p \|v_n^+\|_p^p\right) = 0. \tag{3.29}$$

Let $t_n = R/\|u_n\|_*$. Using (3.21) and (3.28)–(3.29), one has

$$\begin{aligned} d &\ge \Phi(u_n) \\ &\ge \Phi(t_n u_n^+) + \frac{t_n^2}{2}\|u_n^-\|_*^2 = \Phi(Rv_n^+) + \frac{R^2}{2}\|v_n^-\|_*^2 \\ &= \frac{R^2}{2}\|v_n\|_*^2 - \int_{\mathbb{R}^N} F(x, Rv_n^+)\mathrm{d}x \\ &= \frac{R^2}{2} + o(1) > d + 1 + o(1), \end{aligned}$$

which is a contradiction. Thus $\delta > 0$. Passing to a $\mathbb{Z}^N$-transformation and using the same argument as in [21, Lemma 3.5], one can show that $\{\|u_n\|_*\}$ is bounded. By (H2) and $u_n \in \mathcal{N}^-$, we have

$$\begin{aligned} \|u_n^+\|_*^2 - \|u_n^-\|_*^2 &= \int_{\mathbb{R}^N} f(x, u_n)u_n \mathrm{d}x \\ &\ge c_1\Big(\int_{|u_n|<1} |u_n|^\varrho \mathrm{d}x + \int_{|u_n|\ge 1} |u_n|^2 \mathrm{d}x\Big). \end{aligned} \tag{3.30}$$

Then we deduce from (2.9)–(2.10) that

$$\begin{aligned} \|u_n^-\|_\varrho^\varrho &\le C_3\Big[\|u_n^-\|_*^\varrho + \int_{|u_n|<1} |u_n^-|^\varrho \mathrm{d}x + \Big(\int_{|u_n|\ge 1} |u_n^-|^2 \mathrm{d}x\Big)^{\varrho/2}\Big] \\ &\le C_4\Big[\|u_n^-\|_*^\varrho + \int_{|u_n|<1} |u_n^+|^\varrho \mathrm{d}x + \int_{|u_n|<1} |u_n|^\varrho \mathrm{d}x \\ &\quad + \Big(\int_{|u_n|\ge 1} |u_n^+|^2 \mathrm{d}x + \int_{|u_n|\ge 1} |u_n|^2 \mathrm{d}x\Big)^{\varrho/2}\Big] \le C_5. \end{aligned} \tag{3.31}$$

This shows that $\{\|u_n^-\|_\varrho\}_n$ is also bounded. By (2.8) we have the boundedness of $\{\|u_n\|\}$. ☐

Lemma 3.8 ([3, Corollary 2.3])**.** *Suppose that* (H1) *is satisfied. If* $u \subset E$ *is a weak solution of the Schrödinger equation*

$$-\triangle u + V(x)u = f(x, u), \;\; quad x \in \mathbb{R}^N, \tag{3.32}$$

i.e.
$$\int_{\mathbb{R}^N} (\nabla u \nabla \psi + V(x) u \psi)\, \mathrm{d}x = \int_{\mathbb{R}^N} f(x,u)\psi\, \mathrm{d}x, \quad \forall \psi \in C_0^\infty(\mathbb{R}^N), \tag{3.33}$$
then $u(x) \to 0$ *as* $|x| \to \infty$.

Proof of Theorem 1.1. By Lemmas 3.3, 3.6 and Theorem 3.1-(a), there is a sequence $\{u_n\}$ satisfying
$$u_n \in \mathcal{N}^-, \quad \Phi(u_n) \to c_0 = \inf_{\mathcal{N}^-} \Phi, \quad \|\Phi'(u_n)\|_{E^*} \to 0. \tag{3.34}$$
Applying Lemma 3.7, we see that $\{u_n\}$ is bounded in E, thus $\|u_n\|_\varrho^\varrho + \|u_n\|_p^p$ is also bounded. If
$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |u_n^+|^2\, \mathrm{d}x = 0,$$
then by Lions's concentration compactness principle, $u_n^+ \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. We deduce from (H2), (2.9), (2.15), (2.16) and (3.34) that
$$\begin{aligned}
2c_0 + o(1) &= \|u_n^+\|^2 - \|u_n^-\|_*^2 - 2\int_{\mathbb{R}^N} F(x,u_n)\, \mathrm{d}x \\
&\le \|u_n^+\|^2 = \int_{\mathbb{R}^N} f(x,u_n)u_n^+\, \mathrm{d}x + \langle \Phi'(u_n), u_n^+\rangle \\
&\le c_2 \int_{\mathbb{R}^N} (|u_n|^{\varrho-1} + |u_n|^{p-1})|u_n^+|\, \mathrm{d}x + o(1) \\
&\le c_2(\|u_n\|_\varrho^{\varrho-1}\|u_n^+\|_\varrho + \|u_n\|_p^{p-1}\|u_n^+\|_p) + o(1) = o(1),
\end{aligned}$$
which is a contradiction since $c_0 > 0$. Thus $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that
$$\int_{B(k_n, 1+\sqrt{N})} |u_n^+|^2\, \mathrm{d}x > \frac{\delta}{2}.$$
Let us define $v_n(x) = u_n(x + k_n)$ so that
$$\int_{B(0, 1+\sqrt{N})} |v_n^+|^2\, \mathrm{d}x > \frac{\delta}{2}. \tag{3.35}$$
Since $V(x)$ and $f(x,u)$ are periodic in x, we have $\|v_n\| = \|u_n\|$ and
$$\Phi(v_n) \to c_0, \quad \|\Phi'(v_n)\|_{E^*} \to 0. \tag{3.36}$$
Passing to a subsequence, we have $v_n \rightharpoonup v_0$ in E, $v_n \to v_0$ in $L^s_{loc}(\mathbb{R}^N)$ for $2 \le s < 2^*$ and $v_n \to v_0$ a.e. on $\mathbb{R}^N$. (3.35) implies that $v_0^+ \ne 0$, so $v_0 \ne 0$. By a standard argument, we show that $\langle \Phi'(v_0), \psi\rangle = 0$, for all $\psi \in C_0^\infty(\mathbb{R}^N)$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in E, we can conclude $\Phi'(v_0) = 0$. Thus $v_0 \in \mathcal{N}^-$ and $\Phi(v_0) \ge c_0$. On the other hand, by (H7), (2.15), (2.16), (3.36) and Fatou's Lemma, we have
$$\begin{aligned}
c_0 &= \lim_{n\to\infty} \big[\Phi(v_n) - \frac{1}{2}\langle \Phi'(v_n), v_n\rangle\big] = \lim_{n\to\infty} \int_{\mathbb{R}^N} \big[\frac{1}{2} f(x,v_n)v_n - F(x,v_n)\big]\, \mathrm{d}x \\
&\ge \int_{\mathbb{R}^N} \lim_{n\to\infty} \big[\frac{1}{2} f(x,v_n)v_n - F(x,v_n)\big]\, \mathrm{d}x = \int_{\mathbb{R}^N} \big[\frac{1}{2} f(x,v_0)v_0 - F(x,v_0)\big]\, \mathrm{d}x \\
&= \Phi(v_0) - \frac{1}{2}\langle \Phi'(v_0), v_0\rangle = \Phi(v_0).
\end{aligned}$$
This shows $\Phi(v_0) \le c_0$, thus $\Phi(v_0) = c_0 = \inf_{\mathcal{N}^-} \Phi$, which together with Lemma 3.8 implies that v_0 is a ground state solution of problem (1.1). $\square$

4. Palais-Smale sequence for the superlinear case

In this section, we consider the superlinear case and make use of following assumptions instead of (H16)–(H17).

(H19) $\|u^+\| + I(u) \to \infty$ as $\|u\| \to \infty$;

(H20) $I(t_n u_n)/t_n^2 \to \infty$ if $t_n \to \infty$ and $u_n^+ \to u^+$ for some $u^+ \neq 0$ as $n \to \infty$;

(H21) For each $u \in \mathcal{N}^-$, $v \in E^-$ and $t \geq 0$,
$$\frac{t^2-1}{2}\langle I'(u), u\rangle + t\langle I'(u), v\rangle + I(u) - I(tu+v) \leq 0,$$
the strict inequality holds if in addition $t \in (0,1]$ and $v \neq 0$, or $t \in (0,1)$, $v = 0$ and $|u| \leq \tau_0$;

(H21') $\frac{t^2-1}{2}\langle I'(u), u\rangle + t\langle I'(u), v\rangle + I(u) - I(tu+v) < 0$ for every $u \in \mathcal{N}^-$, $t \geq 0$ and $v \in E^-$ such that $u \neq tu+v$;

(H22) $I(u) < I(u+v)$ for every $v \in E^- \setminus \{0\}$ and $u \in \mathcal{M}$, where
$$\mathcal{M} = \{u \in E : \Phi'(u)|_{E^-} = 0\} = \{u \in E : I'(u)|_{E^-} = 0\}. \tag{4.1}$$

It follows from (3.26) that (H21) is satisfied under (H7). By [4, Proposition 4.2], (H16)–(H17) hold for all $u \in E \setminus E^-$, under (H12), (H13), (H15), (H19), (H20) and (H21'). Combining the proof of Lemma 3.6 with the argument of [4, Proposition 4.2], we see that (H16) and (H17) hold for all $u \in E \setminus E^-$, also using (H21') instead of (H21). We deduce from the definition of $\mathcal{M}$ and (3.26) that (H22) holds under the following strict convexity condition:

(H23) $F(x, t+s) - F(x,t) - f(x,t)s > 0$ for all $t, s \in \mathbb{R},\ x \in \mathbb{R}^N$ and $s \neq 0$.

By (3.23) we see that (H23) holds if (H5) and (H2) are satisfied.

In light of (H12) and (H22), $u \notin E^-$ if $u \in \mathcal{M} \setminus \{0\}$. By the definition of $\mathcal{M}$ and a similar argument as in [4, Proposition 4.2], following conditions (H24) and (H25) are satisfied under (H13), (H19), (H20) and (H22).

(H24) for any $u \in E^+$, there is a unique nontrivial critical point $m(u)$ of $\Phi|_{\{u\}\oplus E^-}$. Moreover, $m(u)$ is the unique global minimum of $\Phi|_{\{u\}\oplus E^-}$;

(H25) for each compact subset $\mathcal{W} \subset E^+$, there exists a constant $C_{\mathcal{W}} > 0$ such that $\|m(u)\| \leq C_{\mathcal{W}}$ for all $u \in \mathcal{W}$.

Similarly, we can define mapping
$$m : E^+ \to \mathcal{M}, \quad u \mapsto m(u). \tag{4.2}$$
The following critical point theorem was established in [5].

Theorem 4.1 ([5, Theorem 4.1, 4.3]). *Let $\Phi \in C^1(E, \mathbb{R})$ satisfy* (H12)–(H15), (H19), (H20).

(I) If (H21) *is satisfied, then $c_0 = \inf_{\mathcal{N}^-} \Phi > 0$ and Φ has a $(PS)_{c_0}$ sequence in $\mathcal{N}^-$.*

(II) If (H22) *is satisfied, then following statements hold.*

(i) *The mapping $m : E^+ \to \mathcal{M}$ is a homeomorphism with inverse $\mathcal{M} \ni u \mapsto u^+ \in E^+$.*

(ii) *Φ has a $(PS)_{\hat{c}_0}$ sequence in $\mathcal{M}$ and*
$$\hat{c}_0 = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \Phi(\gamma(t)) > 0, \tag{4.3}$$
where
$$\Gamma = \{\gamma \in C([0,1], \mathcal{M}) : \gamma(0) = 0,\ \|\gamma(1)^+\| > r,\ \Phi(\gamma(1)) < 0\}, \tag{4.4}$$

and $r > 0$ satisfying $\inf_{u\in E^+:\|u\|=r}\Phi(u) > 0$.

(iii) *If in addition* (H21) *is satisfied, then* $\hat{c}_0 \le c_0$*, and if* $\hat{c}_0$ *is achieved by a critical point then* $c_0 = \hat{c}_0$.

The functional $\Phi\circ m$ has the classical mountain pass geometry. In fact, (H15) and (H22) imply that

$$\Phi\circ m(u) \ge \Phi(u) \ge \kappa > 0 \quad \text{for } u \in E^+,\ \|u\| = r, \tag{4.5}$$

moreover, for $u \in E^+\setminus\{0\}$ write $m(tu) = tu + v_t$ with $v_t \in E^-$ and set $u_t = u + \frac{1}{t}v_t = \frac{1}{t}m(tu)$, then by (H20),

$$\frac{1}{t^2}I(m(tu)) = \frac{1}{t^2}I(tu_t) \to \infty \quad \text{as } t\to\infty,$$

this implies that

$$\Phi\circ m(tu) = \frac{t^2}{2}\|u\|^2 - I(m(tu)) \to -\infty \text{ as } t\to\infty. \tag{4.6}$$

Then the classical mountain pass geometry follows. Set

$$\Sigma := \{\sigma \in C([0,1],E^+):\ \sigma(0)=0,\ \|\sigma(1)\| > r,\ \text{and}\ \ \Phi\circ m(\sigma(1)) < 0\}, \tag{4.7}$$

the mountain pass value of $\Phi\circ m$ can be written by

$$\hat{c}_0 = \inf_{\sigma\in\Sigma}\sup_{t\in[0,1]}\Phi\circ m(\sigma(t)) \ge \kappa > 0. \tag{4.8}$$

Remark 4.2. By (4.6), there is $t_0 > 0$ such that $\Phi(m(t_0u)) < 0$ for any $u \in E^+\setminus\{0\}$. Therefore the path $\gamma_u(t) := m(tt_0u^+)$, $t\in[0,1]$, lies in Γ. Assuming that (H5) holds, it follows from (3.21) that for any $u \in \mathcal{N}^-$,

$$\sup_{t\in[0,1]}\Phi(\gamma_u(t)) \le \Phi(u) \Longrightarrow \hat{c}_0 \le c_0. \tag{4.9}$$

If $\hat{c}_0$ is achieved by a critical point $u_0 \in E$, then one deduces from (3.3) and (4.8) that $u_0^+ \ne 0$ and $u_0 \in \mathcal{N}^-$. This yields $\hat{c}_0 = \Phi(u_0) \ge c_0$, and so $\hat{c}_0 = \Phi(u_0) = c_0$.

Lemma 4.3. *Let* (H1)–(H3) *be satisfied. Then* (H12)–(H15), (H19), (H20) *hold. Moreover,* (H21) *is satisfied under* (H7)*, and* (H22) *holds under* (H23).

Proof. It follows from Lemma 3.3 that (H12)–(H15) hold. In view of (3.26), (H23) implies (H22), and (H7) yields (H21).

To prove (H19), assume that $\|u_n\|\to\infty$. By (3.2),

$$\begin{aligned}
&\|u_n^+\| + I(u_n)\\
&= \|u_n^+\| + \frac{1}{2}\|u_n^-\|_*^2 + \int_{\mathbb{R}^N} F(x,u_n)\,\mathrm{d}x\\
&\ge \|u_n^+\| + \frac{1}{2}\|u_n^-\|_*^2 + \frac{c_1}{\varrho}\Big(\int_{|u_n|<1}|u_n|^{\varrho}\,\mathrm{d}x + \int_{|u_n|\ge 1}|u_n|^2\,\mathrm{d}x\Big).
\end{aligned} \tag{4.10}$$

If $\|u_n\|_* \to \infty$, the conclusion holds. Suppose $\|u_n\|_*$ is bounded, then $\|u_n^-\|_\varrho \to \infty$. By (2.9) and (2.10), one has

$$\begin{aligned}
\|u_n^-\|_\varrho^\varrho &\le C_6\Big[\|u_n^-\|_*^\varrho + \int_{|u_n|<1} |u_n^-|^\varrho \,\mathrm{d}x + \Big(\int_{|u_n|\ge 1} |u_n^-|^2 \,\mathrm{d}x\Big)^{\varrho/2}\Big] \\
&\le C_7\Big[\|u_n^-\|_*^\varrho + \int_{|u_n|<1} |u_n^+|^\varrho \,\mathrm{d}x + \int_{|u_n|<1} |u_n|^\varrho \,\mathrm{d}x \\
&\quad + \Big(\int_{|u_n|\ge 1} |u_n^+|^2 \,\mathrm{d}x + \int_{|u_n|\ge 1} |u_n|^2 \,\mathrm{d}x\Big)^{\varrho/2}\Big] \\
&\le C_8\Big[1 + \int_{|u_n|<1} |u_n|^\varrho \,\mathrm{d}x + 2^{(\varrho-2)/2}\Big(\int_{|u_n|\ge 1} |u_n|^2 \,\mathrm{d}x\Big)^{\varrho/2}\Big],
\end{aligned} \tag{4.11}$$

this and (4.10) imply that $\|u_n^+\| + I(u_n) \to \infty$.

To show (H20), suppose that there are sequences $t_n \to \infty$ and $u_n^+ \to u^+ \neq 0$ such that $I(t_n u_n)/t_n^2$ is bounded. By (4.10) and (4.11), one sees that $\{\|u_n^-\|_* + \|u_n^-\|_\varrho\}_n$ is bounded. Passing to a subsequence, we may assume that $u_n^- \rightharpoonup u^-$ in E^-. Let $\Omega := \{x \in \mathbb{R}^N : \ u^+ + u^- \neq 0\}$. If $|\Omega| > 0$, then we deduce from (H3) and Fatou's lemma that

$$\frac{I(t_n u_n)}{t_n^2} = \frac{1}{2}\|u_n^-\|_*^2 + \int_{\mathbb{R}^N} \frac{F(x, t_n^2 u_n)}{t_n^2} \,\mathrm{d}x \to \infty,$$

a contradiction. Thus $|\Omega| = 0$, i.e. $u^+ = -u^-$ a.e. on $\mathbb{R}^N$. By (2.7), $(u^+, u^-)_* = 0$. Then $u^+ = 0$ which is also a contradiction. Therefore $I(t_n u_n)/t_n^2 \to \infty$. $\square$

5. Perturbation method and proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3 by considering a perturbed problem associated with (1.1). For any $\epsilon > 0$, define $F_\epsilon(x, u) = F(x, u) + \epsilon|u|^\varrho$ and

$$\Phi_\epsilon(u) = \frac{1}{2}\|u^+\|^2 - I_\epsilon(u), \quad u \in E, \tag{5.1}$$

where

$$I_\epsilon(u) = \frac{1}{2}\|u^-\|_*^2 + \int_{\mathbb{R}^N} F_\epsilon(x, u)\mathrm{d}x. \tag{5.2}$$

Similarly, we define

$$\mathcal{N}_\epsilon^- := \big\{u \in E \setminus E^- : \Phi_\epsilon'(u)|_{\hat{E}(u)} = 0\big\}, \quad \mathcal{M}_\epsilon := \{u \in E : I_\epsilon'(u)|_{E^-} = 0\}. \tag{5.3}$$

Let

$$c_\epsilon = \inf_{u \in \mathcal{N}_\epsilon^-} \Phi_\epsilon(u), \quad \hat{c}_\epsilon = \inf_{\gamma \in \Gamma_\epsilon} \sup_{t \in [0,1]} \Phi_\epsilon(\gamma(t)) = \inf_{\sigma \in \Sigma_\epsilon} \sup_{t \in [0,1]} \Phi_\epsilon \circ m_\epsilon(\sigma(t)), \tag{5.4}$$

where

$$\Gamma_\epsilon = \{\gamma \in C([0,1], \mathcal{M}_\epsilon) : \gamma(0) = 0,\ \|\gamma(1)^+\| > r_0,\ \Phi_\epsilon(\gamma(1)) < 0\},$$
$$\Sigma_\epsilon := \{\sigma \in C([0,1], E^+) : \sigma(0) = 0,\ \|\sigma(1)\| > r_0,\ \Phi_\epsilon \circ m_\epsilon(\sigma(1)) < 0\},$$

$r_0 > 0$ satisfies $\inf_{u \in E^+,\ \|u\| = r_0} \Phi_\epsilon(u) > 0$ for all $\epsilon \in (0, 1]$, and $m_\epsilon : E^+ \to \mathcal{M}_\epsilon$.

Note that F_ϵ satisfies (H7) (actually (H4) is also satisfied) and (H23) provided that F satisfies (H5) and (H9), respectively. $\square$

Lemma 5.1. *Let* (H2), (H5) *be satisfied. Then for every* $\epsilon \in (0, 1]$*, there exists a* $\kappa_0 > 0$ *independent of* ϵ *such that* $c_\epsilon \ge \kappa_0$.

Proof. For any $u \in \mathcal{N}_\epsilon^-$, by a similar argument as in (3.21) it holds

$$\Phi_\epsilon(u) \geq \Phi_\epsilon(tu+v) + \frac{1}{2}\|v\|_*^2, \quad \forall v \in E^-,\ t \geq 0. \tag{5.5}$$

In view of (H2) and (2.9),

$$\begin{aligned}
\Phi_\epsilon(u) \geq \Phi_\epsilon(tu^+) &= \frac{t^2}{2}\|u^+\| - \int_\Omega F(x,tu^+)\mathrm{d}x - \epsilon t^\varrho \|u^+\|_\varrho^\varrho \\
&\geq \frac{t^2}{2}\|u^+\|^2 - \big(\frac{c_2}{\varrho}+1\big)t^\varrho\|u^+\|_\varrho^\varrho - \frac{c_2}{p}t^p\|u^+\|_p^p \\
&\geq \frac{t^2}{2}\|u^+\|^2 - \big(\frac{c_2}{\varrho}+1\big)\gamma_\varrho^\varrho t^\varrho\|u^+\|^\varrho - \frac{c_2}{p}\gamma_p^p t^p\|u^+\|^p \quad \forall u \in \mathcal{N}_\epsilon^-,\ t \geq 0.
\end{aligned}$$

Note that $2 < \varrho \leq p$. Thus for $\|tu^+\|$ small, there exists a $\kappa_0 > 0$ independent of ϵ such that $c_\epsilon \geq \kappa_0$. □

Lemma 5.2. *Let* (H2) *and* (H9) *be satisfied. Then for every* $\epsilon \in (0,1]$*, there exists a* $\hat{\kappa} > 0$ *independent of* ϵ *such that* $\hat{c}_\epsilon \geq \hat{\kappa}$.

Proof. Since F_ϵ satisfies (H23), we see that (H22) holds for I_ϵ. Then for any $u \in \mathcal{M}_\epsilon$,

$$\Phi_\epsilon(u) \geq \Phi_\epsilon(u^+).$$

We deduce from (H2) and (2.9) that for any $u \in E^+$,

$$\begin{aligned}
\Phi_\epsilon(m_\epsilon(u)) \geq \Phi_\epsilon(u) &= \Phi(u) - \epsilon\|u\|_\varrho^\varrho \\
&\geq \frac{1}{2}\|u\|^2 - \big(\frac{c_2}{\varrho}+1\big)\|u\|_\varrho^\varrho - \frac{c_2}{p}\|u\|_p^p \\
&\geq \frac{1}{2}\|u\|^2 - \big(\frac{c_2}{\varrho}+1\big)\gamma_\varrho^\varrho\|u\|^\varrho - \frac{c_2}{p}\gamma_p^p\|u\|^p.
\end{aligned} \tag{5.6}$$

By (5.4) and $2 < \varrho \leq p$, for $\|u\|$ small there exists a $\hat{\kappa} > 0$ independent of ϵ such that $\hat{c}_\epsilon \geq \hat{\kappa}_\epsilon$. □

Proof of Theorem 1.3. Under the conditions of Theorem 1.3-(a), it is not difficult to verify that F_ϵ satisfies (H2), (H3) and (H7). In view of Theorem 4.1-(I) and Lemma 4.3, for any $\epsilon > 0$, there exists a sequence $\{u_n^\epsilon\}$ satisfying

$$u_n^\epsilon \in \mathcal{N}_\epsilon^-, \quad \Phi_\epsilon(u_n^\epsilon) \to c_\epsilon, \quad \|\Phi_\epsilon'(u_n^\epsilon)\|_{E^*} \to 0. \tag{5.7}$$

Lemma 5.1 yields that $c_\epsilon \geq \kappa_0 > 0$. Let $\{\epsilon_n\}$ be a sequence such that $\epsilon_n \searrow 0$ as $n \to \infty$, and

$$u_n^{\epsilon_n} \in \mathcal{N}_{\epsilon_n}^-, \quad \Phi_{\epsilon_n}(u_n^{\epsilon_n}) \to \bar{c} \geq \kappa_0 > 0, \quad \|\Phi_{\epsilon_n}'(u_n^{\epsilon_n})\|_{E^*} \to 0. \tag{5.8}$$

For the sake of notational simplicity, write $u_n = u_n^{\epsilon_n}$. The rest of the proof is divided into three steps.

Step 1. Prove that $\{u_n\}$ is bounded in E. First we prove that $\{\|u_n\|_*\}$ is bounded. To this end, arguing by contradiction, suppose that $\|u_n\|_* \to \infty$. Let $v_n = u_n/\|u_n\|_*$, then $\|v_n\|_* = 1$. If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |v_n^+|^2 \mathrm{d}x = 0,$$

then by Lions's concentration compactness principle, $v_n^+ \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. Fix $R > [2(1+\bar{c})]^{1/2}$; it follows from (H2) that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} F_{\epsilon_n}(x, Rv_n^+)\mathrm{d}x \le \lim_{n\to\infty}(c_2+\epsilon_n)\left(R^{\varrho}\|v_n^+\|_{\varrho}^{\varrho} + R^p\|v_n^+\|_p^p\right) = 0. \tag{5.9}$$

Let $t_n = R/\|u_n\|_*$. Using (5.5), (5.8) and (5.9), one has

$$\begin{aligned}
\bar{c} + o(1) &= \Phi_{\epsilon_n}(u_n)\\
&\ge \Phi_{\epsilon_n}(t_n u_n^+) + \frac{t_n^2}{2}\|u_n^-\|_*^2 = \Phi_{\epsilon_n}(Rv_n^+) + \frac{R^2}{2}\|v_n^-\|_*^2\\
&= \frac{R^2}{2}\|v_n\|_*^2 - \int_{\mathbb{R}^N} F_{\epsilon_n}(x, Rv_n^+)\mathrm{d}x\\
&= \frac{R^2}{2} + o(1) > \bar{c} + 1 + o(1),
\end{aligned}$$

which is a contradiction. Thus $\delta > 0$. Passing to a $\mathbb{Z}^N$-transformation and using the same argument as in [30, Lemma 4.4], one can show that $\{\|u_n\|_*\}$ is bounded. By (3.31), $\{u_n\}$ is bounded in E.

Step 2. Verify that $\Phi'(v_0) = 0$ and $\Phi(v_0) \le \bar{c}$ for some $v_0 \in E \setminus \{0\}$. By (5.8) and the same argument as in the Proof of Theorem 1.1, we can show that there is a $v_0 \in E$ with $v_0^+ \neq 0$ such that $\Phi'(v_0) = 0$ and $\Phi(v_0) \le \bar{c}$. By Lemma 3.8, v_0 is a nontrivial solution of problem (1.1).

Step 3. Show that $\Phi(v_0) = \bar{c} = c_0$. We infer form Step 2 and $v_0 \in \mathcal{N}^-$ that $\bar{c} \ge \Phi(v_0) \ge c_0$. Next, we prove that $\bar{c} \le c_0$.

Let ϵ be any positive number. Then there exists a $u_\epsilon \in \mathcal{N}^-$ such that $\Phi(u_\epsilon) < c_0 + \epsilon$, and

$$\langle \Phi'(u_\epsilon), u_\epsilon\rangle = 0, \quad \langle \Phi'(u_\epsilon), v\rangle = 0, \quad \forall v \in E^-. \tag{5.10}$$

Applying [19, Lemma 3.3], there exist $t_n > 0$ and $v_n \in E^-$ such that $t_n u_\epsilon + v_n \in \mathcal{N}_{\epsilon_n}^-$. Then

$$\begin{aligned}
c_{\epsilon_n} &\le \Phi_{\epsilon_n}(t_n u_\epsilon + v_n) = \frac{t_n^2}{2}\|u_\epsilon^+\|^2 - I_{\epsilon_n}(t_n u_\epsilon + v_n)\\
&= \frac{t_n^2}{2}\|u_\epsilon^+\|^2 - \frac{1}{2}\|t_n u_\epsilon^- + v_n\|_*^2 - \int_{\mathbb{R}^N} F(t_n u_\epsilon + v_n)\mathrm{d}x - \epsilon_n\|t_n u_\epsilon + v_n\|_{\varrho}^{\varrho}
\end{aligned} \tag{5.11}$$

which, together with (H20), imply that there exists a $K_\epsilon > 0$ such that $0 < t_n < K_\epsilon$ and $\|v_n\| < K_\epsilon$. By Lemma 2.2-(i), $\|v_n\|_{\varrho} \le C_9 K_\epsilon$. Using (3.3), (3.20), (5.1), (5.10) and Hölder inequality, we are led to

$$
\begin{aligned}
& c_0+\epsilon \\
& >\Phi(u_\epsilon)=\Phi_{\epsilon_n}(u_\epsilon)+\epsilon_n\|u_\epsilon\|_\varrho^\varrho \\
& \geq \Phi_{\epsilon_n}(t_n u_\epsilon+v_n)+\frac{1}{2}\|v_n\|_*^2+\frac{1-t_n^2}{2}\langle\Phi_{\epsilon_n}'(u_\epsilon),u_\epsilon\rangle-t_n\langle\Phi_{\epsilon_n}'(u_\epsilon),v_n\rangle \\
& \geq c_{\epsilon_n}+\frac{1}{2}\|v_n\|_*^2+\frac{1-t_n^2}{2}\langle\Phi_{\epsilon_n}'(u_\epsilon),u_\epsilon\rangle-t_n\langle\Phi_{\epsilon_n}'(u_\epsilon),v_n\rangle \\
& =c_{\epsilon_n}+\frac{1}{2}\|v_n\|_*^2-\frac{1-t_n^2}{2}\epsilon_n\varrho\|u_\epsilon\|_\varrho^\varrho+t_n\epsilon_n\varrho(|u_\epsilon|^{\varrho-2}u_\epsilon,v_n)_2 \\
& \geq c_{\epsilon_n}-\frac{\epsilon_n\varrho}{2}\|u_\epsilon\|_\varrho^\varrho-K_\epsilon\epsilon_n\varrho\|u_\epsilon\|_\varrho^{\varrho-1}\|v_n\|_\varrho \\
& \geq c_{\epsilon_n}-\frac{\epsilon_n\varrho}{2}\|u_\epsilon\|_\varrho^\varrho-C_9K_\epsilon^2\epsilon_n\varrho\|u_\epsilon\|_\varrho^{\varrho-1},
\end{aligned}
\tag{5.12}
$$

this yields

$$\bar{c}=\lim_{n\to\infty}c_{\epsilon_n}\leq c_0+\epsilon. \tag{5.13}$$

Since $\epsilon>0$ is arbitrary, one has $\bar{c}\leq c_0$. Thus $c_0=\bar{c}\geq\kappa_0$, i.e.,

$$\Phi(v_0)=c_0=\inf_{u\in\mathcal{N}^-}\Phi(u).$$

Theorem 1.3-(a) is proved.

Under the assumptions of Theorem 1.3-(b), one can verify that F_ϵ satisfies (H2), (H8) and (H23). In view of Theorem 4.1-(ii) and Lemma 4.3, for any $\epsilon>0$ there exists a sequence $\{u_n^\epsilon\}$ satisfying

$$u_n^\epsilon\in\mathcal{M}_\epsilon,\quad \Phi_\epsilon(u_n^\epsilon)\to\hat{c}_\epsilon,\quad \|\Phi_\epsilon'(u_n^\epsilon)\|_{E^*}\to 0. \tag{5.14}$$

Lemma 5.2 yields that $\hat{c}_\epsilon\geq\hat{\kappa}_\epsilon>0$. Let $\{\epsilon_n\}$ be a sequence such that $\epsilon_n\searrow 0$ as $n\to\infty$, and

$$u_n^{\epsilon_n}\in\mathcal{M}_{\epsilon_n},\quad \Phi_{\epsilon_n}(u_n^{\epsilon_n})\to\tilde{c}\geq 0,\quad \|\Phi_{\epsilon_n}'(u_n^{\epsilon_n})\|_{E^*}\to 0. \tag{5.15}$$

For the sake of notational simplicity, write $u_n=u_n^{\epsilon_n}$. Similar to the proof of Theorem 1.3-(a), the rest of the proof is also divided into three steps.

Step 1. Prove that $\{u_n\}$ is bounded in E. Since (3.20) does not hold under (H9), the argument used in the Proof of Theorem 1.3-(a) is not applicable. We modify it as follows.

Arguing by contradiction, suppose that $\|u_n\|\to\infty$. (5.15) implies the existence of an $M>0$ such that $\Phi_{\epsilon_n}(u_n)\geq -M$ for all $n\in\mathbb{N}$. By (H2) and (5.1), we have

$$
\begin{aligned}
-2M\leq 2\Phi_{\epsilon_n}(u_n)&=\|u_n^+\|^2-\|u_n^-\|_*^2-2\int_{\mathbb{R}^N}F_{\epsilon_n}(x,u_n)\mathrm{d}x \\
&\leq\|u_n^+\|^2-\|u_n^-\|_*^2-\frac{2c_1}{\varrho}\Big(\int_{|u_n|<1}|u_n|^\varrho\mathrm{d}x+\int_{|u_n|\geq 1}|u_n|^2\mathrm{d}x\Big).
\end{aligned}
\tag{5.16}
$$

From (2.9), (2.10) and (5.16), one sees that

$$
\begin{aligned}
\|u_n^-\|_\varrho^\varrho &\le C_{10}\Big[\|u_n^-\|_*^\varrho + \int_{|u_n|<1} |u_n^-|^\varrho \mathrm{d}x + \Big(\int_{|u_n|\ge 1} |u_n^-|^2 \mathrm{d}x\Big)^{\varrho/2}\Big] \\
&\le C_{10}\|u_n^-\|_*^\varrho + C_{11}\Big(\int_{|u_n|<1} |u_n^+|^\varrho \mathrm{d}x + \int_{|u_n|<1} |u_n|^\varrho \mathrm{d}x\Big) \\
&\quad + C_{11}\Big(\int_{|u_n|\ge 1} |u_n^+|^2 \mathrm{d}x + \int_{|u_n|\ge 1} |u_n|^2 \mathrm{d}x\Big)^{\varrho/2} \\
&\le C_{10}\|u_n^-\|_*^\varrho + C_{12}\Big(\|u_n^+\|^\varrho + \|u_n^+\|^2 - \|u_n^-\|_*^2 + 2M\Big) \\
&\quad + C_{13}\left(\|u_n^+\|^2 - \|u_n^-\|_*^2 + 2M\right)^{\varrho/2} \\
&\le C_{14}(1 + \|u_n^+\|^\varrho + \|u_n^+\|^2),
\end{aligned} \tag{5.17}
$$

this with (2.6), (2.8) and (5.16) imply that

$$
\begin{aligned}
\|u_n\|^2 &= \|u_n^+\|^2 + \|u_n^-\|_*^2 + \|u_n^-\|_\varrho^2 \\
&\le 2\|u_n^+\|^2 + 2M + C_{14}\left(1 + \|u_n^+\|^\varrho + \|u_n^+\|^2\right)^{2/\varrho}.
\end{aligned}
$$

Thus

$$
\|u_n^+\| \to \infty \text{ as } n \to \infty, \quad \text{and} \quad \|u_n\| \le C_{15}(1 + \|u_n^+\|). \tag{5.18}
$$

By (H8) and (5.1)–(5.2), one has

$$
\begin{aligned}
&\Phi_{\epsilon_n}(u_n) - \frac{1}{2}\langle \Phi'_{\epsilon_n}(u_n), u_n\rangle \\
&= \int_{\mathbb{R}^N} \big[\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\big]\,\mathrm{d}x + \Big(\frac{\varrho}{2} - 1\Big)\epsilon_n \|u_n\|_\varrho^\varrho \\
&\ge \frac{\mu-2}{2\mu}\int_{\mathbb{R}^N} f(x,u_n)u_n\,\mathrm{d}x + \Big(\frac{\varrho}{2} - 1\Big)\epsilon_n \|u_n\|_\varrho^\varrho;
\end{aligned} \tag{5.19}
$$

then setting $\theta_n = \|\Phi'_{\epsilon_n}(u_n)\|_{E^*}$,

$$
\epsilon_n \|u_n\|_\varrho^\varrho \le C_{16}(1 + \theta_n\|u_n\|), \quad \text{and} \quad \theta_n \to 0 \text{ as } n \to \infty. \tag{5.20}
$$

Let $\psi \in C^\infty(\mathbb{R},\mathbb{R})$ be such that $0 \le \psi(t) \le 1$ and $\psi(t) = 0$ if $|t| \le 1$, $\psi(t) = 1$ if $|t| \ge 2$. Set

$$
f_1(x,t) := \psi(t)f(x,t), \quad \text{and} \quad f_2(x,t) := f(x,t) - f_1(x,t) = (1-\psi(t))f(x,t). \tag{5.21}
$$

Then by (H2) we obtain with $\varrho' = \frac{\varrho}{\varrho-1}$, $p' = \frac{p}{p-1}$

$$
C_{17}|f_1(x,t)|^{p'} \le tf_1(x,t), \quad C_{17}|f_2(x,t)|^{\varrho'} \le tf_2(x,t). \tag{5.22}
$$

Using (5.15) and (5.19), we see that

$$
\|f_1(x,u_n)\|_{p'}^{p'} + \|f_2(x,u_n)\|_{\varrho'}^{\varrho'} \le C_{18}(1 + \theta_n\|u_n\|).
$$

Moreover, the Hölder inequality yields

$$
\begin{aligned}
&\int_{\mathbb{R}^N} f(x,u_n)u_n^+\,\mathrm{d}x \le \|f_1(x,t)\|_{p'}\|u_n^+\|_p + \|f_2(x,t)\|_{\varrho'}\|u_n^+\|_\varrho \\
&\le C_{19}(1 + \theta_n\|u_n\|)^{1/p'}\|u_n^+\|_p + C_{19}(1 + \theta_n\|u_n\|)^{1/\varrho'}\|u_n^+\|_\varrho.
\end{aligned} \tag{5.23}
$$

Then we deduce from (2.9), (5.15), (5.18), (5.20) and $\epsilon_n \to 0$ as $n \to \infty$ that

$$\begin{aligned} \|u_n^+\|^2 &= \langle \Phi'_{\epsilon_n}(u_n), u_n^+ \rangle + \int_{\mathbb{R}^N} f(x,u_n)u_n^+ \,\mathrm{d}x + \epsilon_n \varrho (|u_n|^{\varrho-2}u_n, u_n^+)_{L^2} \\ &\le C_{20}\Big(1 + \|u_n\|^{1/p'} + \|u_n\|^{1/\varrho'}\Big)\|u_n^+\| + \varrho \epsilon_n^{1/\varrho}\Big(\epsilon_n^{1/\varrho}\|u_n\|_\varrho\Big)^{\varrho-1}\|u_n^+\|_\varrho \\ &\le C_{21}\Big(1 + \|u_n\|^{1/p'} + \|u_n\|^{1/\varrho'}\Big)\|u_n^+\| \\ &\le C_{22}\Big(1 + \|u_n^+\|^{1/p'} + \|u_n^+\|^{1/\varrho'}\Big)\|u_n^+\|. \end{aligned} \tag{5.24}$$

This is a contradiction since $1/\varrho' < 1$ and $1/p' < 1$. Thus $\{\|u_n\|\}$ is bounded.

Step 2. Verify that $\Phi'(\tilde{v}) = 0$ and $\Phi(\tilde{v}) \le \tilde{c}$ for some $\tilde{v} \in E \setminus \{0\}$. By (5.15) and the same argument as in the Proof of Theorem 1.1, we can certify that there is $\tilde{v} \in E$ with $\tilde{v}^+ \neq 0$ such that $\Phi'(\tilde{v}) = 0$ and $\Phi(\tilde{v}) \le \tilde{c}$. This and Lemma 3.8, imply that $\tilde{v}$ is a nontrivial solution of problem (1.1).

Step 3. Show that $\Phi(\tilde{v}) = \hat{c}_0 = c_0$ if (H5) is additionally satisfied. Note that if (H23) is satisfied, it does not need to consider the perturbation term $\epsilon|u|^\varrho$, and $\Phi_\epsilon, \hat{c}_\epsilon$ in (5.14) should be replaced by Φ and $\hat{c}_0$, respectively. Then by Step 2 there is a $\tilde{v} \in E$ with $\tilde{v}^+ \neq 0$ such that $\Phi'(\tilde{v}) = 0$ and $\Phi(\tilde{v}) \le \hat{c}_0$. This yields that $c_0 \le \Phi(\tilde{v}) \le \hat{c}_0$ since $\tilde{v} \in \mathcal{N}^-$. It follows from (4.9) that $\hat{c}_0 \le c_0$. Thus $\Phi(\tilde{v}) = \hat{c}_0 = c_0 \ge \kappa_0 > 0$. In view of (3.23), (H23) holds under (H2) and (H5). This completes the proof. □

6. Appendix

Here we show statements (i)–(v) in the proof of Theorem 3.1.

(i) Suppose that $u_n \in (E_0^+ \oplus E^-) \setminus E^-$, $u_n \to u \in (E_0^+ \oplus E^-) \setminus E^-$. By (H17), $\{\hat{n}(u_n)\}$ is bounded. It suffices to show that $\hat{n}(u_n) \to \hat{n}(u)$ after passing to a subsequence. Write $\hat{n}(u_n) = t_n u_n + v_n$ with $v_n \in E^-$. Passing to a subsequence, we may assume that $t_n \to t$ and $v_n \rightharpoonup v$ in E^-. Then $\hat{n}(u_n) \xrightarrow{\tau} tu + v$. Setting $\hat{n}(u) = \bar{t}u + \xi$, it follows from (H16) that

$$\Phi(\hat{n}(u_n)) \ge \Phi\big(\bar{t}u_n + \xi\big) \to \Phi\big(\bar{t}u + \xi\big) = \Phi(\hat{n}(u)),$$

and hence, using the τ-sequentially lower semicontinuity of I,

$$\begin{aligned} \Phi(\hat{n}(u)) &\le \limsup_{n\to\infty} \Phi(\hat{n}(u_n)) \\ &= \limsup_{n\to\infty} \Big(\frac{1}{2}t_n^2\|u_n^+\|^2 - I\left(\hat{n}(u_n)\right)\Big) \\ &\le \frac{1}{2}t^2\|u^+\|^2 - I\big(tu + v\big) \\ &= \Phi(tu + v) \le \Phi(\hat{n}(u)). \end{aligned}$$

Hence the inequalities above must be equalities. It follows that $\Phi(tu+v) = \Phi(\hat{n}(u))$ and $I\left(\hat{n}(u_n)\right) \to I(tu+v)$. By (H14), $\hat{n}(u_n) \to tu+v$ and so $v_n \to v$. On the other hand, the uniqueness property (H16) yields that $\hat{n}(u) = tu+v$. Thus $\hat{n}(u_n) \to \hat{n}(u)$.

Define $\check{n} : \mathcal{N}^- \to S^+$, $z \mapsto z^+/\|z^+\|$, we first certify that $\check{n}(z) \in E_0^+$. Since $z \in L^2(\mathbb{R}^N)$, $\mathrm{meas}\{x \in \mathbb{R}^N : |z(x)| \le \alpha_0\} = \infty$, it follows from (1.11) that

$$\int_{\mathbb{R}^N} \frac{f(x,z)f_\infty(x,z)}{V_\infty(x)} < 0. \tag{6.1}$$

By the same argument as in [22, Lemma 3.2], for any $u \in E$ it holds

$$\begin{aligned} \langle \Phi'(u), u+2v\rangle \geq \|u^+\|^2 - \|u^- + v\|_*^2 + \|v\|_*^2 - \int_{\mathbb{R}^N} V_\infty(x)\left|u+v\right|^2 \mathrm{d}x \\ + \int_{\mathbb{R}^N} \frac{V_\infty(x) f(x,u)u - |f(x,u)|^2}{V_\infty(x)} \mathrm{d}x, \quad \forall v \in E^-. \end{aligned} \tag{6.2}$$

From (6.1), (6.2) and the fact $z \in \mathcal{N}^-$, we deduce for any $v \in E^-$ that

$$\begin{aligned} &\|\check{n}(z)\| - \|v\|_*^2 - \int_{\mathbb{R}^N} V_\infty(x) |\check{n}(z) + v|^2 \mathrm{d}x \\ &= \frac{\|z^+\|^2}{\|z^+\|^2} - \|v\|_*^2 - \int_{\mathbb{R}^N} V_\infty(x) \Big| \frac{z}{\|z^+\|} - \frac{z^-}{\|z^+\|} + v\Big|^2 \mathrm{d}x \\ &\leq \frac{1}{\|z^+\|^2} \langle \Phi'(z), z + 2(\|z^+\| v - z^-)\rangle - \frac{1}{\|z^+\|^2} \| \|z^+\| v - z^-\|_*^2 \\ &\quad - \frac{1}{\|z^+\|^2} \int_{\mathbb{R}^N} \frac{V_\infty(x) f(x,z)\cdot z - |f(x,z)|^2}{V_\infty(x)} \mathrm{d}x \\ &= -\|v - \frac{1}{\|z^+\|} z^-\|_*^2 + \frac{1}{\|z^+\|^2} \int_{\mathbb{R}^N} \frac{f(x,z) f_\infty(x,z)}{V_\infty(x)} < 0. \end{aligned} \tag{6.3}$$

Thus $\check{n}(z) \in E_0^+$. It is easy to see that $\check{n}$ is continuous and $\check{n} = n^{-1}$ (the inverse of n). Then (i) follows immediately.

(ii) Let $u \in E_0^+$, $\phi \in E^+$ and put $\hat{n}(u) = s_u u + \xi_u$, $\xi_u \in E^-$. We claim that $u + t\phi \in E_0^+$ for $|t|$ small. Arguing indirectly, assume that there exists a sequence $\{t_n\}$ such that $u + t_n\phi \notin E_0^+$ and $t_n \to 0$. Then there is $\{v_n\} \subset E^-$ such that

$$\|u + t_n\phi\|^2 - \|v_n\|_*^2 - \int_{\mathbb{R}^N} V_\infty(x) |u + t_n\phi + v_n|^2 \mathrm{d}x \geq 0, \tag{6.4}$$

this with (2.10) imply that $\{v_n\}$ is bounded in E. Passing to a subsequence, we may assume that $v_n \rightharpoonup v_0$ in E^-. Then it follows from (6.4) and the weakly sequentially lower semicontinuity of the norm that

$$\begin{aligned} \|u\|^2 &= \lim_{n\to\infty} \|u + t_n\phi\|^2 \\ &\geq \liminf_{n\to\infty} \left[\|v_n\|_*^2 + \int_{\mathbb{R}^N} V_\infty(x) |u + t_n\phi + v_n|^2 \mathrm{d}x \right] \\ &\geq \|v_0\|_*^2 + \int_{\mathbb{R}^N} V_\infty(x) |u + v_0|^2 \mathrm{d}x, \end{aligned}$$

which contradicts with the fact that $u \in E_0^+$. Thus $u + t\phi \in E_0^+$ for $|t|$ small.

By (H16) and the mean value theorem, we obtain

$$\begin{aligned} \Phi(\hat{n}\left(u + t\phi\right)) - \Phi(\hat{n}(u)) &= \Phi\left(s_{u+t\phi}(u + t\phi) + \xi_{u+t\phi}\right) - \Phi\left(s_u u + \xi_u\right) \\ &\leq \Phi\left(s_{u+t\phi}(u + t\phi) + \xi_{u+t\phi}\right) - \Phi\left(s_{u+t\phi} u + \xi_{u+t\phi}\right) \\ &= \langle \Phi'\big(s_{u+t\phi} u + \xi_{u+t\phi} + t\tau_t s_{u+t\phi}\phi\big), t s_{u+t\phi}\phi\rangle \end{aligned}$$

for some $\tau_t \in (0,1)$. Similarly,

$$\begin{aligned} \Phi(\hat{n}\left(u + t\phi\right)) - \Phi(\hat{n}(u)) &\geq \Phi\left(s_u(u + t\phi) + \xi_u\right) - \Phi\left(s_u u + \xi_u\right) \\ &= \langle \Phi'\left(s_u u + \xi_u + t\eta_t s_u\phi\right), t s_u\phi\rangle \end{aligned}$$

for some $\eta_t \in (0,1)$. Since the map $u \mapsto \hat{n}(u)$ is continuous according to (i), we see combining two inequalities that

$$\begin{aligned}\langle (\Phi \circ \hat{n})'(u), \phi \rangle &= \lim_{t\to 0} \frac{\Phi(\hat{n}\,(u+t\phi)) - \Phi(\hat{n}(u))}{t} = s_u \langle \Phi'\,(s_u u + \xi_u)\,, \phi\rangle \\ &= \frac{\|\hat{n}(u)^+\|}{\|u\|} \langle \Phi'(\hat{n}(u), \phi\rangle\end{aligned}$$

Hence the Gâteaux derivative of $\Phi \circ \hat{n}$ is bounded linear in ϕ and continuous in u. It follows that $\Phi \circ \hat{n}$ is of class C^1, see e.g. [35, Proposition 1.3]. Then (ii) holds. (iii), (iv) and (v) are easy consequences of (ii) and the definition of $\hat{n}$.

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