EXISTENCE OF SOLUTIONS TO BURGERS EQUATIONS IN A NON-PARABOLIC DOMAIN

YASSINE BENIA, BOUBAKER-KHALED SADALLAH

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Abstract. In this article, we study the semilinear Burgers equation with time variable coefficients, subject to boundary condition in a non-parabolic domain. Some assumptions on the boundary of the domain and on the coefficients of the equation will be imposed. The right-hand side of the equation is taken in $L^2(\Omega)$. The method we used is based on the approximation of the non-parabolic domain by a sequence of subdomains which can be transformed into regular domains. This paper is an extension of the work [2].

1. Introduction

The Burgers equation is a fundamental partial differential equation in modeling many physical phenomena, such as fluid mechanics, acoustics, turbulence [3, 6], traffic flow, growth of interfaces, and financial mathematics [7, 12].

In [11], the author studied a linear parabolic equation in a domain similar to the one considered in this work. Other references on the analysis of linear parabolic problems in non-regular domains are discussed for example in [1, 5, 8, 9].

The work by Clark et al. [4] is devoted to the homogeneous Burgers equation in non-parabolic domains which can be transformed into rectangle. In the same domains, we have established the existence, uniqueness and the optimal regularity of the solution to the non-homogeneous Burgers equation with time variable coefficients in an anisotropic Sobolev space (see [2]). The present paper is an extension of this last work to another type of non-regular domains.

Let $\Omega \subset \mathbb{R}^2$ be the “triangular” domain

$$\Omega = \{(t, x) \in \mathbb{R}^2; \ 0 < t < T, \ x \in I_t\},$$

where $T$ is a positive number and

$$I_t = \{x \in \mathbb{R}; \ \varphi_1(t) < x < \varphi_2(t), \ t \in (0, T)\},$$

with

$$\varphi_1(0) = \varphi_2(0).$$

The functions $\varphi_1, \varphi_2$ are defined on $[0, T]$, and belong to $\mathcal{C}^1(0, T)$.

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The most interesting point of the problem studied here is the fact that \( \varphi_1(0) = \varphi_2(0) \), because the domain is not rectangular and cannot be transformed into a regular domain without the appearance of some degenerate terms in the equation.

In \( \Omega \), we consider the boundary-value problem for the non-homogeneous Burgers equation with variable coefficient

\[
\begin{align*}
\partial_t u(t, x) + c(t)u(t, x)\partial_x u(t, x) - \partial_x^2 u(t, x) &= f(t, x) \quad (t, x) \in \Omega, \\
u(t, \varphi_1(t)) = u(t, \varphi_2(t)) &= 0 \quad t \in (0, T), \\
\end{align*}
\]

(1.2)

where \( f \in L^2(\Omega) \) and \( c(t) \) is given.

We look for some conditions on the functions \( c(t) \), \( \varphi_1(t) \) and \( \varphi_2(t) \) such that (1.2) admits a unique solution \( u \) belonging to the anisotropic Sobolev space

\[
H^{1,2}(\Omega) = \{ u \in L^2(\Omega); \partial_t u, \partial_x u, \partial_x^2 u \in L^2(\Omega) \}.
\]

In the sequel, we assume that there exist positive constants \( c_1 \) and \( c_2 \), such that

\[
c_1 \leq c(t) \leq c_2, \quad \text{for all } t \in (0, T),
\]

(1.3)

and we note that

\[
\|u\|_{L^2(I_t)} = \left( \int_{\varphi_1(t)}^{\varphi_2(t)} \|u(t, x)\|^2 \, dx \right)^{1/2},
\]

\[
\|u\|_{L^\infty(I_t)} = \sup_{x \in I_t} \|u(t, x)\|.
\]

To establish the existence of a solution to (1.2), we also assume that

\[
|\varphi'(t)| \leq \gamma \quad \text{for all } t \in [0, T],
\]

(1.4)

where \( \gamma \) is a positive constant and \( \varphi(t) = \varphi_2(t) - \varphi_1(t) \) for all \( t \in [0, T] \).

**Remark 1.1.** Once problem (1.2) is solved, we can deduce the solution of the problem

\[
\begin{align*}
\partial_t u(t, x) + a(t)u(t, x)\partial_x u(t, x) - b(t)\partial_x^2 u(t, x) &= f(t, x) \quad (t, x) \in \Omega, \\
u(t, \varphi_1(t)) = u(t, \varphi_2(t)) &= 0 \quad t \in (0, T), \\
\end{align*}
\]

(1.5)

Indeed, consider the case where \( a(t) \) and \( b(t) \) are positive and bounded functions for all \( t \in [0, T] \). Let \( h \) be defined by \( h : [0, T] \to [0, T'] \)

\[
h(t) = \int_0^t b(s) \, ds,
\]

we put \( \psi_i = \varphi_i \circ h^{-1} \) where \( i = 1, 2 \). Using the change of variables \( t' = h(t) \), \( v(t', x) = u(t, x) \), (1.5) becomes equivalent to (1.2), because it may be written as follows

\[
\begin{align*}
\partial_{t'} v(t', x) + c(t')v(t', x)\partial_x v(t', x) - \partial_x^2 v(t', x) &= g(t', x) \quad (t', x) \in \Omega', \\
v(t', \psi_1(t')) = v(t', \psi_2(t')) &= 0, \quad t' \in (0, T'),
\end{align*}
\]

where \( c(t') = \frac{a(t)}{b(t)} \), \( g(t', x) = \frac{f(t,x)}{b(t)} \), \( \Omega' = \{(t', x) \in \mathbb{R}^2; \ 0 < t' < T', \ x \in I_{t'} \} \) and \( T' = \int_0^T b(s) \, ds \).
For the study of problem (1.2) we will follow the method used in [11], which consists in observing that this problem admits a unique solution in domains that can be transformed into rectangles, i.e., when \( \varphi_1(0) \neq \varphi_2(0) \).

The paper is organized as follows. In the next section we study problem (1.2) in domain that can be transformed into a rectangle. When \( \varphi_1 \) and \( \varphi_2 \) are monotone on \((0,T)\), we solve in Section 3 the problem in a triangular domain: We approximate this domain by a sequence of subdomains \((\Omega_n)_{n\in\mathbb{N}}\). Then we establish an a priori estimate of the type

\[
\|u_n\|_{H^1(\Omega_n)}^2 \leq K\|f_n\|_{L^2(\Omega_n)}^2 \leq K\|f\|_{L^2(\Omega)}^2,
\]

where \( u_n \) is the solution of (1.2) in \( \Omega_n \) and \( K \) is a constant independent of \( n \). This inequality allows us to pass to the limit in \( n \). Finally, Section 4 is devoted to problem (1.2) in the case when \( \varphi_1 \) and \( \varphi_2 \) are monotone only near 0.

Our main result is as follows.

**Theorem 1.2.** Assume that \( c \) and \((\varphi_i(t))_{i=1,2}\) satisfy the conditions (1.1), (1.3) and (1.4). Then, the problem

\[
\partial_t u(t,x) + c(t)u(t,x)\partial_x u(t,x) - \partial^2_x u(t,x) = f(t,x) \quad (t,x) \in \Omega,
\]

\[
u(t,\varphi_1(t)) = u(t,\varphi_2(t)) = 0 \quad t \in (0,T),
\]

admits in the triangular domain \( \Omega \) a unique solution \( u \in H^{1,2}(\Omega) \) in the following cases:

**Case 1.** \( \varphi_1 \) (resp \( \varphi_2 \)) is a decreasing (resp increasing) function on \((0,T)\).

**Case 2.** \( \varphi_1 \) (resp \( \varphi_2 \)) is a decreasing (resp increasing) function only near 0.

Theses cases will be proved in Section 3 and Section 4 respectively.

2. **Solution in a domain that can be transformed into a rectangle**

Let \( \Omega \subset \mathbb{R}^2 \) be the domain

\[\Omega = \{(t,x) \in \mathbb{R}^2 : 0 < t < T, \ x \in I_t\},\]

\[I_t = \{x \in \mathbb{R} : \varphi_1(t) < x < \varphi_2(t), \ t \in (0,T)\}.\]

In this section, we assume that \( \varphi_1(0) \neq \varphi_2(0) \). In other words

\( \varphi_1(t) < \varphi_2(t) \) for all \( t \in [0,T] \).

**Theorem 2.1.** If \( f \in L^2(\Omega) \) and \( c(t), (\varphi_i)_{i=1,2} \) satisfy the assumptions (1.3), (1.4) and (2.1), then the problem

\[
\partial_t u(t,x) + c(t)u(t,x)\partial_x u(t,x) - \partial^2_x u(t,x) = f(t,x) \quad (t,x) \in \Omega,
\]

\[
u(0,x) = 0 \quad x \in J = (\varphi_1(0),\varphi_2(0)),
\]

\[
u(t,\varphi_1(t)) = u(t,\varphi_2(t)) = 0 \quad t \in (0,T),
\]

admits a solution \( u \in H^{1,2}(\Omega) \).

**Proof.** The change of variables: \( \Omega \to R \)

\[
(t,x) \mapsto (t,y) = \left( t, \frac{x - \varphi_1(t)}{\varphi_2(t) - \varphi_1(t)} \right)
\]
transforms $\Omega$ into the rectangle $R = (0, T) \times (0, 1)$. Putting $u(t, x) = v(t, y)$ and $f(t, x) = g(t, y)$, problem (2.2) becomes

$$
\frac{\partial}{\partial t} v(t, y) + p(t) v(t, y) \frac{\partial}{\partial y} v(t, y) - q(t) \partial_y^2 v(t, y) + r(t, y) \partial_y v(t, y)
= g(t, y) \quad (t, y) \in R,
$$

$$
v(0, y) = 0 \quad y \in (0, 1),
$$

$$
v(t, 0) = v(t, 1) = 0 \quad t \in (0, T),
$$

where

$$
\varphi(t) = \varphi_2(t) - \varphi_1(t), \quad p(t) = \frac{c(t)}{\varphi(t)},
$$

$$
q(t) = \frac{1}{\varphi^2(t)}, \quad r(t, y) = -\frac{y \varphi'(t) + \varphi'(t)}{\varphi(t)}.
$$

This change of variables preserves the spaces $H^{1,2}$ and $L^2$. In other words

$$
f \in L^2(\Omega) \iff g \in L^2(R),
$$

$$
u \in H^{1,2}(\Omega) \iff v \in H^{1,2}(R).
$$

According to (1.3) and (1.4), the functions $p, q$ and $r$ satisfy the following conditions

$$
\alpha < p(t) < \beta, \quad \forall t \in [0, T],
$$

$$
\alpha < q(t) < \beta, \quad \forall t \in [0, T],
$$

$$
|\partial_y r(t, y)| \leq \beta, \quad \forall (t, y) \in R,
$$

where $\alpha$ and $\beta$ are positive constants.
So, problem (2.2) is equivalent to problem (2.3), and by [2] problem (2.3) admits a solution \( v \in H^{1,2}(R) \). Then, problem (2.2) in the domain \( \Omega \) admits a solution \( u \in H^{1,2}(\Omega) \).

3. Proof of Theorem 1.2, Case 1

Let

\[
\Omega = \{(t,x) \in \mathbb{R}^2 : 0 < t < T, \ x \in I_t\},
\]

\[
I_t = \{x \in \mathbb{R} : \phi_1(t) < x < \phi_2(t), \ t \in (0,T)\},
\]

with \( \phi_1(0) = \phi_2(0) \) and \( \phi_1(T) < \phi_2(T) \).

![Non-parabolic domain](image)

**Figure 2.** Non-parabolic domain.

For each \( n \in \mathbb{N}^* \), we define

\[
\Omega_n = \{(t,x) \in \mathbb{R}^2 : \frac{1}{n} < t < T, \ x \in I_t\},
\]

and we set \( f_n = f|_{\Omega_n} \), where \( f \) is given in \( L^2(\Omega) \). By Theorem 2.1, there exists a solution \( u_n \in H^{1,2}(\Omega_n) \) of the problem

\[
\begin{align*}
\partial_t u_n(t,x) + c(t)u_n(t,x)\partial_x u_n(t,x) - \partial_x^2 u_n(t,x) &= f_n(t,x) \quad (t,x) \in \Omega_n, \\
\phi_1(\frac{1}{n}) < x < \phi_2(\frac{1}{n}), \\
u_n(\frac{1}{n},x) = 0, \\
u_n(t,\phi_1(t)) = u_n(t,\phi_2(t)) = 0 & t \in [\frac{1}{n},T],
\end{align*}
\]

(3.1)

in \( \Omega_n \).

To prove Case 1 of Theorem 1.2, we have to pass to the limit in (3.1). For this purpose we need the following result.
Proposition 3.1. There exists a positive constant $K$ independent of $n$ such that
\[\|u_n\|_{H^1(T_n)}^2 \leq K\|f_n\|_{L^2(\Omega_n)}^2 \leq K\|f\|_{L^2(\Omega)}^2.\]

To prove this proposition we need some preliminary results.

Lemma 3.2. There exists a positive constant $K_1$ independent of $n$ such that
\begin{align*}
\|u_n\|_{L^2(\Omega_n)}^2 &\leq K_1\|\partial_x u_n\|_{L^2(\Omega_n)}^2, \quad (3.2) \\
\|\partial_x u_n\|_{L^2(\Omega_n)}^2 &\leq K_1\|f_n\|_{L^2(\Omega_n)}^2, \quad (3.3)
\end{align*}

Proof. We have
\[|u_n|^2 = \left| \int_{\varphi_1(t)}^{x} \partial_x u_n \, ds \right|^2 \leq (x - \varphi_1(t)) \int_{\varphi_1(t)}^{x} |\partial_x u_n|^2 \, ds,\]
integrating from $\varphi_1(t)$ to $\varphi_2(t)$, we obtain
\[\int_{\varphi_1(t)}^{\varphi_2(t)} |u_n|^2 \, dx \leq \int_{\varphi_1(t)}^{\varphi_2(t)} \left( (x - \varphi_1(t)) \int_{\varphi_1(t)}^{x} |\partial_x u_n|^2 \, ds \right) \, dx,
\]
hence
\[\int_{\varphi_1(t)}^{\varphi_2(t)} |u_n|^2 \, dx \leq (\varphi_2(t) - \varphi_1(t)) \int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x u_n|^2 \, dx,
\]
and
\[\int_{\varphi_1(t)}^{\varphi_2(t)} |u_n|^2 \, dx \leq (\varphi_2(t) - \varphi_1(t))^2 \int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x u_n|^2 \, dx.
\]
Then, there exists a positive constant $K_1$ independent of $n$ such that
\[\|u_n\|_{L^2(I_n)}^2 \leq K_1\|\partial_x u_n\|_{L^2(I_n)}^2,
\]
integrating between $1/n$ and $T$ we obtain inequality (3.2).

Now, multiplying both sides of (3.1) by $u_n$ and integrating between $\varphi_1(t)$ and $\varphi_2(t)$, we obtain
\[\frac{1}{2} \frac{d}{dt} \int_{\varphi_1(t)}^{\varphi_2(t)} (u_n)^2 \, dx + c(t) \int_{\varphi_1(t)}^{\varphi_2(t)} \partial_x u_n u_n^2 \, dx - \int_{\varphi_1(t)}^{\varphi_2(t)} u_n \partial_x^2 u_n \, dx = \int_{\varphi_1(t)}^{\varphi_2(t)} f_n u_n \, dx.
\]
Integration by parts gives
\[c(t) \int_{\varphi_1(t)}^{\varphi_2(t)} \partial_x u_n u_n^2 \, dx = \frac{c(t)}{3} \int_{\varphi_1(t)}^{\varphi_2(t)} \partial_x (u_n)^3 \, dx = 0;
\]
then
\[\frac{1}{2} \frac{d}{dt} \int_{\varphi_1(t)}^{\varphi_2(t)} (u_n)^2 \, dx + \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x u_n)^2 \, dx = \int_{\varphi_1(t)}^{\varphi_2(t)} f_n u_n \, dx. \quad (3.4)
\]
By integrating (3.4) from $1/n$ to $T$, we find that
\[\frac{1}{2} \|u_n(T, x)\|_{L^2(I_n)}^2 + \int_{1/n}^{T} \|\partial_x u_n(s)\|_{L^2(I_n)}^2 \, ds
\leq \int_{1/n}^{T} \|f_n(s)\|_{L^2(I_n)} \|u_n(s)\|_{L^2(I_n)} \, ds.
\]
Using the elementary inequality
\[|rs| \leq \frac{r^2}{2} + \frac{s^2}{2\varepsilon}, \quad \forall r, s \in R, \ \forall \varepsilon > 0,
\]
with \( \varepsilon = K_1 \), we obtain
\[
\frac{1}{2} \| u_n(T, x) \|_{L^2(I_T)}^2 + \int_{1/n}^T \| \partial_x u_n(s) \|_{L^2(I_T)}^2 \, ds \\
\leq \frac{K_1}{2} \int_{1/n}^T \| f_n(s) \|_{L^2(I_T)}^2 \, ds + \frac{1}{2K_1} \int_{1/n}^T \| u_n(s) \|_{L^2(I_T)}^2 \, ds.
\]

Thanks to (3.2), we have
\[
\| u_n(T, x) \|_{L^2(I_T)}^2 + \int_{1/n}^T \| \partial_x u_n(s) \|_{L^2(I_T)}^2 \, ds \leq K_1 \int_{1/n}^T \| f_n(s) \|_{L^2(I_T)}^2 \, ds,
\]
so,
\[
\| \partial_x u_n \|_{L^2(\Omega_n)} \leq K_1 \| f_n \|_{L^2(\Omega_n)}.
\]

**Corollary 3.3.** There exists a positive constant \( K_2 \) independent of \( n \), such that for all \( t \in [1/n, T] \),
\[
\| \partial_x u_n \|_{L^2(I_T)}^2 + \int_{1/n}^T \| \partial_x^2 u_n(s) \|_{L^2(I_T)}^2 \, ds \leq K_2.
\]

**Proof.** Multiplying both sides of (3.1) by \( \partial_x^2 u_n \) and integrating between \( \varphi_1(t) \) and \( \varphi_2(t) \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x u_n)^2 \, dx + \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x^2 u_n)^2 \, dx \\
= - \int_{\varphi_1(t)}^{\varphi_2(t)} f_n \partial_x^2 u_n \, dx + c(t) \int_{\varphi_1(t)}^{\varphi_2(t)} u_n \partial_x u_n \partial_x^2 u_n \, dx.
\]

Using Cauchy-Schwartz inequality, (3.5) with \( \varepsilon = \frac{1}{2} \) leads to
\[
\left| \int_{\varphi_1(t)}^{\varphi_2(t)} f_n \partial_x^2 u_n \, dx \right| \leq \left( \int_{\varphi_1(t)}^{\varphi_2(t)} | \partial_x^2 u_n |^2 \, dx \right)^{1/2} \left( \int_{\varphi_1(t)}^{\varphi_2(t)} | f_n |^2 \, dx \right)^{1/2} \\
\leq \frac{1}{4} \int_{\varphi_1(t)}^{\varphi_2(t)} | \partial_x^2 u_n |^2 \, dx + \int_{\varphi_1(t)}^{\varphi_2(t)} | f_n |^2 \, dx.
\]

Now, we have to estimate the last term of (3.7). An integration by parts gives
\[
\int_{\varphi_1(t)}^{\varphi_2(t)} u_n \partial_x u_n \partial_x^2 u_n \, dx = \int_{\varphi_1(t)}^{\varphi_2(t)} u_n (\frac{1}{2} (\partial_x u_n)^2) \, dx = -\frac{1}{2} \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x u_n)^3 \, dx.
\]

Since \( \partial_x u_n \) satisfies \( \int_{\varphi_1(t)}^{\varphi_2(t)} \partial_x u_n \, dx = 0 \) we deduce that the continuous function \( \partial_x u_n \) is zero at some point \( \xi(t) \in (\varphi_1(t), \varphi_2(t)) \), and by integrating \( 2 \partial_x u_n \partial_x^2 u_n \) between \( \xi(t) \) and \( x \), we obtain
\[
2 \int_{\xi(t)}^{x} \partial_x u_n \partial_x^2 u_n \, ds \int_{\xi(t)}^{x} = \partial_x (\partial_x u_n)^2 \, ds = (\partial_x u_n)^2,
\]
the Cauchy-Schwartz inequality gives
\[
\| \partial_x u_n \|^2_{L^\infty(I_t)} \leq 2 \| \partial_x u_n \|_{L^2(I_t)} \| \partial_x^2 u_n \|_{L^2(I_t)},
\]
but
\[
\| \partial_x u_n \|_{L^3(I_t)} \leq \| \partial_x u_n \|^2_{L^2(I_t)} \| \partial_x u_n \|_{L^\infty(I_t)},
\]
so, (3.3) yields

$$\left| \int_{\varphi_1(t)}^{\varphi_2(t)} c(t)u_n \partial_x u_n \partial_x^2 u_n \, dx \right| \leq \left( \int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x^2 u_n|^2 \, dx \right)^{1/4} \left( c_2^{4/5} \int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x u_n|^2 \, dx \right)^{5/4}.$$

Finally, by Young’s inequality $|AB| \leq \frac{|A|^p}{p} + \frac{|B|^{p'}}{p'}$, with $1 < p < \infty$ and $p' = \frac{p}{p-1}$. Choosing $p = 4$ (then $p' = \frac{4}{3}$)

$$A = \left( \int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x^2 u_n|^2 \, dx \right)^{1/4}, \quad B = \left( c_2^{4/5} \int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x u_n|^2 \, dx \right)^{5/4},$$

the estimate of the last term of (3.7) becomes

$$\left| \int_{\varphi_1(t)}^{\varphi_2(t)} c(t)u_n \partial_x u_n \partial_x^2 u_n \, dx \right| \leq \frac{1}{4} \int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x^2 u_n|^2 \, dx + \frac{3}{4} c_2^{4/3} \int_{\varphi_1(t)}^{\varphi_2(t)} |\partial_x u_n|^2 \, dx \right)^{5/3}. \quad (3.9)$$

Let us return to (3.7): By integrating between $\frac{1}{n}$ and $t$, from the estimates (3.8) and (3.9), we obtain

$$\|\partial_x u_n\|_{L^2(I_t)}^2 + \int_{1/n}^{t} \|\partial_x^2 u_n(s)\|_{L^2(I_t)}^2 \, ds \leq 2 \int_{1/n}^{t} \|f_n(s)\|_{L^2(I_t)}^2 \, ds + \frac{3}{2} c_2^{4/3} \int_{1/n}^{t} \left( \|\partial_x u_n(s)\|_{L^2(I_t)}^2 \right)^{5/3} \, ds.$$

If $f_n \in L^2(\Omega_n)$, then there exists a constant $c_3$ such that

$$\|\partial_x u_n\|_{L^2(I_t)}^2 + \int_{1/n}^{t} \|\partial_x^2 u_n(s)\|_{L^2(I_t)}^2 \, ds \leq c_3 + \frac{3}{2} c_2^{4/3} \int_{1/n}^{t} \left( \|\partial_x u_n(s)\|_{L^2(I_t)}^2 \right)^{2/3} \|\partial_x u_n(s)\|_{L^2(I_t)}^2 \, ds.$$

Consequently, the function

$$\varphi(t) = \|\partial_x u_n\|_{L^2(I_t)}^2 + \int_{1/n}^{t} \|\partial_x^2 u_n(s)\|_{L^2(I_t)}^2 \, ds$$

satisfies the inequality

$$\varphi(t) \leq c_3 + \int_{1/n}^{t} \left( \frac{3}{2} c_2^{4/3} \|\partial_x u_n(s)\|_{L^2(I_t)}^{4/3} \right) \varphi(s) \, ds, \quad (4.1)$$

Gronwall’s inequality shows that

$$\varphi(t) \leq c_3 \exp \left( \int_{1/n}^{t} \left( \frac{3}{2} c_2^{4/3} \|\partial_x u_n(s)\|_{L^2(I_t)}^{4/3} \right) \, ds \right).$$

According to Lemma 3.2, the integral $\int_{1/n}^{T} \|\partial_x u_n\|_{L^2(I_t)}^{4/3} \, ds$ is bounded by a constant independent of $n$. So there exists a positive constant $K_2$ such that

$$\|\partial_x u_n\|_{L^2(I_t)}^2 + \int_{1/n}^{T} \|\partial_x^2 u_n(s)\|_{L^2(I_t)}^2 \, ds \leq K_2.$$
Lemma 3.4. There exists a constant $K_3$ independent of $n$ such that

$$
\|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|\partial^2_x u_n\|^2_{L^2(\Omega_n)} \leq K_3\|f_n\|^2_{L^2(\Omega_n)}.
$$

Then Theorem 3.1 is a direct consequence of Lemmas 3.2 and 3.4.

Proof. To prove Lemma 3.4 we develop the inner product in $L^2(\Omega_n)$,

$$
\|f_n\|^2_{L^2(\Omega_n)} = (\partial_t u_n + c(t)u_n\partial_x u_n - \partial^2_x u_n, \partial_t u_n + c(t)u_n\partial_x u_n - \partial^2_x u_n)_{L^2(\Omega_n)}
$$

$$
= \|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|\partial^2_x u_n\|^2_{L^2(\Omega_n)} + \|c(t)u_n\|_{L^2(\Omega_n)}^2
$$

$$
- 2(\partial_t u_n, \partial^2_x u_n)_{L^2(\Omega_n)} + 2(\partial_t u_n, c(t)u_n\partial_x u_n)_{L^2(\Omega_n)}
$$

$$
- 2(c(t)u_n\partial_x u_n, \partial^2_x u_n)_{L^2(\Omega_n)},
$$

so,

$$
\|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|\partial^2_x u_n\|^2_{L^2(\Omega_n)}
$$

$$
= \|f_n\|^2_{L^2(\Omega_n)} - \|c(t)u_n\|_{L^2(\Omega_n)}^2 + 2(\partial_t u_n, \partial^2_x u_n)_{L^2(\Omega_n)}
$$

$$
- 2(\partial_t u_n, c(t)u_n\partial_x u_n)_{L^2(\Omega_n)} + 2(\partial_t u_n, \partial^2_x u_n)_{L^2(\Omega_n)}. \tag{3.10}
$$

Using (1.3) and (3.5(292,645),(397,662)(296,645),(397,662)(296,645),(397,662)(296,645),(397,662)(296,645),(397,662)(296,645),(397,662) with $\varepsilon = 1/2$, we obtain

$$
\left| -2(\partial_t u_n, c(t)u_n\partial_x u_n)_{L^2(\Omega_n)} \right| \leq \frac{1}{2} \|\partial_t u_n\|^2_{L^2(\Omega_n)} + 2\varepsilon_2^2\|u_n\partial_x u_n\|^2_{L^2(\Omega_n)}, \tag{3.11}
$$

and

$$
\left| 2(\partial_t u_n, \partial^2_x u_n)_{L^2(\Omega_n)} \right| \leq 2\varepsilon_2^2\|u_n\partial_x u_n\|^2_{L^2(\Omega_n)} + \frac{1}{2} \|\partial^2_x u_n\|^2_{L^2(\Omega_n)}. \tag{3.12}
$$

Now calculating the last term of (3.10),

$$
(\partial_t u_n, \partial^2_x u_n)_{L^2(\Omega_n)} = -\int_{1/n}^T \int_{\varphi_1(t)}^{\varphi_2(t)} \partial_t (\partial_x u_n)\partial_x u_n \, dx \, dt + \int_{1/n}^T \int_{\varphi_1(t)}^{\varphi_2(t)} \partial_x u_n \partial^2_x u_n \, dx \, dt
$$

$$
= -\frac{1}{2} \int_{1/n}^T \int_{\varphi_1(t)}^{\varphi_2(t)} \partial_t (\partial_x u_n)^2 \, dx \, dt + \int_{1/n}^T \int_{\varphi_1(t)}^{\varphi_2(t)} \partial_x u_n \partial^2_x u_n \, dx \, dt
$$

$$
= -\frac{1}{2} \left[ \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x u_n)^2 \, dx \right]_{1/n}^T + \int_{1/n}^T \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x u_n)^2 \, dx \, dt
$$

$$
= -\frac{1}{2} \int_{1/n}^T (\partial_x u_n)^2(T,x) \, dx + \frac{1}{2} \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x u_n)^2 \left( \frac{1}{n}, x \right) \, dx
$$

$$
+ \int_{1/n}^T \partial_x u_n(t,\varphi_2(t))\partial_x u_n(t,\varphi_1(t)) \, dt
$$

$$
- \int_{1/n}^T \partial_x u_n(t,\varphi_1(t))\partial_x u_n(t,\varphi_1(t)) \, dt.
$$

According to the boundary conditions, we have

$$
\partial_t u_n(t,\varphi_i(t)) + \varphi_i'(t)\partial_x u_n(t,\varphi_i(t)) = 0, \quad i = 1, 2,
$$

so

$$
(\partial_t u_n, \partial^2_x u_n)_{L^2(\Omega_n)} = -\frac{1}{2} \int_{\varphi_1(T)}^{\varphi_2(T)} (\partial_x u_n)^2(T,x) \, dx - \int_{1/n}^T \varphi_2'(t)(\partial_x u_n(t,\varphi_2(t)))^2 \, dt
$$
there exists a constant zero of

We know that a solution \( u \) that

it follows that

Then

□

This completes the proof.

\[ \tilde{u} \] implies that

On the other hand, using the injection of \( H^1(I_t) \) in \( L^\infty(I_t) \), we obtain

According to Corollary 3.3, \( \| u_n \|_{H^2(\Omega_n)}^2 \) is bounded, then by (3.3) and (3.14), there exists a constant \( K \) independent of \( n \), such that

However,

then, from lemmas 3.2 and 3.4, there exists a constant \( K \) independent of \( n \), such that

This completes the proof. \( \square \)

Existence and uniqueness. Choose a sequence \( (\Omega_n)_{n \in N} \) of the domains defined previously, such that \( \Omega_n \subset \Omega \), as \( n \to +\infty \) then \( \Omega_n \to \Omega \).

Consider \( u_n \in H^{1,2}(\Omega_n) \) the solution of

\[ c(t) u_n(t,x) \partial_x u_n(t,x) - \partial_x^2 u_n(t,x) = f_n(t,x) \quad (t,x) \in \Omega_n, \]

\[ u_n(\frac{1}{n},x) = 0 \quad \varphi_1(\frac{1}{n}) < x < \varphi_2(\frac{1}{n}), \]

\[ u_n(t,\varphi_1(t)) = u_n(t,\varphi_2(t)) = 0 \quad t \in [\frac{1}{n},1]. \]

We know that a solution \( u_n \) exists by the Theorem 2.1. Let \( \tilde{u}_n \) be the extension by zero of \( u_n \) outside \( \Omega_n \). From the proposition 3.1 results the inequality

This implies that \( \tilde{u}_n, \partial_t \tilde{u}_n \) and \( \partial_{x}^j \tilde{u}_n, j = 1,2 \) are bounded in \( L^2(\Omega_n) \), from Corollary 3.3 \( \tilde{u}_n \partial_x u_n \) is bounded in \( L^2(\Omega_n) \). So, it is possible to extract a subsequence from \( u_n \), still denoted \( u_n \) such that

\[ \partial_t \tilde{u}_n \to \partial_t u \quad \text{weakly in} \quad L^2(\Omega_n), \]

\[ \partial_x^j \tilde{u}_n \to \partial_x^j u \quad \text{weakly in} \quad L^2(\Omega_n), \]

\[ \tilde{u}_n \partial_x \tilde{u}_n u_n \to u \partial_x u \quad \text{weakly in} \quad L^2(\Omega_n). \]

Then \( u \in H^{1,2}(\Omega) \) is solution to problem (1.2).
For the uniqueness, let us observe that any solution $u \in H^{1,2}(\Omega)$ of problem (1.2) is in $L^\infty(0, T, H^1(I))$. Indeed, by the same way as in Corollary 3.3 we prove that there exists a positive constant $K_2$ such that for all $t \in [0, T]$

$$\|\partial_x u\|_{L^2(I_t)}^2 + \int_0^T \|\partial_x^2 u(s)\|_{L^2(I_t)}^2 \, ds \leq K_2.$$

Let $u_1, u_2 \in H^{1,2}(\Omega)$ be two solutions of (1.2). We put $u = u_1 - u_2$. It is clear that $u \in L^\infty(0, T, H^1(I))$. The equations satisfied by $u_1$ and $u_2$ leads to

$$\int_{\varphi_1(t)} \left[ \partial_t uw + c(t)uw\partial_x u_1 + c(t)u_2 w\partial_x u + \partial_x u_1 w\right] \, dx = 0.$$

Taking, for $t \in [0, T]$, $w = u$ as a test function, we deduce that

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2(I_t)}^2 + \|\partial_x u\|_{L^2(I_t)}^2 = -c(t) \int_{\varphi_1(t)} u^2 \partial_x u_1 \, dx - c(t) \int_{\varphi_1(t)} u_2 u \partial_x u \, dx. \tag{3.15}$$

An integration by parts gives

$$c(t) \int_{\varphi_1(t)} u^2 \partial_x u_1 \, dx = -2c(t) \int_{\varphi_1(t)} u \partial_x uu_1 \, dx,$$

then (3.15) becomes

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2(I_t)}^2 + \|\partial_x u\|_{L^2(I_t)}^2 = \int_{\varphi_1(t)} c(t)(2u_1 - u_2) u \partial_x u \, dx.$$

By (1.3) and inequality (3.5) with $\varepsilon = 2$, we obtain

$$\left| \int_{\varphi_1(t)} c(t)(2u_1 - u_2) u \partial_x u \, dx \right| \leq \frac{1}{4} c_2^2 (2\|u_1\|_{L^\infty(0, T, H^1(I_t))} + \|u_2\|_{L^\infty(0, T, H^1(I_t))})^2 \|u\|_{L^2(I_t)}^2 + \|\partial_x u\|_{L^2(I_t)}^2.$$

So, we deduce that there exists a non-negative constant $D$, such as

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2(I_t)}^2 \leq D\|u\|_{L^2(I_t)}^2,$$

and Gronwall’s lemma leads to $u = 0$. This completes the proof of Theorem 1.2 Case 1.

4. PROOF OF THEOREM 1.2 CASE 2

In this case we set $\Omega = Q_1 \cup Q_2 \cup \Sigma_{T_1}$ where

$$Q_1 = \{(t, x) \in \mathbb{R}^2 : 0 < t < T_1, \ x \in I_t\},$$

$$Q_2 = \{(t, x) \in \mathbb{R}^2 : T_1 < t < T, \ x \in I_t\},$$

$$\Sigma_{T_1} = \{(T_1, x) \in \mathbb{R}^2 : x \in I_{T_1}\},$$

with $T_1$ small enough. $f \in L^2(\Omega)$ and $f_i = f_{|Q_i}$, $i = 1, 2$.

Theorem 1.2 Case 1, applied to the domain $Q_1$, shows that there exists a unique solution $u_1 \in H^{1,2}(Q_1)$ of the problem

$$\partial_t u_1(t, x) + c(t)u_1(t, x)\partial_x u_1(t, x) - \partial_x^2 u_1(t, x)$$

$$+ \int_0^T \|\partial_x^2 u(s)\|_{L^2(I_t)} \, ds \leq K_2.$$
Lemma 4.2. If we deduce from Lemma 4.1 the following result.

The above lemma is a special case of [10, Theorem 2.1, Vol. 2]. Using the transformation $[T_1, T] \times [0, 1] \rightarrow Q_2$,

$$(t, x) \mapsto (t, y) = (t, (\varphi_2(t) - \varphi_1(t))x + \varphi_1(t))$$

we deduce from Lemma 4.1 the following result.

**Lemma 4.2.** If $u \in H^{1,2}(Q_2)$, then $u|_{\Gamma_{T_1}} \in H^1(\Gamma_{T_1})$.

We denote the trace $u_{1|\Gamma_{T_1}}$ by $u_0$ which is in the Sobolev space $H^1(\Gamma_{T_1})$ because $u_1 \in H^{1,2}(Q_1)$.

Theorem 2.1 applied to the domain $Q_2$, shows that there exists a unique solution $u_2 \in H^{1,2}(Q_2)$ of the problem

$$\partial_t u_2(t, x) + c(t)u_2(t, x)\partial_x u_2(t, x) - \partial^2_x u_2(t, x) = f_2(t, x) \quad (t, x) \in Q_2,$$

$$u_2(0, x) = u_0(x) \quad \varphi_1(T_1) < x < \varphi_2(T_1),$$

$$u_2(t, \varphi_1(t)) = u_2(t, \varphi_2(t)) = 0 \quad t \in [T_1, T],$$

putting

$$u = \begin{cases} u_1 & \text{in } Q_1, \\ u_2 & \text{in } Q_2, \end{cases}$$

we observe that $u \in H^{1,2}(\Omega)$ because $u_{1|\Gamma_{T_1}} = u_{2|\Gamma_{T_1}}$ and is a solution of the problem

$$\partial_t u(t, x) + c(t)u(t, x)\partial_x u(t, x) - \partial^2_x u(t, x) = f(t, x) \quad (t, x) \in \Omega,$$

$$u(t, \varphi_1(t)) = u(t, \varphi_2(t)) = 0 \quad t \in (0, T).$$

We prove the uniqueness of the solution by the same way as in Case 1.

**References**


YASSINE BENIA  
DEPT OF MATHEMATICS AND INFORMATICS, UNIVERSITY OF BENVYUCEF BENKHEDDA (ALGER 1), 16000, ALGIERS, ALGERIA  
E-mail address: benia.yacine@yahoo.fr

BOUBAKER-KHALED SADALLAH  
LAB. PDE & HIST MATHS; DEPT OF MATHEMATICS, E.N.S., 16050, KOUBA, ALGIERS, ALGERIA  
E-mail address: sadallah@ens-kouba.dz