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BOUNDARY CONTROLLABILITY AND OBSERVABILITY OF COUPLED WAVE EQUATIONS WITH MEMORY

TI-JUN XIAO, ZHE XU

Abstract. In this article we consider the controllability for a system of coupled wave equations with memory. We reduce the control problem to a moment problem which can be solved by showing the Riesz property of the associated families of functions. In that way, we obtain (direct or indirect) boundary observability inequalities and boundary controllability of the system.

1. Introduction

This article concerns the controllability and observability of the system

$$\begin{aligned} &u_{tt}(x,t) - u_{xx}(x,t) - \int_0^t (u_{xx}(x,s) - \beta y(x,s))N(t-s)ds + \alpha y(x,t) = 0, \\ &y_{tt}(x,t) - y_{xx}(x,t) - \int_0^t (y_{xx}(x,s) - \beta u(x,s))N(t-s)ds + \alpha u(x,t) = 0, \end{aligned} \tag{1.1}$$

subject to initial and Dirichlet-Neumann boundary conditions

$$\begin{gathered} u(x,0) = u_0, \quad y(x,0) = y_0, \quad u_t(x,0) = u_1, \quad y_t(x,0) = y_1, \\ u(0,t) = g(t), \quad y(0,t) = f(t), \quad u_x(\pi,t) = y_x(\pi,t) = 0, \end{gathered} \tag{1.2}$$

where $\alpha, \beta \in \mathbb{R}$ (the set of real numbers) are the coupling coefficients, $N(t)$ the memory kernel, and $f(t), g(t)$ the control functions. Models of this type are of interest in vibrating problems in relation to viscoelastic material, see for example [11, 12, 13, 14, 15].

When the memory terms are absent (i.e. $N(\cdot) = 0$), controllability properties of the coupled equations (1.1) with Dirichlet boundary conditions are discussed in [3]; with an explicit analytic condition (which is necessary and sufficient) on the coupling coefficient, the authors use the method of moments to establish indirect exact boundary controllability of the system. See also the earlier work [1] for the boundary controllability of an abstract system of two coupled second order evolution equations (without memory either) by means of a two level energy method, under the smallness condition on the coupling coefficient.

The controllability of single equations with memory has been studied in many papers. We would like to mention specially the papers [2, 12, 14, 15], where the controllability problems are reduced to moment problems. By showing the Riesz

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property of associated function families, the authors prove the controllability of the systems.

The control problems for coupled $string - beam$ equations with memory are investigated in [13]. In their model, the memory kernel is of the exponential form and the controls act on the boundary points of both string and beam. Reachability results are obtained by writing the solutions of the system as Fourier series and then showing Ingham type estimates.

In this article, we combine and adapt the ideas and methods from [2, 3, 12, 14, 15] to study the controllability of the coupled memory system (1.1). We consider two cases; one involves two controls f and g, and the other only involves one control g. We will illustrate the difference between the two cases.

As will be seen, we consider the existence of solutions in a weak form, because the control functions may be rough. For general study on the existence of solutions to equations with memory, we refer to some related results in [5, 6, 7, 8, 9, 16]. It is known that exact controllability implies stabilizability in linear cases. We also refer the reader to [7, 8, 9, 10, 11] for more information about the stability and perturbation results for equations with memory.

This article is organized as follows. In Section 2, we state our main theorems. In Section 3, we reduce the control problem of system (1.1)-(1.2) to a moment problem and give proofs of Theorems 2.4 and 2.5 for controllability and observability when we have two control functions. In Section 4, we prove Theorems 2.6 and 2.7 regarding the case when we have only one control function.

2. Main results

Let

$$H = L^2(0,\pi), \quad V = \{v \in H^1(0,\pi), v(0) = 0\}. \tag{2.1}$$

There is a natural continuous embedding $V \subset H$, which leads to the natural embedding of H into the dual space V'.

First we give a lemma for defining the weak solution of (1.1)-(1.2) with controls on the boundary.

Lemma 2.1. *Suppose that $u_1, y_1 \in V$ and*

$$u_0, y_0 \in H^2(0,\pi) \quad \text{with } u_0(0) = y_0(0) = u_0'(\pi) = y_0'(\pi) = 0.$$

Let (u, y) be the classical solution of system (1.1)-(1.2) with $f(t), g(t) \equiv 0$. Then

$$\|u_x(0,t)\|^2_{L^2(0,T)} + \|y_x(0,t)\|^2_{L^2(0,T)} \le C(\|u_0\|^2_V + \|u_1\|^2_H + \|y_0\|^2_V + \|y_1\|^2_H),$$
$$\|u_x(0,t)\|^2_{L^2(0,T)} \le C(\|u_0\|^2_V + \|u_1\|^2_H + \|y_0\|^2_H + \|y_1\|^2_{V'}),$$

where C is a constant independent of the initial data.

We present the proof at the end of Section 3. Now consider the dual problem

$$
\begin{aligned}
&\overline{u}_{tt}(x,t)-\overline{u}_{xx}(x,t)-\int_t^T \overline{u}_{xx}(x,s)N(s-t)ds\\
&+\alpha\overline{y}(x,t)+\beta\int_t^T \overline{y}(x,s)N(s-t)ds=0,\\
&\overline{y}_{tt}(x,t)-\overline{y}_{xx}(x,t)-\int_t^T \overline{y}_{xx}(x,s)N(s-t)ds\\
&+\alpha\overline{u}(x,t)+\beta\int_t^T \overline{u}(x,s)N(s-t)ds=0,\\
&\overline{u}(x,T)=\overline{u}_0,\quad \overline{u}_t(x,T)=\overline{u}_1, \overline{y}(x,T)=\overline{y}_0,\quad \overline{y}_t(x,T)=\overline{y}_1,\\
&\overline{u}(0,t)=\overline{u}_x(\pi,t)=\overline{y}(0,t)=\overline{y}_x(\pi,t)=0.
\end{aligned}
\tag{2.2}
$$

Let u,y be the solutions of (1.1)-(1.2) with null initial data, multiply the equations in $\overline{u},\overline{y}$ by u,y respectively, and integrate them over $(0,T)\times(0,\pi)$. After some computations, we obtain

$$
-\int_0^\pi u(T)\overline{u}_1+y(T)\overline{y}_1-u_t(T)\overline{u}_0-y_t(T)\overline{y}_0dx=\int_0^T g(t)\varphi_1(t)+f(t)\varphi_2(t)dt,
$$

where

$$
\begin{aligned}
\varphi_1(t)&:=\overline{u}_x(0,t)+\int_t^T N(s-t)\overline{u}_x(0,s)ds,\\
\varphi_2(t)&:=\overline{y}_x(0,t)+\int_t^T N(s-t)\overline{y}_x(0,s)ds.
\end{aligned}
$$

This suggests the following definition (see [9] for weak solutions of the systems without memory).

Definition 2.2. Let $T>0$, and $g,f\in L^2(0,T)$. We say that (u,y) is a (weak) solution of (1.1)-(1.2) with null initial data, if $u,y\in C([0,T];H)\cap C^1([0,T];V')$ and satisfy

$$
\begin{aligned}
&-(u(S),\overline{u}_1)-(y(S),\overline{y}_1)+\langle u_t(S),\overline{u}_0\rangle+\langle y_t(S),\overline{y}_0\rangle\\
&=\int_0^S g(t)\varphi_1(t)dt+\int_0^S f(t)\varphi_2(t)dt,
\end{aligned}
\tag{2.3}
$$

for any $S>0$ and any functions $\overline{u}_0,\overline{u}_1,\overline{y}_0,\overline{y}_1\in C_c^\infty(0,\pi)$, where $(\overline{u},\overline{y})$ is the solution of (2.2), and $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ denote the inner product in H and the duality pairing between V and V' respectively.

From (2.3), we can deduce (by applying Lemma 2.1 to the dual system (2.2)) that for any $T>0$, there is a constant $C>0$ such that

$$
\begin{aligned}
&\sup_{t\in[0,T]}(\|u(t)\|_H+\|u_t(t)\|_{V'}+\|y(t)\|_H+\|y_t(t)\|_{V'})\\
&\le C(\|g\|_{L^2(0,T)}+\|f\|_{L^2(0,T)}),
\end{aligned}
$$

and, when $f(t)\equiv 0$,

$$
\sup_{t\in[0,T]}(\|u(t)\|_H+\|u_t(t)\|_{V'}+\|y(t)\|_V+\|y_t(t)\|_H)\le C\|g\|_{L^2(0,T)}.
$$

The above observation leads to the following existence and uniqueness theorem.

Theorem 2.3. *Let $T > 0$, and $g, f \in L^2(0,T)$. Then system* (1.1)-(1.2) *with null initial data has a unique (weak) solution (u, y). Moreover, if $f(t) \equiv 0$, we have $y \in C([0,T]; V) \cap C^1([0,T]; H)$.*

Now we state our theorem for the controllability of the system with two controls.

Theorem 2.4. *Let $T \geq 2\pi$ and $N(t) \in H^2(0,T)$. Then* (1.1)-(1.2) *is exactly controllable on the time interval $[0,T]$ and the control space is $H \times V' \times H \times V'$. That is, for any given final state $(\xi_0, \xi_1, \eta_0, \eta_1) \in H \times V' \times H \times V'$ we can choose suitable $g(t), f(t) \in L^2(0,T)$ such that the solution of* (1.1)-(1.2) *with null initial data satisfies $(u(T), u_t(T), y(T), y_t(T)) = (\xi_0, \xi_1, \eta_0, \eta_1)$.*

We can also get the observability of the system.

Theorem 2.5. *Let $T \geq 2\pi$, and (u, y) solve* (1.1)-(1.2) *with*

$$(u_0, u_1, y_0, y_1) \in V \times H \times V \times H, \quad \text{and} \quad f(t), g(t) \equiv 0.$$

Then we have the observability inequality

$$\begin{aligned} &\|u_x(0,\cdot)\|^2_{L^2(0,T)} + \|y_x(0,\cdot)\|^2_{L^2(0,T)} \\ &\geq C\|u_0\|^2_V + \|u_1\|^2_H + \|y_0\|^2_V + \|y_1\|^2_H, \end{aligned}$$

where C is a positive constant independent of the initial data.

The following two theorems are for the case of only involving one control. As can be seen, the control space for y (resp. u) is smaller (resp. the same), and the least control time is larger.

We assume that the coupling coefficients α and β are equal, and

$$(n - \frac{1}{2})^2 + \alpha \neq (m - \frac{1}{2})^2 - \alpha, \tag{2.4}$$

for any $m, n \in \mathbb{N}$.

Theorem 2.6. *Let* (2.4) *hold and $f(t) \equiv 0$. Suppose that $T \geq 4\pi$ and $N(t) \in H^3(0,T)$. Then* (1.1)-(1.2) *is exactly controllable on time interval $[0,T]$ and the control space is $H \times V' \times V \times H$. That is, for any given final state $(\xi_0, \xi_1, \eta_0, \eta_1) \in H \times V' \times V \times H$ we can choose suitable $g(t) \in L^2(0,T)$ such that the solution of* (1.1)-(1.2) *with null initial data satisfies $(u(T), u_t(T), y(T), y_t(T)) = (\xi_0, \xi_1, \eta_0, \eta_1)$.*

Theorem 2.7. *Let assumption* (2.4) *hold, $T \geq 4\pi$, and u, y solve* (1.1)-(1.2) *with*

$$(u_0, u_1, y_0, y_1) \in V \times H \times H \times V', \text{ and } f(t), g(t) \equiv 0.$$

Then we have the observability inequality

$$\|u_x(0,t)\|^2 \geq C\|u_0\|^2_V + \|u_1\|^2_H + \|y_0\|^2_H + \|y_1\|^2_{V'},$$

where C is a positive constant independent of the initial data.

3. Reduction to a moment problem

In this section, we transform the control problem to a moment problem. First, we define an operator A on H by

$$A = -\frac{d^2}{dx^2}, \quad \text{with } D(A) := H^2(0,\pi) \cap V.$$

Write

$$\phi_n(x) := \sqrt{\frac{2}{\pi}} \sin(n - \frac{1}{2})x, \quad \lambda_n := (n - \frac{1}{2})^2, \quad \mu_n := n - \frac{1}{2},$$

for $n \in \mathbb{N}$ (the set of positive integers). It is easy to see that λ_n and $\phi_n(x)$ are respectively the eigenvalues and the corresponding normalized eigenfunctions of operator A, and $\{\phi_n\}_{n\in\mathbb{N}}$ forms an orthogonal basis in H. Hence, letting (u, y) be the solution of (1.1) with null initial data, we see that there exist $\{\omega_n\}_{n\in\mathbb{N}}$, $\{\omega_n'\}_{n\in\mathbb{N}}$, $\{\upsilon_n\}_{n\in\mathbb{N}}$ and $\{\upsilon_n'\}_{n\in\mathbb{N}}$ such that

$$\begin{gathered} u(x,t) = \sum_{n=1}^{+\infty} \omega_n(t)\phi_n(x), \quad u_t(x,t) = \sum_{n=1}^{+\infty} \omega_n'(t)\phi_n(x), \\ y(x,t) = \sum_{n=1}^{+\infty} \upsilon_n(t)\phi_n(x), \quad y_t(x,t) = \sum_{n=1}^{+\infty} \upsilon_n'(t)\phi_n(x). \end{gathered} \tag{3.1}$$

Observe that

$$\begin{gathered} (Au, \phi_n)_H = -\mu_n \sqrt{\frac{2}{\pi}} g(t) + \lambda_n \omega_n(t), \quad n \geq 1, \\ (Ay, \phi_n)_H = -\mu_n \sqrt{\frac{2}{\pi}} f(t) + \lambda_n \upsilon_n(t), \quad n \geq 1, \end{gathered}$$

by integration by parts and the boundary condition in (1.2). Then, multiplying the equations in (1.1) by $\phi_n(x)$ and using (3.1), we know that $\omega_n(t)$ and $\upsilon_n(t)$ satisfy

$$\begin{gathered} \omega_n''(t) + \lambda_n \omega_n(t) + \lambda_n \int_0^t N(t-s)\omega_n(s)ds + \alpha \upsilon_n(t) \\ + \beta \int_0^t N(t-s)\upsilon_n(s)ds = \mu_n \widetilde{g}(t), \\ \omega_n(0) = \omega'(0) = 0, \\ \upsilon_n''(t) + \lambda_n \upsilon_n(t) + \lambda_n \int_0^t N(t-s)\upsilon_n(s)ds + \alpha \omega_n(t) \\ + \beta \int_0^t N(t-s)\omega_n(s)ds = \mu_n \widetilde{f}(t), \\ \upsilon_n(0) = \upsilon'(0) = 0, \end{gathered} \tag{3.2}$$

where

$$\begin{gathered} \widetilde{g}(t) := \sqrt{\frac{2}{\pi}} \Big(g(t) + \int_0^t N(t-s) g(s) ds \Big), \\ \widetilde{f}(t) := \sqrt{\frac{2}{\pi}} \Big(f(t) + \int_0^t N(t-s) f(s) ds \Big). \end{gathered} \tag{3.3}$$

Clearly, for any given $T > 0$, the above two maps $g \to \widetilde{g}$ and $f \to \widetilde{f}$ are both bounded and boundedly invertible in $L^2(0, T)$.

Set

$$a_n(t) = \omega_n(t) + \upsilon_n(t), \; b_n(t) = \omega_n(t) - \upsilon_n(t), \tag{3.4}$$

$\widehat{g}(t) = \widetilde{g}(t) + \widetilde{f}(t)$, and $\widehat{f}(t) = \widetilde{g}(t) - \widetilde{f}(t)$. Obviously

$$\begin{gathered} a_n''(t) + \lambda_{1n} a_n(t) + (\lambda_n + \beta) \int_0^t N(t-s) a_n(s) ds = \mu_n \widehat{g}(t), \\ a_n(0) = a_n'(0) = 0, \end{gathered} \tag{3.5}$$

$$\begin{gathered} b_n''(t)+\lambda_{2n}b_n(t)+(\lambda_n-\beta)\int_0^t N(t-s)b_n(s)ds=\mu_n\widehat{f}(t),\\ b_n(0)=b_n'(0)=0, \end{gathered} \tag{3.6}$$

where $\lambda_{1n}:=\lambda_n+\alpha$, $\lambda_{2n}:=\lambda_n-\alpha$ for all $n\geq 1$. Therefore,

$$\begin{gathered} a_n(t)=\int_0^t e_{1n}(t-s)\widehat{g}(s)ds, \quad a_n'(t)=\int_0^t e_{2n}(t-s)\widehat{g}(s)ds,\\ b_n(t)=\int_0^t s_{1n}(t-s)\widehat{f}(s)ds, \quad b_n'(t)=\int_0^t s_{2n}(t-s)\widehat{f}(s)ds, \end{gathered}$$

where $e_{2n}(t):=e_{1n}'(t)$, $s_{2n}(t):=s_{1n}'(t)$, and $e_{1n}(t)$ and $s_{1n}(t)$ are respectively the solutions to the corresponding homogeneous equations of (3.5) and (3.6) with initial data

$$e_{1n}(0)=0,\ e_{1n}'(0)=\mu_n, \quad s_{1n}(0)=0,\ s_{1n}'(0)=\mu_n.$$

Next, we show the connection of the controllability of system (1.1)-(1.2) with some moment problem. Let $T>0$. For any given final states $\xi_0(x),\eta_0(x)\in H$ and $\xi_1(x),\eta_1(x)\in V'$, we have

$$\begin{gathered} u(x,T)=\xi_0(x)=\sum_{n=1}^{+\infty}\xi_{0n}\phi_n(x), \quad u_t(x,T)=\xi_1(x)=\sum_{n=1}^{+\infty}\xi_{1n}\phi_n(x),\\ y(x,T)=\eta_0(x)=\sum_{n=1}^{+\infty}\eta_{0n}\phi_n(x), \quad y_t(x,T)=\eta_1(x)=\sum_{n=1}^{+\infty}\eta_{1n}\phi_n(x), \end{gathered}$$

where

$$\begin{gathered} \xi_{0n}:=(\xi_0,\phi_n)_{L^2(0,\pi)}, \quad \xi_{1n}:=(\xi_1,\phi_n)_{L^2(0,\pi)},\\ \eta_{0n}:=(\eta_0,\phi_n)_{L^2(0,\pi)}, \quad \eta_{1n}:=(\eta_1,\phi_n)_{L^2(0,\pi)}. \end{gathered}$$

It is easy to see that $\{\xi_{0n}\}$, $\{\frac{\xi_{1n}}{\mu_n}\}$, $\{\beta_{0n}\}$, $\{\frac{\beta_{1n}}{\mu_n}\}$ are all in l^2. We write

$$\begin{gathered} e_{\pm n}(t)=\frac{1}{\mu_n}e_{2n}(t)\pm e_{1n}(t)i, \quad z_{\pm 1n}=\frac{1}{\mu_n}\xi_{1n}\pm\xi_{0n}i,\\ s_{\pm n}(t)=\frac{1}{\mu_n}s_{2n}(t)\pm s_{1n}(t)i, \quad z_{\pm 2n}=\frac{1}{\mu_n}\xi_{1n}\pm\eta_{0n}i. \end{gathered}$$

Then, $e_{\pm n}(t)$ and $s_{\pm n}(t)$ solve respectively the equations

$$e_{\pm n}''(t)+\lambda_{1n}e_{\pm n}(t)+(\lambda_n+\beta)\int_0^t N(t-s)e_{\pm n}(s)ds=0, \tag{3.7}$$

$$s_{\pm n}''(t)+\lambda_{2n}s_{\pm n}(t)+(\lambda_n-\beta)\int_0^t N(t-s)s_{\pm n}(s)ds=0, \tag{3.8}$$

with the initial conditions

$$e_{\pm n}(0)=s_{\pm 0}(0)=1, \quad e_{\pm n}'(0)=s_{\pm 0}'(0)=\pm\mu_n i. \tag{3.9}$$

This leads to the moment problem

$$z_{\pm 1n}+z_{\pm 2n}=\int_0^T e_{\pm n}(T-s)\widehat{g}(s)ds, \tag{3.10}$$

$$z_{\pm 1n}-z_{\pm 2n}=\int_0^T s_{\pm n}(T-s)\widehat{f}(s)ds. \tag{3.11}$$

So, proving that system (1.1)-(1.2) is controllable and the control space is $H \times V' \times H \times V'$ is equivalent to finding g, f such that (3.10) and (3.11) are satisfied, i.e., the above moment problem being solvable.

To deal with the moment problem (3.10)-(3.11), it suffices to show the Riesz property of families $\{e_{\pm n}(t)\}$ and $\{s_{\pm n}(t)\}$. There are several equivalent ways to define Riesz sequences, we use the following.

Definition 3.1. Let $\{\beta_n\}$ be a sequence in a Hilbert space H. We say $\{\beta_n\}$ is a Riesz sequence, if there exist $m, M > 0$ such that

$$m\|\alpha_n\|_{l^2} \le \|\sum \alpha_n\beta_n\|_H \le M\|\alpha_n\|_{l^2}$$

whenever $\{\alpha_n\} \in l^2$. If $\{\beta_n\}$ is complete in H, we call it a *Riesz basis*.

Note that $\{e_{\pm n}(t)\}$ and $\{s_{\pm n}(t)\}$ solve the equations (3.7), (3.8) with initial conditions in (3.9). The Riesz property of the families which solve similar equations has been studied in [12, 2], and the following result can be obtained in the same way.

Proposition 3.2. *For any given $T \ge 2\pi$, $\{e_{\pm n}(t)\}$ and $\{s_{\pm n}(t)\}$ are both Riesz sequences in $L^2(0,T)$.*

Clearly, this proposition justifies Theorem 2.4. To derive Theorem 2.5, we shall use the property of Riesz sequences. When (u, y) is the solution of (1.1)-(1.2), we know that $\omega_n(t)$ and $\upsilon_n(t)$ solve (3.2) with the null initial data replaced by

$$\omega_n(0) = \alpha_{0n}, \quad \omega_n'(0) = \alpha_{1n}, \quad \upsilon_n(0) = \beta_{0n}, \quad \upsilon_n'(0) = \beta_{1n},$$

where

$$\begin{aligned}\alpha_{0n} &:= (u_0, \phi_n)_{L^2(0,\pi)}, \quad \alpha_{1n} := (u_1, \phi_n)_{L^2(0,\pi)},\\ \beta_{0n} &:= (y_0, \phi_n)_{L^2(0,\pi)}, \quad \beta_{1n} := (y_1, \phi_n)_{L^2(0,\pi)},\end{aligned}$$

and that

$$\begin{aligned}&\|u_0\|_V^2 + \|u_1\|_H^2 + \|y_0\|_V^2 + \|y_1\|_H^2\\ &\asymp \|\mu_n\alpha_{0n}\|_{l^2}^2 + \|\alpha_{1n}\|_{l^2}^2 + \|\mu_n\beta_{0n}\|_{l^2}^2 + \|\beta_{1n}\|_{l^2}^2.\end{aligned}$$

Noticing the definitions of $e_{\pm n}(t)$ and $s_{\pm n}(t)$, we deduce that

$$\begin{aligned}&\omega_n(t)\\ &= \frac{1}{4}\Big\{\Big[(\alpha_{0n} + \beta_{0n}) - \frac{(\alpha_{1n} + \beta_{1n})}{n}i\Big]e_n(t) + \Big[(\alpha_{0n} + \beta_{0n}) + \frac{(\alpha_{1n} + \beta_{1n})}{n}i\Big]e_{-n}(t)\\ &\quad + \Big[(\alpha_{0n} - \beta_{0n}) - \frac{(\alpha_{1n} - \beta_{1n})}{n}i\Big]s_n(t) + \Big[(\alpha_{0n} - \beta_{0n}) + \frac{(\alpha_{1n} - \beta_{1n})}{n}i\Big]s_{-n}(t)\Big\},\end{aligned}$$

and

$$\begin{aligned}&\upsilon_n(t)\\ &= \frac{1}{4}\Big\{\Big[(\alpha_{0n} + \beta_{0n}) - \frac{(\alpha_{1n} + \beta_{1n})}{n}i\Big]e_n(t) + \Big[(\alpha_{0n} + \beta_{0n}) + \frac{(\alpha_{1n} + \beta_{1n})}{n}i\Big]e_{-n}(t)\\ &\quad - \Big[(\alpha_{0n} - \beta_{0n}) - \frac{(\alpha_{1n} - \beta_{1n})}{n}i\Big]s_n(t) - \Big[(\alpha_{0n} - \beta_{0n}) + \frac{(\alpha_{1n} - \beta_{1n})}{n}i\Big]s_{-n}(t)\Big\}.\end{aligned}$$

Accordingly, when $T \ge 2\pi$, using the Riesz property of $e_{\pm n}(t)$ and $s_{\pm n}(t)$ we obtain

$$m(\|u_0\|_V^2 + \|u_1\|_H^2 + \|y_0\|_V^2 + \|y_1\|_H^2)$$

$$\begin{aligned}&\le \int_0^T |u_x(0,t)|^2 + |y_x(0,t)|^2 dt\\ &\le M(\|u_0\|_V^2 + \|u_1\|_H^2 + \|y_0\|_V^2 + \|y_1\|_H^2).\end{aligned}$$

It is clear that the second inequality is still valid when $T < 2\pi$; thus we have shown Theorem 2.5 and the first part of Lemma 2.1.

4. Proofs of Theorems 2.6 and 2.7

First, we state the moment problem related to the control problem when $f(t) \equiv 0$. It is easy to see that in this case, under Assumption (2.4), $e_{\pm n}(t)$ and $s_{\pm n}(t)$ satisfy

$$e''_{\pm n}(t) + \lambda_{1n} e_{\pm n}(t) + \lambda_{1n} \int_0^t N(t-s) e_{\pm n}(s) ds = 0, \tag{4.1}$$

$$s''_{\pm n}(t) + \lambda_{2n} s_{\pm n}(t) + \lambda_{2n} \int_0^t N(t-s) s_{\pm n}(s) ds = 0, \tag{4.2}$$

with the initial conditions

$$e_{\pm n}(0) = s_{\pm 0}(0) = 1, \quad e'_{\pm n}(0) = s'_{\pm 0}(0) = \pm \mu_n i. \tag{4.3}$$

By hypothesis, $\xi_0(x) \in H$, $\xi_1(x) \in V'$, and $\eta_0(x) \in V$, $\eta_1(x) \in H$; so $\{z_{\pm 1n}, \mu_n z_{\pm 2n}\} \in l^2$. Thus the moment problem is as follows

$$z_{\pm 1n} = \frac{1}{2} \int_0^T (e_{\pm n}(T-s) + s_{\pm n}(T-s)) \widetilde{g}(s) ds, \tag{4.4}$$

$$\mu_n z_{\pm 2n} = \frac{1}{2} \int_0^T \mu_n (e_{\pm n}(T-s) - s_{\pm n}(T-s)) \widetilde{g}(s) ds. \tag{4.5}$$

Since we have only one control $g(t)$, we have to prove the Riesz property of the function family

$$\Big\{\frac{1}{2}(e_{\pm n}(t) + s_{\pm n}(t)), \frac{\mu_n}{2}(e_{\pm n}(t) - s_{\pm n}(t))\Big\}.$$

Theorem 4.1. *Let assumption 2.4 be satisfied, $\{e_{\pm n}(t)\}$ and $\{s_{\pm n}(t)\}$ solve the equations* (4.1) *and* (4.2) *respectively with the initial condition* (4.3)*. Then*

$$\{\frac{1}{2}(e_{\pm n}(t) + s_{\pm n}(t)), \frac{\mu_n}{2}(e_{\pm n}(t) - s_{\pm n}(t))\}$$

is a Riesz sequence in $L^2(0,T)$ when $T \ge 4\pi$.

Remark 4.2. As in Section 3, we can prove Theorem 2.7 and the second inequality in Lemma 2.1 by using Theorem 4.1.

Before proving Theorem 4.1, we state two definitions.

Definition 4.3. We say two sequences $\{e_n\}$, $\{z_n\}$ in a Hilbert space are quadratically close when

$$\sum \|e_n - z_n\|^2 < +\infty.$$

Definition 4.4. A sequence $\{e_n\}$ in a Hilbert space is said to be ω-independent if

$$\{e_n\} \in l^2 \quad \text{and} \quad \sum \alpha_n e_n = 0$$

implies $\alpha_n = 0$ for all n.

Our proof is based on Bari's theorem [4, 17], which is as follows (with a slight changes).

Theorem 4.5. *Let $\{u_n\}_{n\geq 1}$ be an ω-independent sequence in H, and let $\{v_n\}_{n\geq n_0}$ (for some $n_0 \in \mathbb{N}$) is a Riesz sequence in H. If $\{u_n\}_{n\geq n_0}$ is quadratically close to $\{v_n\}_{n\geq n_0}$, then $\{u_n\}_{n\geq 1}$ is also a Riesz sequence in H.*

By Assumption 2.4, it is clear that $\lambda_{1n} \neq \lambda_{2m}$ and there exists $n_0 \in \mathbb{N}$ such that $\lambda_{1n}, \lambda_{2n} > 0$ when $n \geq n_0$.

Now we begin our proof of Theorem 4.1. First we prove the quadratic closeness for $n \geq n_0$. Set

$$v = \frac{N(0)}{2}, \quad w_{1n} = \sqrt{\lambda_{1n}}, \quad \delta_{\pm 1n}(t) := e_{\pm n}(t) - e^{\pm i w_{1n} t + vt}, \quad n \geq n_0.$$

By equation (4.1) with initial condition (4.3), when $n \geq n_0$ we obtain

$$e_n(t) = e^{iw_{1n}t} + c_{1n} e^{iw_{1n}t} - c_{1n} e^{-iw_{1n}t} - w_{1n} \int_0^t \sin w_{1n}(t-s) \int_0^s N(s-r) e_n(r)\, dr\, ds,$$

where

$$c_{1n} := \frac{\mu_n - w_{1n}}{2w_{1n}}.$$

Thus, integration by parts gives

$$e_n(t) = e^{iw_{1n}t} + p_{1n}(t) + \int_0^t N_{1n}^*(t-s) e_n(s) ds,$$

where

$$p_{1n}(t) := c_{1n} e^{iw_{1n}t} - c_{1n} e^{-iw_{1n}t},$$

$$N_{1n}^*(t) := N(0) \cos w_{1n} t - N(t) + \int_0^t \cos w_{1n}(t-s) N'(s) ds.$$

Using Gronwall's inequality we have

$$|e_n(t)| \leq C_1, \quad \forall t \in [0, T]. \tag{4.6}$$

Also, we see that

$$\delta_{1n}(t) = p_{1n}(t) + q_{1n}(t) + r_{1n}(t) + \int_0^t N_{1n}^*(t-s) \delta_{1n}(s) ds,$$

where

$$\begin{aligned} q_{1n}(t) := &\frac{v(e^{(iw_{1n}+v)t} - e^{-iw_{1n}t})}{2iw_{1n} + v} - \frac{N(0)}{iw_{1n} + v} e^{(iw_{1n}+v)t} \\ &+ \frac{N(t)}{iw_{1n} + v} - \frac{1}{iw_{1n} + v} \int_0^t N'(t-s) e^{(iw_{1n}+v)s} ds, \\ r_{1n}(t) := &\int_0^t \int_0^{t-s} \cos w_{1n}(t-s-r) N'(r) dr e^{(iw_{1n}+v)s} ds. \end{aligned}$$

It is clear that

$$|p_{1n}(t) + q_{1n}(t) + r_{1n}(t)| \leq \frac{C_2}{\mu_n}, \quad \forall t \in [0, T].$$

Therefore,

$$|\delta_{1n}(t)| \leq \frac{C_3}{\mu_n}, \quad \forall t \in [0, T],\ n \geq n_0. \tag{4.7}$$

The same can be done for $\delta_{-1n}(t)$.

For $s_{\pm n}(t)$, we set

$$w_{2n} = \sqrt{\lambda_{2n}},\ \delta_{\pm 2n} = s_{\pm n}(t) - e^{\pm iw_{2n}t+vt}, \quad n \geq n_0.$$

Obviously,

$$\begin{aligned} s_n(t) = e^{iw_{2n}t} + c_{2n}e^{iw_{2n}t} - c_{2n}e^{-iw_{2n}t} \\ - w_{2n}\int_0^t \sin w_{2n}(t-s)\int_0^s N(s-r)s_n(r)\,dr\,ds, \end{aligned}$$

and

$$\delta_{2n}(t) = p_{2n}(t) + q_{2n}(t) + r_{2n}(t) + \int_0^t N^*_{2n}(t-s)\delta_{2n}(s)ds,$$

where

$$\begin{aligned} c_{2n} &:= \frac{\mu_n - w_{2n}}{2w_{2n}}, \quad p_{2n}(t) := c_{2n}e^{iw_{2n}t} - c_{2n}e^{-iw_{2n}t}, \\ q_{2n}(t) &:= \frac{v(e^{(iw_{2n}+v)t} - e^{-iw_{2n}t})}{2iw_{2n}+v} - \frac{N(0)}{iw_{2n}+v}e^{(iw_{2n}+v)t} \\ &\quad + \frac{N(t)}{iw_{2n}+v} - \frac{1}{iw_{2n}+v}\int_0^t N'(t-s)e^{(iw_{2n}+v)s}ds, \\ r_{2n}(t) &:= \int_0^t\int_0^{t-s}\cos w_{2n}(t-s-r)N'(r)dre^{(iw_{2n}+v)s}ds, \\ N^*_{2n}(t) &:= N(0)\cos w_{2n}t - N(t) + \int_0^t \cos w_{2n}(t-s)M'(s)ds. \end{aligned}$$

Then we obtain

$$|s_{\pm n}(t)| \leq C_4, \quad |\delta_{\pm 2n}(t)| \leq \frac{C_5}{\mu_n},\ \forall t\in[0,T], \quad n\geq n_0. \tag{4.8}$$

Next, we focus on the sequence

$$\Big\{\frac{1}{2}(e_{\pm n}(t) + s_{\pm n}(t)), \frac{\mu_n}{2}(e_{\pm n}(t) - s_{\pm n}(t))\Big\}.$$

For convenience, we set

$$\begin{aligned} z_{\pm 1n}(t) &= \frac{1}{2}(e_{\pm n}(t) + s_{\pm n}(t)), \\ z_{\pm 2n}(t) &= \frac{\mu_n}{2}(e_{\pm n}(t) - s_{\pm n}(t)), \\ \epsilon_{\pm 1n}(t) &= z_{\pm 1n}(t) - e^{\pm(iw_{1n}+v)t}, \\ \epsilon_{\pm 2n}(t) &= z_{\pm 2n}(t) - \frac{\mu_n}{2}\Big(e^{\pm(iw_{1n}+v)t} - e^{(\pm iw_{2n}+v)t}\Big), \quad n\geq n_0. \end{aligned} \tag{4.9}$$

It is easy to see that when $n \geq n_0$,

$$\begin{aligned} \epsilon_{1n}(t) &= \frac{1}{2}\Big(\delta_{1n}(t) + \delta_{2n}(t) + e^{(iw_{2n}+v)t} - e^{(iw_{1n}+v)t}\Big), \\ \epsilon_{2n}(t) &= \frac{\mu_n}{2}(\delta_{1n}(t) - \delta_{2n}(t)). \end{aligned}$$

Since $w_{1n} - w_{2n} = O(\frac{1}{\mu_n})$, we have

$$|\epsilon_{1n}(t)| \leq \frac{C_6}{\mu_n}, \quad \forall t\in[0,T],\ n\geq n_0. \tag{4.10}$$

To estimate $\epsilon_{2n}(t)$, we observe that

$$\begin{aligned}\epsilon_{2n}(t) =& \frac{\mu_n}{2}(p_{1n}(t) - p_{2n}(t) + q_{1n}(t) - q_{2n}(t) + r_{1n}(t) - r_{2n}(t)) \\ &+ \int_0^t N^*_{1n}(t-s)\epsilon_{2n}(t) + \frac{\mu_n}{2}\int_0^t (N^*_{1n}(t-s) - N^*_{2n}(t-s))\delta_{2n}(s)ds.\end{aligned}$$

Using (4.8) we have

$$|\epsilon_{2n}(t)| \le \frac{C_7}{\mu_n}, \quad \forall t \in [0,T],\ n \ge n_0. \tag{4.11}$$

The same can be done for $\epsilon_{-1n}(t)$ and $\epsilon_{-2n}(t)$.

Let $T \ge 4\pi$. From [3, page 706], one knows that under assumption (2.4),

$$\Big\{\sin w_{1n}t,\ \frac{\sin w_{1n}t - \sin w_{2n}t}{w_{1n} - w_{2n}},\ \cos w_{1n}t,\ \frac{\cos w_{1n}t - \cos w_{2n}t}{w_{1n} - w_{2n}}\Big\}_{n\ge n_0}$$

forms a Riesz sequence in $L^2(0,T)$. Accordingly, noting $w_{1n} - w_{2n} = O(\frac{1}{\mu_n})$, we see that

$$\Big\{e^{\pm iw_{1n}t},\ \frac{\mu_n}{2}(e^{\pm iw_{1n}t} - e^{\pm iw_{2n}t})\Big\}_{n\ge n_0}$$

is also a Riesz sequence in $L^2(0,T)$; hence so is

$$\Big\{e^{(\pm iw_{1n}+v)t},\ \frac{\mu_n}{2}(e^{(\pm iw_{1n}+v)t} - e^{(\pm iw_{2n}+v)t})\Big\}_{n\ge n_0}.$$

This family and $\{z_{\pm 1n}, z_{\pm 2n}\}_{n\ge n_0}$ are quadratically close by (4.10) and (4.11). The following lemma (due to Paley and Wiener) will be helpful.

Lemma 4.6. *If $\{e_n\}$ is quadratically close to a Riesz sequence $\{z_n\}$, then there exists $N \ge 1$ such that $\{e_n : n \ge N\}$ is also a Riesz sequence.*

According to this lemma, there exists $N_0 \ge n_0$ such that $\{z_{\pm 1n}, z_{\pm 2n}\}_{n\ge N_0}$ is a Riesz sequence in $L^2(0,T)$ when $T \ge 4\pi$.

Next we derive asymptotic representations for $\epsilon'_{\pm 1n}(t)$ and $\epsilon'_{\pm 2n}(t)$. First we calculate the derivative of $\delta_{1n}(t)$:

$$\begin{aligned}p'_{1n}(t) &= iw_{1n}c_{1n}e^{iw_{1n}t} + iw_{1n}c_{1n}e^{-iw_{1n}t},\\ q'_{1n}(t) &= -\frac{3N(0)}{4}e^{(iw_{1n}+v)t} + \frac{N(0)}{4}e^{-iw_{1n}t} + \frac{v^2}{4iw_{1n} + N(0)}e^{(iw_{1n}+v)t}\\ &\quad - \frac{v^2}{4iw_{1n} + N(0)}e^{-iw_{1n}t} - \int_0^t N'(t-s)e^{(iw_{1n}+v)s}ds,\\ r'_{1n}(t) &= \frac{N'(0)}{N(0)}\Big(e^{(iw_{1n}+v)t} - e^{iw_{1n}t}\Big) + \frac{N'(0)}{4iw_{1n} + N(0)}\Big(e^{(iw_{1n}+v)t} - e^{-iw_{1n}t}\Big)\\ &\quad + \int_0^t \cos w_{1n}(t-s)\int_0^s N''(s-r)e^{(iw_{1n}+v)r}\,dr\,ds.\end{aligned}$$

Then, we find

$$\begin{aligned}\delta'_{1n}(t) &= D_1e^{iw_{1n}t} + D_2e^{-iw_{1n}t} + D_3e^{(iw_n+v)t}\\ &+ \int_0^t N^*_{1n}(t-s)\delta'_{1n}(s)ds + \chi_{1n}(t),\end{aligned} \tag{4.12}$$

where D_1, D_2, D_3 are constants, and $\chi_{1n}(t)$ is such that

$$|\chi_{1n}(t)| \le \frac{C_8}{\mu_n}, \quad \forall t \in [0,T],\ n \ge n_0.$$

Next we divide $\delta'_{1n}(t)$ into three parts:

$$\delta'_{1n}(t) = D_1\sigma_{1n}(t) + D_2\eta_{1n}(t) + D_3\varsigma_{1n}(t) + \zeta_{1n}(t),$$

where

$$\zeta_{1n}(t) := \chi_{1n}(t) + \int_0^t N^*_{1n}(t-s)\zeta_{1n}(s)ds, \tag{4.13}$$

$$\sigma_{1n}(t) := e^{iw_{1n}t} + \int_0^t N^*_{1n}(t-s)\sigma_{1n}(s)ds, \tag{4.14}$$

$$\eta_{1n}(t) := e^{-iw_{1n}t} + \int_0^t N^*_{1n}(t-s)\eta_{1n}(s)ds, \tag{4.15}$$

$$\varsigma_{1n}(t) := e^{(iw_{1n}+v)t} + \int_0^t N^*_{1n}(t-s)\varsigma_{1n}(s)ds. \tag{4.16}$$

It is clear that

$$|\zeta_{1n}(t)| \le \frac{C_9}{\mu_n}, \quad \forall t \in [0,T],\ n \ge n_0. \tag{4.17}$$

Since $\sigma_{1n}(t)$ and $\eta_{1n}(t)$ have the same form as $\delta_{1n}(t)$, we have

$$D_1\sigma_{1n}(t) + D_2\eta_{1n}(t) = D_1e^{(iw_{1n}+v)t} + D_2e^{(-iw_{1n}+v)t} + \kappa_{1n}(t),$$

where $\kappa_{1n}(t)$ satisfies

$$|\kappa_{1n}(t)| \le \frac{C_{10}}{\mu_n}, \quad \forall t \in [0,T],\ n \ge n_0,$$

and $\varsigma_{1n}(t)$ has the estimate

$$|\varsigma_{1n}(t)| \le C_{11}, \quad \forall t \in [0,T],\ n \ge n_0. \tag{4.18}$$

Computing $\varsigma_{1n}(t)$ term by term we have

$$\varsigma_{1n}(t) = \varsigma^1_{1n}(t) + \varsigma^2_{1n}(t) + \varsigma^3_{1n}(t),$$

where

$$\varsigma^1_{1n}(t) := -\int_0^t N(t-s)\varsigma_{1n}(s)ds,$$

$$\varsigma^2_{1n}(t) := \int_0^t \int_0^{t-s} \cos w_{1n}(t-s-r)N'(r)dr\varsigma_{1n}(s)ds,$$

$$\varsigma^3_{1n}(t) := e^{(iw_{1n}+v)t} + N(0)\int_0^t \cos w_{1n}(t-s)\varsigma_{1n}(s)ds.$$

By (4.18),

$$|\varsigma^1_{1n}(t)| \le C_{12}, \quad |\varsigma^2_{1n}(t)| \le \frac{C_{12}}{\mu_n}, \quad \forall t \in [0,T],\ n \ge n_0. \tag{4.19}$$

Noting the above estimates we deduce that

$$\varsigma^3_{1n}(t) = e^{(iw_{1n}+v)t} + N(0)\int_0^t \cos w_{1n}(t-s)\varsigma^3_{1n}(s)ds + \varrho_{1n}(t),$$

where

$$\varrho_{1n}(t) := N(0)\int_0^t \cos w_{1n}(t-s)(\varsigma^1_{1n}(s)+\varsigma^2_{1n}(s))ds.$$

Hence, combining (4.18) and (4.19) yields

$$|\varrho_{1n}| \le \frac{C_{13}}{\mu_n}, \quad \forall t\in[0,T],\ n\ge n_0.$$

Now, separate $\varsigma^3_{1n}(t)$ into two parts. One of them has the order $O(\frac{1}{\mu_n})$, and the other satisfies

$$\theta_{1n}(t) = e^{(iw_{1n}+v)t} + N(0)\int_0^t \cos w_{1n}(t-s)\theta_n(s)ds.$$

To find out the exact form of $\theta_{1n}(t)$, we introduce the integral operator

$$P_{1n}(p) := N(0)\int_0^t \cos w_{1n}(t-s)p(s)ds.$$

Clearly,

$$(I-P_{1n})\theta_{1n} = e^{(iw_{1n}+v)t};$$

then we can construct $\theta_{1n}(t)$ by the convergent iteration series

$$\theta_{1n}(t) = \sum_{k=0}^{+\infty} P^k_{1n} e^{(iw_{1n}+v)t}, \tag{4.20}$$

with the help of consecutive applications of the operator P_n. Then, we see that

$$\begin{aligned}\theta_n(t) &= O(\frac{1}{\mu_n}) + e^{iw_{1n}t}\sum_{i=0}^{+\infty}\sum_{j=i}^{+\infty}\frac{(vt)^j}{j!}\\ &= O(\frac{1}{\mu_n}) + e^{iw_{1n}t}(\sum_{i=0}^{+\infty}\frac{(vt)^i}{i!}(i+1))\\ &= O(\frac{1}{\mu_n}) + e^{iw_{1n}t}(e^{vt}+vte^{vt}).\end{aligned}$$

For $\varsigma^1_{1n}(t)$ we have

$$\varsigma^1_{1n}(t) = -\int_0^t N(t-s)\varsigma^1_{1n}(s)ds - \int_0^t N(t-s)\left(e^{(iw_{1n}+v)s}+vse^{(iw_{1n}+v)s}\right)ds + O(\frac{1}{\mu_n});$$

thus

$$|\varsigma^1_{1n}(t)| \le \frac{C_{14}}{\mu_n}, \quad \forall t\in[0,T],\ n\ge n_0.$$

By now we have made every term of $\delta'_{1n}(t)$ clear, and $\delta'_{1n}(t)$ can be written in the form

$$\delta'_{1n}(t) = Q_1e^{(-iw_{1n}+v)t} + Q_2e^{(iw_{1n}+v)t} + Q_3e^{(iw_{1n}+v)t}t + O(\frac{1}{\mu_n}), \quad n\ge n_0 \tag{4.21}$$

(Q_1, Q_2, Q_3 being constants); the same can be done for $\delta'_{2n}(t)$, and we can get

$$\delta'_{2n}(t) = Q_1e^{(-iw_{2n}+v)t} + Q_2e^{(iw_{2n}+v)t} + Q_3e^{(iw_{2n}+v)t}t + O(\frac{1}{\mu_n}), \quad n\ge n_0. \tag{4.22}$$

Thus,

$$\epsilon'_{1n}(t) = Q_1e^{(-iw_{1n}+v)t} + Q_2e^{(iw_{1n}+v)t} + Q_3e^{(iw_{1n}+v)t}t$$

$$+\frac{\mu_n}{2}\left(e^{(iw_{2n}+v)t}-e^{(iw_{1n}+v)t}\right)+O(\frac{1}{\mu_n}),\quad n\geq n_0.$$

To give the asymptotic representations for $\epsilon'_{2n}(t)$, we estimate

$$\begin{aligned}\delta'_{1n}(t)-\delta'_{2n}(t)&=D_1(\sigma_{1n}(t)-\sigma_{2n}(t))+D_2(\eta_{1n}(t)-\eta_{2n}(t))\\&\quad+D_3(\varsigma_{1n}(t)-\varsigma_{2n}(t))+\zeta_{1n}(t)-\zeta_{2n}(t).\end{aligned}$$

Noticing (4.14), (4.15), we infer that

$$\sigma_{1n}(t)-\sigma_{2n}(t)=e^{(iw_{1n}+v)t}-e^{(iw_{2n}+v)t}+O(\frac{1}{\mu_n^2}),$$

$$\eta_{1n}(t)-\eta_{2n}(t)=e^{(-iw_{1n}+v)t}-e^{(-iw_{2n}+v)t}+O(\frac{1}{\mu_n^2}).$$

Computing $\chi_{1n}(t)$ in (4.12) and combining (4.13), we obtain

$$\zeta_{1n}(t)-\zeta_{2n}(t)=O(\frac{1}{\mu_n^2}).$$

We estimate $\varsigma_{1n}(t)-\varsigma_{2n}(t)$ step by step similarly to the estimate for $\varsigma_{1n}(t)$. First, since

$$\varsigma_{1n}(t)=e^{(iw_{1n}+v)t}+vte^{(iw_{1n}+v)t}+O(\frac{1}{\mu_n}),$$

$$\varsigma_{2n}(t)=e^{(iw_{2n}+v)t}+vte^{(iw_{2n}+v)t}+O(\frac{1}{\mu_n}),$$

it follows that

$$|\varsigma_{1n}(t)-\varsigma_{2n}(t)|=O(\frac{1}{\mu_n}),\tag{4.23}$$

$$|\varsigma^1_{1n}(t)-\varsigma^1_{2n}(t)|=O(\frac{1}{\mu_n}).\tag{4.24}$$

Then we have

$$\begin{aligned}\varsigma^2_{1n}(t)-\varsigma^2_{2n}(t)&=\int_0^t\int_0^{t-s}(\cos w_{1n}(t-s-r)-\cos w_{2n}(t-s-r))N'(r)dr\varsigma_{1n}(s)ds\\&\quad+\int_0^t\int_0^{t-s}\cos w_{2n}(t-s-r)N'(r)dr(\varsigma_{1n}(s)-\varsigma_{1n}(s))ds;\end{aligned}$$

it is clear that the second term is $O(\frac{1}{\mu_n^2})$. Also,

$$\begin{aligned}&\int_0^t\int_0^{t-s}(\cos w_{1n}(t-s-r)-\cos w_{2n}(t-s-r))N'(r)dr\varsigma_{1n}(s)ds\\&=\int_0^t\Big(\frac{1}{w_{1n}}\sin w_{1n}(t-s)-\frac{1}{w_{2n}}\sin w_{2n}(t-s)\Big)\Big\{N'(0)\varsigma_{1n}(s)\\&\quad+\int_0^t N''(s-r)\varsigma_{1n}(r)dr\Big\}ds.\end{aligned}$$

Therefore,

$$|\varsigma^2_{1n}(t)-\varsigma^2_{2n}(t)|=O(\frac{1}{\mu_n^2}).\tag{4.25}$$

Thus, using (4.23)-(4.25) we obtain

$$\varsigma^3_{1n}(t)-\varsigma^3_{2n}(t)=\theta_{1n}(t)-\theta_{2n}(t)+O(\frac{1}{\mu_n^2}).$$

Computing the convergent iteration series (4.20) and omitting the terms of the order $O(\frac{1}{\mu_n^2})$; we have

$$\begin{aligned} \theta_{1n}(t)-\theta_{2n}(t) &= e^{(iw_{1n}+v)t}-e^{(iw_{2n}+v)t}+vt(e^{(iw_{1n}+v)t} \\ &\quad -e^{(iw_{2n}+v)t})+O(\frac{1}{\mu_n^2}). \end{aligned} \tag{4.26}$$

Using (4.24)-(4.26) we have

$$|\varsigma_{1n}^1(t)-\varsigma_{2n}^1(t)| = O(\frac{1}{\mu_n^2}). \tag{4.27}$$

The above estimates imply that

$$\begin{aligned} \varsigma_{1n}(t)-\varsigma_{2n}(t) &= e^{(iw_{1n}+v)t}-e^{(iw_{2n}+v)t} \\ &\quad + vt\Big(e^{(iw_{1n}+v)t}-e^{(iw_{2n}+v)t}\Big)+O(\frac{1}{\mu_n^2}). \end{aligned} \tag{4.28}$$

Therefore,

$$\begin{aligned} &\epsilon_{2n}'(t) \\ &= S_1\mu_n\Big(e^{(-iw_{1n}+v)t}-e^{(-iw_{2n}+v)t)}\Big)+S_2\mu_n\Big(e^{(iw_{1n}+v)t}-e^{(iw_{2n}+v)t)}\Big) \\ &\quad + vtS_3\mu_n\Big(e^{(iw_{1n}+v)t}-e^{(iw_{2n}+v)t}\Big)+O(\frac{1}{\mu_n}), \end{aligned} \tag{4.29}$$

where S_1, S_2, S_3 are constants. Since the memory kernel $N(t) \in H^3(0,T)$, we can also estimate $\epsilon_{1n}^{(2)}(t), \epsilon_{2n}^{(2)}(t)$ and $\epsilon_{1n}^{(3)}(t), \epsilon_{2n}^{(3)}(t)$ $(:= \epsilon_{2n}'''(t))$. The asymptotic representations are given by

$$\begin{aligned} \epsilon_{1n}^{(k)}(t) &= \mu_n^{k-1}\Big(Q_{1k}e^{(-iw_{1n}+v)t}+Q_{2k}e^{(iw_{1n}+v)t}+Q_{3k}e^{(iw_{1n}+v)t}t\Big) \\ &\quad + Q_{4k}\frac{\mu_n}{2}\Big(e^{(iw_{2n}+v)t}-e^{(iw_{1n}+v)t}\Big)+O(\frac{1}{\mu_n}), \quad n \ge n_0, \\ \epsilon_{2n}^{(k)}(t) &= \mu_n^{k-1}\Big(S_{1k}\mu_n(e^{(-iw_{1n}+v)t}-e^{(-iw_{2n}+v)t)}\Big) \\ &\quad + S_{2k}\mu_n\Big(e^{(iw_{1n}+v)t}-e^{(iw_{2n}+v)t)}\Big) \\ &\quad + vtS_{3k}\mu_n\Big(e^{(iw_{1n}+v)t}-e^{(iw_{2n}+v)t}\Big)+O(\frac{1}{\mu_n}), \quad n \ge n_0; \end{aligned} \tag{4.30}$$

where, $k = 2, 3$, and Q_{ik}, S_{jk} $(i = 1, \dots, 4;\ j = 1, \dots 3)$ are constants.

Also, the same results can be proved for $\epsilon_{-1n}(t)$ and $\epsilon_{-2n}(t)$. Next we prove the ω-independence of $\{z_{\pm 1n}(t), z_{\pm 2n}(t)\}$. Let $\{\alpha_{\pm n}, \beta_{\pm n}\} \in l^2$ satisfy

$$\sum \alpha_{\pm n} z_{\pm 1n}(t) + \beta_{\pm n} z_{\pm 2n}(t) = 0.$$

Then

$$\sum \gamma_{\pm n} e_{\pm n}(t) + \eta_{\pm n} s_{\pm n}(t) = 0,$$

where

$$\gamma_{\pm n} := \frac{1}{2}\alpha_{\pm n} + \frac{\mu_n}{2}\beta_{\pm n}, \quad \eta_{\pm n} := \frac{1}{2}\alpha_{\pm n} - \frac{\mu_n}{2}\beta_{\pm n},$$

and

$$\alpha_{\pm n} = \beta_{\pm n} = 0 \quad \Longleftrightarrow \quad \gamma_{\pm n} = \eta_{\pm n} = 0.$$

We set

$$\widetilde{z}_{\pm 1n}(t) = e^{(iw_{1n}+v)t}, \quad \widetilde{z}_{\pm 2n}(t) = \mu_n(e^{(\pm iw_{1n}+v)t} - e^{(\pm iw_{2n}+v)t}), \quad n \geq n_0,$$

and

$$\begin{aligned} G(t) &= \sum_{n\geq n_0} \alpha_{\pm n}\widetilde{z}_{\pm 1n}(t) + \beta_{\pm n}\widetilde{z}_{\pm 2n}(t) \\ &= -\sum_{n\geq n_0} \alpha_{\pm n}\epsilon_{\pm 1n}(t) + \beta_{\pm n}\epsilon_{\pm 2n}(t) - \sum_{n<n_0} \alpha_{\pm n}z_{\pm 1n}(t) + \beta_{\pm n}z_{\pm 2n}(t). \end{aligned}$$

Using the representations for $\{\epsilon_{\pm 1n}(t), \epsilon_{\pm 2n}(t)\}_{n\geq n_0}$ we have

$$G(t) \in L^2(0,T) \quad \text{and} \quad \{\alpha_{\pm n}, \beta_{\pm n}\} \in l^2.$$

Recall that when $T \geq 4\pi$, $\{\widetilde{z}_{\pm 1n}(t), \widetilde{z}_{\pm 2n}(t)\}_{n\geq n_0}$ forms a Riesz sequence in $L^2(0,T)$, and use the representation for $\{\epsilon'_{\pm 1n}(t), \epsilon'_{\pm 2n}(t)\}_{n\geq n_0}$; then we obtain

$$\begin{aligned} G'(t) = \sum_{n\geq n_0} &((\pm iw_{1n}+v)\alpha_{\pm n} \pm i(w_{1n}-w_{2n})\mu_n\beta_{\pm n})\widetilde{z}_{\pm 1n}(t) \\ &+ (\pm iw_{2n}+v)\beta_{\pm n}\widetilde{z}_{\pm 2n}(t) \quad \in L^2(0,T), \end{aligned}$$

which implies $\{\mu_n\alpha_{\pm n}, \mu_n\beta_{\pm n}\} \in l^2$.

By the derived representations of $\{\epsilon^{(k)}_{\pm 1n}(t), \epsilon^{(k)}_{\pm 2n}(t)\}$ for $k = 2,3$, we deduce that

$$\{\mu_n^3\alpha_{\pm n}, \mu_n^3\beta_{\pm n}\} \in l^2 \quad \text{and} \quad \{\mu_n^2\gamma_{\pm n}, \mu_n^2\eta_{\pm n}\} \in l^2.$$

Now, using the equations (4.1) and (4.2) for $\{e_{\pm n}(t), s_{\pm n}(t)\}$ yields

$$\begin{aligned} &\sum \lambda_{1n}\gamma_{\pm n}e_{\pm n}(t) + \lambda_{2n}\eta_{\pm n}s_{\pm n}(t) \\ &+ \int_0^t N(t-s)\sum \lambda_{1n}\gamma_{\pm n}e_{\pm n}(s) + \lambda_{2n}\eta_{\pm n}s_{\pm n}(s)ds = 0. \end{aligned}$$

This implies

$$\sum \lambda_{1n}\gamma_{\pm n}e_{\pm n}(t) + \lambda_{2n}\eta_{\pm n}s_{\pm n}(t) = 0,$$

and so

$$\sum (\lambda_{1n}-\lambda_{11})\gamma_{\pm n}e_{\pm n}(t) + (\lambda_{2n}-\lambda_{11})\eta_{\pm n}s_{\pm n}(t) = 0.$$

Setting $\gamma^{(1)}_{\pm n} = (\lambda_{1n}-\lambda_{11})\gamma_{\pm n}$ and $\eta^{(1)}_{\pm n} = (\lambda_{2n}-\lambda_{11})\eta_{\pm n}$, we have

$$\begin{gathered} \sum \gamma^{(1)}_{\pm n}e_{\pm n}(t) + \eta^{(1)}_{\pm n}s_{\pm n}(t) = 0, \\ \sum \alpha^{(1)}_{\pm n}z_{\pm 1n}(t) + \beta^{(1)}_{\pm n}z_{\pm 2n}(t) = 0, \end{gathered}$$

where

$$\alpha^{(1)}_{\pm n} := \gamma^{(1)}_{\pm n} + \eta^{(1)}_{\pm n}, \quad \beta^{(1)}_{\pm n} := \frac{1}{\mu_n}(\gamma^{(1)}_{\pm n} - \eta^{(1)}_{\pm n}).$$

Similarly, we obtain

$$\sum (\lambda_{1n}-\lambda_{21})\gamma^{(1)}_{\pm n}e_{\pm n}(t) + (\lambda_{2n}-\lambda_{21})\eta^{(1)}_{\pm n}s_{\pm n}(t) = 0.$$

Thus for any $k \geq 1$, we can construct

$$\gamma^{(2k)}_{\pm n} = \prod_{i=1,2,\, j\leq k} (\lambda_{1n}-\lambda_{ij})\gamma_{\pm n}, \quad \eta^{(2k)}_{\pm n} = \prod_{i=1,2,\ j\leq k} (\lambda_{2n}-\lambda_{ij})\eta_{\pm n},$$

such that

$$\sum_{n>k} \gamma_{\pm n}^{(2k)} e_{\pm n}(t) + \eta_{\pm n}^{(2k)} s_{\pm n}(t) = 0. \tag{4.31}$$

Let $k = N_0$ in (4.31). Recalling that $\{z_{\pm 1n}(t), z_{\pm 2n}(t)\}_{n \geq N_0}$ is a Riesz sequence in $L^2(0,T)$, using (4.9), and noting assumption (2.4), we can deduce that $\gamma_{\pm n} = \eta_{\pm n} = 0$ for all $n > N_0$. Hence

$$\sum_{n \leq N_0} \gamma_{\pm n} e_{\pm n}(t) + \eta_{\pm n} s_{\pm n}(t) = 0.$$

The linear independence of the finite sequence $\{e_{\pm n}(t), s_{\pm n}(t)\}_{n \leq N_0}$ is easy to see; so we have

$$\gamma_{\pm n} = \eta_{\pm n} = 0, \ \forall n \leq N_0.$$

Thus, we derive the $\omega-$independence of $\{z_{\pm 1n}(t), z_{\pm 2n}(t)\}$. The proof is complete.

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Ti-Jun Xiao (corresponding author)
Shanghai Key Laboratory for Contemporary Applied Mathematics, School of Mathematical Sciences, Fudan University, Shanghai 200433, China
E-mail address: tjxiao@fudan.edu.cn

Zhe Xu
Shanghai Key Laboratory for Contemporary Applied Mathematics, School of Mathematical Sciences, Fudan University, Shanghai 200433, China
E-mail address: 13110180046@fudan.edu.cn